工程紊(湍)流力学

翟庆良 著

东北大学出版社
Northeastern University Press

工程紊(湍)流力学

ISBN 978-7-81102-751-8

9 787811 027518 >

定价：58.00元

工程紊(湍)流力学

翟庆良 著

东 北 大 学 出 版 社

· 沈 阳 ·

图书在版编目（CIP）数据

工程紊(湍)流力学 / 翟庆良著．—沈阳：东北大学出版社，2009.9
ISBN 978-7-81102-751-8

Ⅰ.工…　Ⅱ.翟　…　Ⅲ.湍流理论　Ⅳ.O357.5

中国版本图书馆 CIP 数据核字(2009)第 177425 号

内容简介

本书阐明湍流形成的机理，提出了“边层流”与“近壁流”理论。按不可压缩与可压缩流体运动，分别依直角坐标系、柱坐标系和球坐标系建立湍流连续性微分方程、动量微分方程、动量矩微分方程和能量微分方程。创立了非线性偏微分方程工程解法，得到确定理想与实际流体运动分界线的工程方法。

对直角坐标系下的湍流管道、进口段、明渠、边界层等问题，利用其对应的微分方程，解出其速度分布、涡旋产生地带、涡旋速度、涡旋直径和涡旋体积分数，使问题从本质上得到深层次的解决，也为其他有关学科的发展打下了理论基础。

本书适合湍流研究工作者以及与流体力学有关的老师及学生。另外，对水利、化工、冶金领域工程师、技术员也有一定的参考价值。

出 版 者：东北大学出版社
地址：沈阳市和平区文化路 3 号巷 11 号
邮编：110004
电话：024—83687331（市场部）　83680267（社务室）
传真：024—83680180（市场部）　83680265（社务室）
网址：http：//www.neupress.com
E-mail：neuph @ neupress.com
印 刷 者：葫芦岛日报社印刷厂
发 行 者：东北大学出版社
幅面尺寸：184mm × 245mm
印　　张：21.75
字　　数：529 千字
出版时间：2009 年 9 月第 1 版
印刷时间：2009 年 9 月第 1 次印刷
责任编辑：刘乃义　　　　责任校对：郎　坤
封面设计：唐敏智　　　　责任出版：杨华宁

ISBN 978-7-81102-751-8　　　　定　价：58.00 元

前　　言

本人自1957年由大连工学院（现为大连理工大学）水利系毕业后，被分配到东北工学院（现为东北大学）流体力学教研室任教，直至1990年10月离休，为本科生讲授“工程流体力学”课程；“文化大革命”后，为本科生讲授“三传理论”课程，为研究生讲授“理论流体力学”课程以及“多相流及其应用”课程。

教课之余，本人从事“多相流理论”的研究工作，多年来写出30余篇文章，其中，有3篇被收入第二、第三、第四届“全国多相流体力学，非牛顿流体力学，物理化学流体力学”学术会议论文集中。另外，《多相流体力学基本方程》一文，刊于1989年《东北工学院学报》第6期；《多相流运动学几个基本公式》一文，刊于1990年《东北工学院学报》第6期；《多相流运动微分方程》一文，刊于1990年《哈尔滨工业大学学报》增刊。

无论在教学上还是在“多相流”学术研究上，我均遇到湍（紊）流问题没有得到解决，而影响其他问题，无法深入研究的情况。所以，立志研究湍（紊）流问题。多年来，在我脑海中形成了几个问题，一起让我思索着，如：湍流形成的机理到底是什么？雷诺数能判别各种情况下的流场的流态吗？普兰特混合场理论作为研究湍流问题的理论，出发点对吗？利用时均值而得到的表达控制湍流运动的雷诺方程，能解决湍流问题吗？带着这些问题，我于1983年写过《管道紊（湍）流形成机理与判别流态新方法》一文，曾被1983年辽宁省力学学会年会选为宣读论文；在1990年写过《Study of the Turbulent Flow Equations》一文，被第四届国际湍流会议选为宣读论文，并纳入论文集中，会后又被于国外出版的《Flow Modeling and Turbulent Measurement》一书选入在111～119页上。

经过多年的努力，我终于于离休后的2001年10月开始动笔写本书，经过7年的时间，于2008年9月暂告定稿。

全书共13章。前2章为理论观点的出发点，第3章为无坐标系积分与微分方程，第4章为直角坐标系流体运动微分方程，第5章到第10章为工程湍流问题的研究解决方法，第11章为柱坐标系流体运动微分方程，第12章为球坐标系流体运动微分方程，第13章为近壁流理论观点。

由于作者水平所限，书中难免有不足之处，恳请有关专家、学者批评指正，本人将万分感激。

作　者

2008年9月28日

于东北大学

目　　录

第1章　边层流

1.1　问题的提出

以层流充分发展管段断面速度分布为例，断面平均速度为 u_0。由于流体有动力黏性系数 μ，紧贴壁面流体微团，接受壁面对其摩擦切力作用，微团平移动能全部转化为旋转动能，其转向依主流方向而成顺时针方向，先转动起来的微团，就带动相邻的微团成逆时针方向转动。这样形成类似主动轮与被动轮的关系，主动轮顺时针方向，被动轮逆时针方向，主动轮直径大，被动轮直径小，出现大小、方向相间，紧贴壁面就地转动的一层流体，如图1-1所示。

由于紧贴壁面相邻两微团就地转动方向相反，不能使流经它们表面的流体微团转动；但因为它们只就地转动，也对来流产生阻力作用，使其平移速度降低很多，形成一层只有平移运动的流层，图1-1中以平行线表示。微团平移层，其速度很小，因而对经过它表面速度大的流体微团产生阻力，此力仍然以剪力方式作用，故使平移运动的来流动能部分地转变为旋转动能，形成既有平移又有旋转运动的流层。依此作用下去，出现如图1-1所示的单一平移以及平移与旋转流层相间的流谱。这样，流区厚度可以通过断面平均速度 u_0 与断面速度分布曲线交点 c 所确定，如图1-2所示，过 c 点作平行线，其线下则是此流区范围，称之为边层流区。

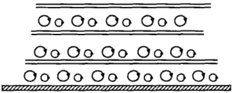


图1-1　边层流内微团运动示意图

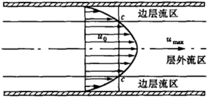


图1-2　边层流区与层外流区图

边层流区中各层流体微团成层状运动，各层流体微团互不混杂。边层流区之外的流区称为层外流区。

根据质量守恒定律，通过管道各断面流量应为常量，而边层流区的速度均小于平均速度 u_0，为保证断面流量不变，必须加大层外流区的流速。流体微团的旋转速度不能显示流量，所以只能加大层外区流体微团的平移速度，由此得出层外区流体微团均具有平移速度而无旋转速度，这也就是层外流区的流谱的特点。

能量守恒定律过去在管道流动中应用过，为保持能量守恒，一切能量损失由压力下降来

提供；在能量互相转化时，压能可以转化为位置，也可以转化为动能，如文德利水表，就是根据压能转化为动能的原理制造的。

现在遇到的问题是壁面对开始就具有断面平均速度 u_0 的微团，施以剪切力作用，使它变成就地旋转而失去平移速度。这当然可以从“一个物体的动能应该包括平移与转动两个动能，而且它们之间可以互相转化”理解为，平移动能全部转化为旋转动能。由质量守恒定律，平移速度加到管中去使它变为 $u_{max}=2u_0$。由此看来，壁面平移速度没有转化为旋转动能，而是改变位置，移到管中来了。经过这样的分析，就出现了能量缺口，为保持能量守恒，只有由压能下降来补上这个缺口。

通过上面的分析，压能不但在断面变化时可以与平移速度动能互相转化，而且在断面不变，在同一个断面上速度分布变化时也能与平移动能互相转化，这种转化是能量守恒、质量守恒、动能守恒、动量守恒互相制约的情况下而实现的转化。

这里说的动量守恒定律，只指断面速度分布是根据动量守恒推导出来的运动微分方程而解得的断面速度分布。

综上所述，层流边层流区的定义是，该区中平移与平移兼有旋转运动流层相间存在，各层之间流体微团互不混杂。层外流区的定义是，该区平移运动流层、各平移流层的微团互不混杂。层外流区所有流体微团只有平移运动而无旋转运动。

谈到这里，读者可能要问，有没有湍流边层流问题与湍流层外流问题？回答是有的，但本章不能谈，只有讲完第 2 章湍流形成机理后，才能把这个问题讲明白。

1.2 边层流、层外流与边界层的区别

边界层是将流场划分为理想流体与实际流体，以便分别应用理想流体与实际流体运动微分方程来研究它们。

边界仅应用于绕流物体与潜体运动的问题，它不能应用于明渠和管道流动。

而边层流、层外流可以应用于任何流场，也就是说，所有流场均存在着边层流与层外流区，即使边界层问题也存在着边层流与层外流区，但在研究边界层内的边层流区的划界上，必须依靠边界层方法解决实际流体与理想流体的分区问题，然后应用实际流体运动微分方程解决边界层内各断面速度分布问题。否则无法定出边层流的范围。

1.3 划分边层流、层外流的意义

边层流、层外流是任何流场中客观存在的，它是根据三守恒定律推导分析出来的，为今后研究流体运动提供方便。

层流边层流为层流转化湍流机理分析提供依据，湍流边层流为建立湍流运动微分方程提供了可能性，为传热传质、多相流理论分析提供了理论基础。

1.4　边层流范围划定方法

流场不同，划分边层流范围方法也有所差异。现分有压管道进口段、平板边界层和明渠三种流场加以讨论。

进口段边界层形成的机理：无论哪种流动情况，均有进口段或开始段流层形成问题，它们的形成过程和机理是一样的。现以平板层流边界成因来说明。

层流平板边界层成因说明：来流速度 u_0，具有动力黏性系数 μ，使它接受壁面对它的剪力阻力作用。流体微团由平移运动转变为就地旋转运动。由于流体是由微团组成的，旋转为顺时针的，必然影响相邻的微团作逆时针方向的旋转，结果壁面上方向相反，相间旋转层就地运动。如图 1-3 所示。

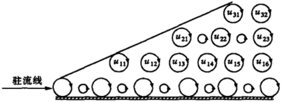


图 1-3　边界层形成机理示意图

顺时针旋转的微团直径大，因为它是直接由壁面剪切作用形成的；而逆时针旋转的微团直径小，因为它是由顺时针旋转带动而引起的，由于能量损耗，它的直径要小一些。

来流微团正对就地旋转层时，形成了驻线。当紧贴驻线上层来流微团，流经几个就地旋转微团阻力影响后，开始明显减速，为 $u_{11} < u_0$。u_{11} 前进，继续累积式地受到壁面上就地旋转微团的阻力作用，使其速度连续地下降，则 $u_{11} > u_{12}$。结果形成一层只有平移运动的微团，而且速度很低。如图 1-4 所示。为什么不旋转呢？

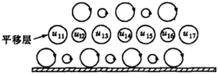


图 1-4　平移层形成原因示意图

壁面上就地旋转的相邻微团转向相反，使得对上层流体微团不能显示转动作用。又因为壁面上就地旋转的微团没有平移速度，对其上层平移速度来流仍然起阻力作用，因它是流体，而不是壁面，形成阻力小一些，结果形成一层只有平移速度很低的流层。

流体微团只有平移运动的流层，其速度低于其上层来流速度，因而对其上层流体起着切力作用。由于是流体，而且有运动，其切力比壁面对流体的作用小一些，结果使来流产生旋转，不能使其失去全部平移速度，出现一层既有平移又有旋转的流层。

根据板长不同，依上述推论，可以得到相应的不同层数的平移流层与平移兼旋转流层相间的边界层。

层流平板边界层内边层流范围划分方法(这里所谈的流动为定常流而且是不可压缩的流

体)：平板边界层为理想流，其速度为 u_0，沿平板长各断面上没有流量不变问题。

这里暂定已知层流边界厚度变化表达式以及层内速度分布函数，并依此将它们绘于图 1-5 中。

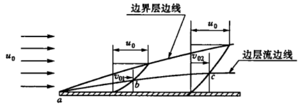


图 1-5　层流平板边界内边层流区示意图

计算出沿板长不同断面上层内的平均流速 v_{01}，v_{02}等。它们各自与所在断面速度分布曲线有一交点，如 b，c 等。连接 a，b，c 等点，得一曲线 abc，其线下就是边层流区，其外就是层外区。

有压圆管层流进口段边层流范围划分方法(这里的讨论限定为不可压缩流体的定常流运动)：理想流体与实际流体分界线，以及实际流体区速度分布规律，在第 7 章均已确定，根据它们可以在相应条件下，绘出理想与实际流体运动分界线，以及各断面的实际流体区的速度分布。管道流动特点是：流量沿流程各断面不变，它与各断面速度分布各有交点。将这些交点连成曲线，即 $abcd$ 曲线，其下则为边层流区，如图 1-6 所示。其外为层外流区，它又分为实际流体区与理想流体区，边层流边线与理想和实际流体分界线之间为实际流体层外流区；理想与实际流体分界线之外为层外流理想流体区。

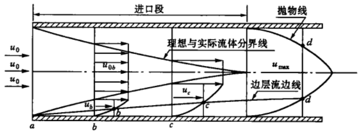


图 1-6　管道进口段边层流范围示意图

明渠层流充分发展段边层流划分方法：壁面对流体运动影响是有限的，边界层理论足以说明这一点，但明渠又与边界层流不同，它有自由面。按此特点将明渠分为低水深与高水深。低水深壁底影响可以到达自由水面，高水深壁底对流体运动的影响不能到达水面。

明渠，在工程上可以分为矩形、梯形等，关于不同断面上速度分布要在第 6 章讨论，这里仅仅定性地示意地绘出来，用以说明边层流区划定方法。

明渠的特点：有自由面，水深一定，流量就一定。绘出断面速度分布，以断面平均速度与其交点 K 作一平行于水面的线。如图 1-7 所示，该线以下为边层流区，其外为层外流区。

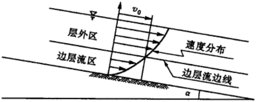


图 1-7　低水位明渠边层流划分示意图

高水深与低水深明渠有所不同，在高水深时，首先要确定渠底对流体运动的影响范围。这个范围可以通过求解进口段边界层曲线而得到。这里仅作已知定性处理。

高水深明渠边层流区划定方法如图 1-8 所示。

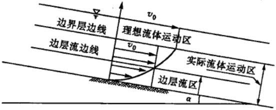


图 1-8　高水深明渠边层流划分示意图

高水深明渠边层流的特点是层外区分为两区：一是边层流边线与边界层边线之间的实际流体区；二是边界层边线与自由水面之间的区，这个区是理想流体区域。

1.5　湍流边层流

以上谈的均为层流边层流问题，同样也有湍流边层流问题。湍流边层流是在层流转化为湍流后而出现的。要把这个问题彻底弄清楚，必须研究完第 2 章。此处仅仅定性地概念性地讲述其定义及其与层流边层流的区别。

湍流边层流是指在湍流的流场情况下，近壁存在的边层流区，其“边”就是指近壁，其“层”是指流体微团运动是成层的。各层间的流体微团互不混杂。湍流边层流厚度随湍流强度增加而变薄。无论怎么薄，至少存在三层，如图 1-9 所示。即紧贴壁面的，就地旋转，相邻两个微团转向相反；贴于其上的流体微团是只有平移运动的流体微团；此层之上是既有平移又有旋转运动的微团，相邻流体微团间转向也相反。凡是顺时针转动的为主动微团，直径比较大；逆时针转动的为被动微团，直径比较小。所有微团运动，无论是否有旋转，均在所在流层中运动，没有互相混杂的现象。从这个意义上来讲，边层流属于层流。

湍流边层流之外，也有层外流区。它与层流时层外流区有本质的差别。根据湍流的相应强度，有相应的具有垂直于主流方向分速度的涡旋微团，从边层流边界上不断地进入到层外流区，形成分散相，而原来的只有平移运动的微团则为连续相。这样，湍流层外流区就类似

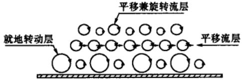


图 1-9 湍流边层流最薄时示意图

于多相流。

湍流边层流区划定方法：划定步骤与层流边层流区划分一样，只是由于湍流断面速度分布比层流时均匀，平均速度 v_0 与最大速度相比要小得多，由此得出边层流层厚度就薄得多。现分几种情况简介如下。

有压湍流充分发展管段边层流划分方法，仍然先根据断面上速度分布与断面平均速度的交点，绘平行于管壁的直线，线下则为边层流区，如图 1-10 所示。

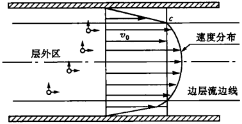


图 1-10 充分发展湍流管段边层流划定示意图

明渠分为水深浅与水深大两种情况，这样分是因为壁底对湍流影响也是有一定范围的，水浅时边层流产生的涡旋可以斜冲到自由水面，而水深大的则不能到达水面，而是走一段距离就消失了。如图 1-11 所示。

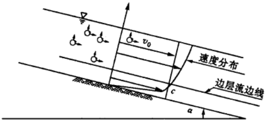


图 1-11 水浅边层流示意图

水深大的情况，层外流区分为两部分：一部分为实际流体区，一部分为理想流体区。而涡旋运动不能进入理想流体区，如图 1-12 所示。

湍流平板边层划分情况如下。

湍流平板前缘总是存在一点层流区，所谓湍流平板，是指能成为湍流的部分。

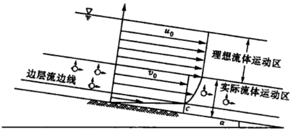


图 1-12　水深大边层流示意图

湍流平板的特点之一是没有固定流量的限制。平板各断面的速度分布是各不相同的，因此必须按各断面速度分布计算出流量，然后求其平均速度 v_{01}，v_{02}，v_{03}等。将它们与各自的断面速度分布交点连成曲线，如图 1-13 所示的 *abcd* 曲线。曲线以下为边层流区；其外则为层外区，层外区又分为实际流体区与理想流体区。而边层流区界面产生的涡旋，只能在实际流体中运动，根本冲不到理想流体区。

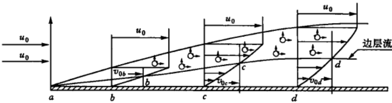


图 1-13　湍流平板边界层中边层流示意图

有压湍流管道进口段边层流划分情况如下。

管道流动的特点是沿管道各断面流量不变。只要绘出几个进口段各断面速度分布，将其各自与断面平均速度的交点连成曲线，如图 1-14 中 *abcd* 曲线所示。该曲线以下为边层流区；其外为层外区，层外区Ⅰ为实际流体区，层外区Ⅱ为理想流体。涡旋微团在边层流界面上间歇地产生后，斜冲入层外流Ⅰ区，在它运动到理想与实际流体分界面处，因其存在条件速度梯度消失而消失，实际是在外界条件改变下，将旋转动能逐渐转化为平移动能。

以上所述中，凡是涉及理想与实际流体分界线、速度分布问题，将在以后各章有关节中分别讨论；关于涡旋运动的形成，将在湍流形成机理一章有关节中加以讨论。

边层流的提出，是运用物理学中三守恒定律对流体运动进行综合分析得出的，因此它的理论基础稳固。

1.6　梳理几个概念

边层流与层外流的提出，使得层流运动的真实流谱得到揭示。层流运动，实际是由边层流区与层外流区两部分组成的。

边层流区是微团平移与微团平移兼旋转流层相间存在的流层。两相邻的旋转微团，转向

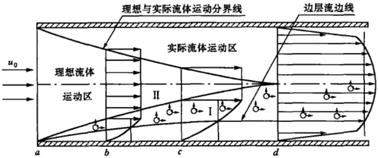


图 1-14　有压湍流进口段边层流区示意图

相反；顺时针转动的微团直径大，逆时针转动的微团直径小。

层外流区，微团成层地只有平移运动。该区虽有速度分布，使得速度梯度存在，但不能产生涡旋运动。

湍流流场也有边层流与层外流两个区域。不过，边层流区厚度比层流时薄得多，而且其层数也随湍流强度加大而变少，但最少也存在着三层。

边层流界面间歇地产生具有垂直于主流方向的速度涡旋，不断地输送到层外流区。

层外流区除具有平移运动微团外，还有相当数量的运动涡旋，而它们又是根据边界条件而消失的，即旋转动能变化为平移动能。

这里所谈，仍然是概念、定性地加以说明。要彻底弄清楚，得在讲完湍流形成机理之后。

1.7　连续性问题

流体本身存在着动力黏度 μ，它使得流体运动呈现连续性，因此可以用数学中的连续函数作为工具来描述流体运动要素的变化规律。

流体是连续介质，是由它本身特性决定的，而不是假设。由于宏观研究流体运动的需要，刻画其组成的最小单元是质点，各质点之间无空隙地紧密相连，一个质点运动，相邻的也跟着动。每个质点都有自己的运动要素值，如压力、速度等。

客观上来讲，流体运动有变形、旋转等现象。为了描述它们，采用比质点大得多的流体微团作为研究对象。微团由几个或若干个流体质点组成，其中各质点可有不同的速度，使整个微团出现变形、旋转与整体平移等现象。根据具体情况的不同，微团可以只有平移，或只有旋转，或兼有之。

无论是质点还是微团，它们在流体运动中都是紧密相邻而无缝隙的。

边层流概念的提出是不会影响连续性的。以管道层流运动为例来具体分析一下。研究断面上速度分布的连续性，是以平移速度为对象的。在壁面上，就地旋转微团平移速度为 0，与其上相邻的流层，具有较低的平移速度，其值不为 0，但接近于 0，没有突变问题；在平移流层之上，是一层既有平移又有旋转的流体微团，而我们只研究平移速度，该速度大于其

下层的平移速度，变化是渐变的。因此，层流管道断面速度分布是连续变化的，可以用连续函数来描述它。

湍流管道边层流与层外流是否对断面速度分布的连续性产生影响呢？从平移速度角度来看，边层流与层流边层流一样，只是其厚度薄而已。速度分布仍然呈连续性。

层外流就不同了。由于从边层流界面上间歇地产生相当数量的涡旋微团，斜冲进来，扰乱原来成层的平行流层，而成为湍流。涡旋微团有沿主流方向的平移速度，但其值较小；也有垂直于主流方向的速度，还有旋转运动。

涡旋微团旋转不能显示流量，只有它平行于主流方向分速度时，才能体现流量，但它的速度比周围的只有平移运动的微团速度低。因此，管道湍流时，断面速度分布如图 1-15 所示。说明断面速度分布已不是连续的规律，无法用数学函数来描述它，另外，它除表明湍流特征之外，在工程上或与其他学科关系上也没有什么可用的价值。

湍流本身或与湍流有关联的学科均重视的问题是湍流边层流界面上，在相应的湍流度下产生多少数量的涡旋，其直径多大，速度如何。要解决这些问题，关键是要知道边层流界面上速度梯度，要求得速度梯度，就必须有一个断面速度连续地分布，如图 1-16 所示，并使这个断面速度分布计算出的流量与图 1-15 所示速度分布算得的流量相等。

由于已知湍流边层流内速度分布满足连续性，既然这个层的界层产生涡旋，就设想有一个对应产生此涡旋的速度梯度。依此建立微分方程，求解出管道断面速度分布，就是如图 1-16 所示的图形。详细讨论需在学完湍流形成机理这章之后，才能弄清楚处理问题的思维方法。

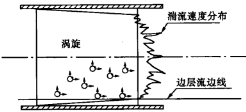


图 1-15　湍流管道速度分布示意图

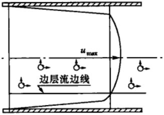


图 1-16　湍流管道速度模拟图

本章所谈问题，是第 2 章研究问题的出发点，也是以后各章研究分析问题的基础。它与第 2 章对全书讨论及处理的问题，起着引导作用。

第 2 章 湍流形成机理与湍流运动

2.1 前 言

本章以边层流的概念为出发点，开始研究湍流形成机理与湍流运动特点。由于外界与边界条件的不同，因此，湍流形成机理与湍流状态的特征也有所差异。下面对有压管道流动充分发展段、管道流动进口段、平板边界流动、明渠流动几种情况分别加以讨论。

本书讨论的问题是基于流体力学中有关概念、公式、定理进行的，还要用到物理学中的一些力学定律，并结合具体的边界条件，综合地进行分析与讨论。

2.2 层流转变为湍流的过程

由流体力学雷诺实验可知，当沿管轴红色流线破碎时，说明整个管道已形成了湍流。这表明，在流动中，有的微团除沿主流方向运动外，还有垂直于主流方向运动的分速度。这个微团是怎么产生的呢？从力学的观点来分析，这个微团一定受到垂直于主流方向的力。当这个力足以克服微团的惯性力时，它就出现了这个分速度。

2.2.1 有压管道充分发展段

管道壁面无论如何光滑，总是有相对的高低不平的，有的壁面本身就存在着高度大小不同的粗糙分布，见图 2-1。

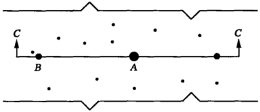


图 2-1 管内壁粗糙分布示意图

将图 2-1 作 $C—C$ 剖面，示于图 2-2 中，可以看清边层流厚度覆盖粗糙情况。为说明方便，将粗糙 A 处流动放大，并标出受力情况。见图 2-3。

边层流中，紧贴壁面一层是就地旋转微团，两相邻微团转向相反；顺时针转动的微团为主动微团，直径大；逆时针转动的微团为被动微团，直径小。紧贴这层之上的一层是只有速度很低的平移运动微团。在它之上的一层是既有旋转又有平移运动的微团层。同样，相邻的

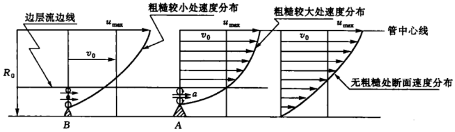


图 2-2　边层流与粗糙分布剖面示意图

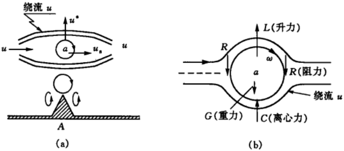


图 2-3　粗糙 A 顶部微团 a 周边流动情况与受力示意图

两个微团转向相反，顺时针的为主动，逆时针的为被动，主动微团直径大，被动微团直径小。依此推下去，一直推到相应的边层流边界上为止。

当雷诺数加大后，断面平均速度 v_0 与断面速度分布均变化，其交点 C 也变化，其变化结果使边层流厚度变薄。在某个雷诺数下，这个厚度恰好接衔壁面粗糙最高顶部，如图 2-2 的粗糙 A 处所示。为研究方便起见，将它取出放大，如图 2-3 所示。

粗糙 A 顶上数第三层，正好为边层流界面，其微团 a 既有平移又有顺时针旋转；并且其周围有绕流，而上边流速高，下边流速低。由伯努利方程可知，上边压力小，下边压力大，则有上下压力差。

由流体力学可知，在流体中的物体有旋转，周围有绕流，则这个物体一定受到升力作用。结合这里的微团 a，其升力可以表示为

$$L = \pi\rho(u - u_s)d_s^3\omega \tag{2-2-1}$$

式中：u——微团 a 的绕流速度；

u_s——微团的平移速度；

d_s——微团的直径；

ρ——流体的密度；

ω——微团的旋转速度。

式(2-2-1)中，微团旋转速度 ω 实际上由两部分组成，即

$$\omega = \omega_1 + \omega_2 \tag{2-2-2}$$

ω_1 是管道断面平均速度 v_0 在壁面引起的剪切力作用下，使其转变为旋转强度值，其表达式为

$$\omega_1=\frac{1.5g}{r_s}\sqrt{v_0^2-u_s^2} \tag{2-2-3}$$

式中：r_s——微团的半径；

u_s——微团所在位置的速度，对于管道而言，其计算公式如下：

$$u_s=\frac{\gamma h_\xi}{\Delta\mu l}(R^2-r^2) \tag{2-2-4}$$

式中：R——管道的半径；

μ——流体动力黏度；

h_ξ——管道 l 长损失水头；

r——流体相对密度。

式(2-2-4)中，r 由下式计算，式中符号如图 2-4 所示：

$$r=R-(\Delta+5r_s) \tag{2-2-5}$$

图 2-4 中，s 由下式表示：

$$s=\Delta+2.5d_s=\Delta+5r_s \tag{2-2-6}$$

式中：Δ——壁面粗糙 A 的高度；

r_s——流体旋转微团半径。

ω_2 为断面速度分布在旋转微团 a 处由速度梯度引起的涡旋强度，对充分发展管道而言，其计算公式为

$$\omega_2=\frac{\gamma h_\xi}{\Delta\mu l} \tag{2-2-7}$$

由图 2-4 可以看出，粗糙 A 的高度对流线产生变弯曲的影响，使得流体微团 a 受到一个瞬时离心力的作用。它由下式计算：

$$c=\frac{mu_s^2}{\Delta+5r_s} \tag{2-2-8}$$

式中：m——流体微团的质量。

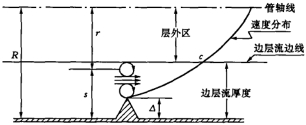


图 2-4　有关公式符号说明

综上所述，流体微团 a 受到升力、瞬时离心力以及向上的微弱压差作用。当这些力的合力足以克服微团本身的惯性力时，就会使它产生垂直于主流方向横向的运动分速度。它一旦运动，就会受到周围流体运动的阻力。由流体力学可知，运动物体的阻力可分为 3 个区：一是斯托克斯区；二是阿连区；三是牛顿区。那么，涡旋微团横向运动时，属于哪个区呢？通过推导的公式可以验证，只有运用牛顿区阻力公式得到的结果才有物理意义。对于涡旋微团而言，其牛顿区阻力公式为

$$R_s = 0.055\rho d_s^2 u_s^{*2} \tag{2-2-9}$$

式中：R_s——涡旋微团受到的阻力；

d_s——涡旋微团的直径；

u_s^*——涡旋微团横向运动速度。

根据图 2-3 中涡旋微团 a 受力分析，利用牛顿第二定律，建立其运动微分方程为

$$\frac{\mathrm{d}u_s^*}{\mathrm{d}t} = 6(u - u_s)\omega - g - \frac{0.33}{d_s}u_s^{*2}$$

$$u_s^*\big|_{t=0} = 0$$

令

$$k_1 = 6(u - u_s)\omega - g \tag{2-2-10}$$

$$k_2 = \frac{0.33}{d_s} \tag{2-2-11}$$

解出

$$u_s^* = \frac{\sqrt{k_1}\left(e^{2t\sqrt{k_1k_2}} - 1\right)}{\sqrt{k_2}\left(e^{2t\sqrt{k_1k_2}} + 1\right)} \tag{2-2-12}$$

对涡旋横向运动速度的分析，当 $6(u - u_s)\omega$ 大于重力加速度 g，$u_s^* = 0$，即相等时，$u_s^* = 0$，这从式(2-2-12)可以看到。

当 $t>0$ 的一个瞬间时，ω 值比较大，使得升力大于重力，使它冲出所在的流层。一旦它离开所在流层，即离开边层流的界面后，其 ω 值只剩下 ω_2，而 ω_1 已经不存在，使得它的升力下降；由式(2-2-7)可以看出，随着 ω_2 上升，γ 值变小，当 $\gamma = 0$ 时，$\omega_2 = 0$，即在管轴线上 $\omega_2 = 0$。

实验室中，雷诺实验管径比较小，当流量达到某一个值时，在边层流界面上，已经有横向运动的涡旋出现，不过它运动的距离小，而是边层流附近的位置，但这部分流动已经破裂为湍流。当实验管中平均速度大到一定程度时，在边层流界面上引起的升力足够大，大大超过重力加速度 g，使横向运动的涡旋可以冲到管轴线，这时整个管道成为湍流。

在某一个流态下，边层流界面首先变为湍流，如图 2-5 所示。

当微团 a 向上运动，离开所在流层时，就说它冲断其顶部只有平移运动的流层，分为左右两部分，右边部分由于惯性作用，继续向前方运动；而原微团所在位置左右两侧微团平移速度比较低，右边惯性作用向前运动，只有左边微团可以充填微团 a 离开后的空穴，但

是它的速度比其层平移速度低，这样，只有微团 a 冲断的平移流层左边的流体下冲空穴，如图 2-5(b)所示。

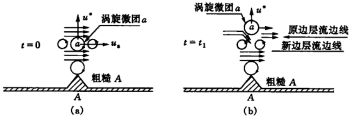


图 2-5　边层流界面首先变为湍流说明图

这种现象说明，边层流界面上方的平移流层中微团冲入边层流界面，使其失去原来成层运动现象，这一出一进说明各层之间有微团交换。说明边层流界面附近首先成为湍流区，使得边流层厚度变薄。

2.2.2　有压管道进口段

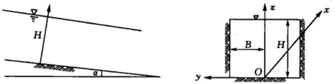
进口段的特点是管中有一锥形理想流体运动区。层流是实际流体，因此层流变为湍流只有在实际流体的层流运动中进行，不会发展到理想流体区域。

进口段层流转变为湍流的机理与充分发展段层流转变为湍流一样。

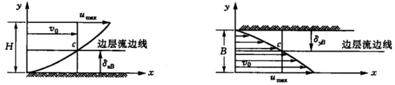
2.2.3　明　渠

明渠的特点是有自由面，因而应分为水深与水浅两种情况分别讨论其层流转化为湍流问题。水浅时，湍流可以到达自由面；而水深时，湍流只有在距渠底、渠侧壁一定范围内出现。渠中间为理想流体，不会出现湍流。

层流矩形明渠断面速度分布如图 2-6 所示。



(a) 矩形明渠断面与坐标选定示意图



(b) $y=0$ 时，xOz 平面边层流边线示意图　　(c) $z=H$ 时，xOy 平面边层流边线示意图

图 2-6　层流矩形明渠断面速度分布示意图

渠底边层流界面处，流经粗糙最高顶部时，顺时针旋转的微团，在瞬时离心力帮助下，在升力作用下，使它产生横向运动，冲断其上部只有平移运动的流层；被冲断左边的流层微团，顺势充填刚斜上升的微团空穴。这就说明原边层流界下移，原边层上边一部分层流层被破坏，这就是湍流出现的特征。

涡旋微团运动速度，在 x 方向为 u_x，在 y 方向为 u^*_{szy}，其表达式为

$$u^*_{szy}=\sqrt{\frac{6(u-u_{sy})\omega_y-g}{\dfrac{0.33}{d_{sy}}}} \tag{2-2-13}$$

$$u_{sy}=\sqrt{u^2-\frac{2}{5}r^2_{sy}\omega^2_y} \tag{2-2-14}$$

式中：u^*_{szy}——ω_y 引起的涡旋微团穿过 z 平面的速度；

u_{sy}——ω_y 引起的涡旋微团在 x 方向的速度；

d_{sy}——ω_y 引起的涡旋直径。

渠底边层流界面，在某一个流态下，产生涡旋横向运动，如图 2-7 所示。

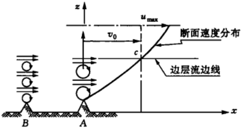


图 2-7　渠底边层流产生湍流示意图

渠侧壁边层流界面产生横向(y 方向)运动涡旋，如图 2-8 所示。

在侧壁边层流界面上产生横向运动涡旋，其速度表达式为

$$u^*_{syz}=\sqrt{\frac{6(u-u_{sz})\omega_z}{\dfrac{0.33}{d_{sz}}}} \tag{2-2-15}$$

$$u_{sz}=\sqrt{u^2-\frac{2}{5}r^2_{sz}\omega^2_z} \tag{2-2-16}$$

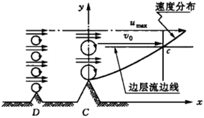


图 2-8　侧壁边层流界面产生横向运动涡旋示意图

式中：u^*_{syz}——ω_z 引起的涡旋微团穿过 y 平面的速度；

u_{sz}——ω_z 引起的涡旋微团在 x 方向的速度；

d_{sz}——ω_z 引起的涡旋微团直径。

2.2.4 平板边界层

层流平板边界层转变为湍流的特点是二维二元流动，其在 x 方向的速度分布如图 2-9 所示，其在 y 方向的速度分布如图 2-10 所示。

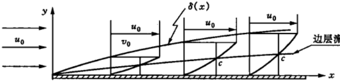


图 2-9　层流边界层内 $u(x,\ y)$ 分布示意图

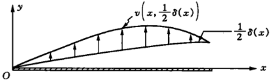


图 2-10　层流平板边界层内 $y=\frac{1}{2}\delta(x)$ 时 $v(x,\ y)$ 分布示意图

层流平板边界层内涡旋强度表达式为

$$\omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \tag{2-2-17}$$

x 方向速度梯度，引起涡旋 a 受到升力的分力作用，为 l_x，形成的分速度为

$$v_s^*=\sqrt{\frac{6(v-v_s)\omega_z^+}{\frac{0.33}{d_s}}} \tag{2-2-18}$$

y 方向速度梯度，引起涡旋 a 受到升力的分力作用，为 l_y，形成的分速度为

$$u_s^*=\sqrt{\frac{6(u-u_s)\omega_z^-}{\frac{0.33}{d_s}}} \tag{2-2-19}$$

式中：

$$\omega_z^+=\frac{1}{2}\frac{\partial v}{\partial x}$$

$$\omega_z^-=\frac{1}{2}\frac{\partial u}{\partial y}$$

$$d_s=\sqrt{\frac{10v}{\omega_z}} \tag{2-2-20}$$

如图 2-11 所示，涡旋 a 是二维二元变化，它在 x 方向的速度为 $u_s(x,\ y)$ 与 $v^*(x,\ y)$ 之和，在 y 方向的速度为 v_s 与 u_s^* 之和。由于 y 方向速度的作用，一旦层流转变为湍流，其边界层厚度要比层流边界厚度大得多。

像前边所谈问题分析一样，层流平板边界层内，首先发生湍流的地方也是在边层流界面

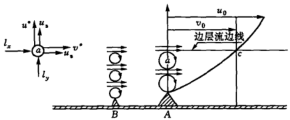


图 2-11　边层流界面微团 a 运动分析示意图

上。在某一个来流速度 u_0 下，具有顺时针旋转的微团 a，经过壁面粗糙最高的点时，由于该处形成的速度梯度大，产生的升力也大，加之粗糙形成的离心力，克服微团 a 的惯性阻力，产生垂直于 x 方向的分速度 u_s^*，使它脱离所在流层，加上其原有 y 方向的速度 v_s，它立刻冲破边层流外的一些平移流层；而其留下的空穴，由冲破平移流层左边平移微团充填。这样，原有的边层流界面被破坏，将其上平移流层也相应地冲断一些，成为一个湍流地带。

2.2.5　小　结

① 实际流体运动区内，存在着边层流区，首先这个界面上就存在发生湍流的地点。

② 层流转变为湍流，开始是局部的，也就是说，层流中有湍流出现在近壁的边层流之外，当流场中平均速度加大时，达到一定程度，湍流才能扩散到整个层流流场，使其成为湍流。

③ 湍流流场在紧贴壁面总有边层流存在，不过很薄而已，它是保证湍流状态的条件。

④ 当外界与边界条件在某一情况之后，层流与湍流是相伴而在的。

⑤ 判别层流中某地方是否为湍流，看该地方有没有垂直于主流方向且具有横向运动分速度的涡旋存在。

⑥ 湍流的范围应该用横向运动涡旋所能达到的地方为界限。

2.3　湍流状态

第 2.2 节讨论层流中出现湍流地带，本节研究湍流状态如何保持；某一点出现涡旋横向运动的频率 f 如何计算；当流动条件改变时，湍流如何发展；涡旋场脉动速度是怎么产生的，其实质是什么；脉动压力是否湍流均存在，什么样的湍流场出现脉动压；湍流场中流线的状态如何。所有这些问题，分别对有压充分发展管道、有压管道进口段、明渠、平板边界层四种情况加以讨论。

2.3.1　有压充分发展管道

在某一个雷诺数下，壁面最高粗糙，恰好在边层流界面以下附近，开始产生涡旋上升，当流态固定时，它何时产生第二次上升的涡旋呢？也就是说，这个粗糙 A，如图 2-12 所示，它的产生上升涡旋频率是什么呢？

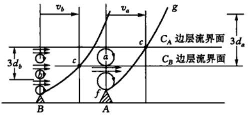


图 2-12　湍流发展示意与频率规定方法

分析认为，涡旋产生点是在点速度梯度下形成的。或者说，通过该点的速度分布是连续的，而且这个连续应有一定的范围。图 2-12 中，最高粗糙 Δ_A 处，其顶部边层流界面 C_A 以下原来就是连续的速度分布；所不同的是，边层流界面 C_A 以上，因为涡旋微团 a 脱离层流层，斜冲而上，造成其上的平移流层连续性破坏，速度分布失去连续性，不能形成速度梯度，也就不能在原处产生涡旋微团 a。当涡旋微团 a 向上走出 $3d_s$ 的距离后，则能使速度分布曲线 $\overset{\frown}{fg}$ 恢复原状。这样，在原处可以重新产生涡旋微团 a。其发生的频率为

$$f=\frac{u_s^*}{3d_s} \tag{2-3-1}$$

式中：u_s^*——涡旋微团垂直于主流方向的分布速度；

d_s——涡旋微团的直径。

当管道流速加大后，断面速度分布改变，平均速度也加大。此刻原来边层流界面已消失；新的边层流界面通过 C_B，即通过粗糙次高 Δ_B 的顶部。这时，流经其顶部具有顺时针的旋转微团 b，受到升力作用，使它达到克服其惯性阻力作用，开始产生横向运动，斜冲到层外流区，如图 2-12 所示。粗糙 Δ_B 的数量比 Δ_A 多，产生的涡旋微团数量也多，说明管道湍流程度加强，涡旋直径变小。那么，粗糙 Δ_A 还会产生横向运动的涡旋吗？分析可知，它已进入边层流以外的层外区，该区由于存在相应数量来自边层流界面处产生的具有横向运动分速度的涡旋，破坏了断面速度分布的连续性，根本无法形成连续的速度分布梯度，所以就不会再产生涡旋；另一方面，它已失去边层流界面以下形成 ω_1 的条件。同理推下去，管道愈加大速度，近壁区速度梯度也加大；边层流厚度变得也愈薄；接近边层界面的粗糙愈多，产生横向运动的涡旋数量也愈大，其直径也愈小，结果管道湍流强度也愈大。

湍流实质上不存在脉动速度。在湍流管道中，在某一点连续测沿轴向速度，其结果如图 2-13 所示。

图 2-13 说明，任一点速度时均值是常数，它满足外界条件不变情况下出现定常流的结论。但时间较短，某点速度是随时间变化的。为什么会出现这种情况呢？原因是在湍流管道中，在某一个流态下，有相应数量来自边层流界面的涡旋，它们是分散相，既有平移速度又有旋转速度，其平移速度有垂直于主流与平行于主流两个方向。测速仪只能测流体微团的平移速度，当流到测点是平移运动的微团时，其值则高；当流到测点是涡旋微团时，其值则低，而且不同微团涡旋平移速度也不同，结果如图 2-13 所示。各涡旋大小不同，其平移速

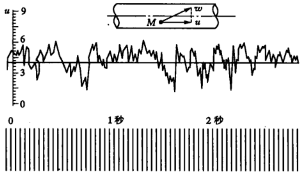


图 2-13　湍流管道轴向速度变化

度也不同，而且均低于相应的连续相平移速度。

在湍流管道中，测 y，z 方向的速度，也会出现脉动现象。由于连续相在这两个方向没有速度，当没有涡旋微团经过测点时，速度为 0；只有当涡旋微团经过测点时，才出现速度。由于涡旋微团是分散相，而且平移速度各涡旋不同，测出的速度有时为 0，有时有值，各值不同，出现了脉动现象。

湍流管道中，流线不是光滑曲线，而是多变的折线，如图 2-14 所示。当 $t=t_1$ 时，通过 A 点，涡旋微团的平移速度比较小，方向为斜上；在它矢径上点 1 处，正好是只有平移运动的微团，其速度是水平的，其值较大；在其矢径上点 2 处，又是斜向上方的涡旋微团，其速度值较小；在其矢径上点 3 处，是由对面壁面边层流界面送来的斜向下涡旋微团，其速度值较小；在其矢径上点 4 处，是水平运动的无旋微团，其速度值较大。当 $t=t_2$ 时，在 B 点，同样也可以示意地绘出通过它的瞬时流线。

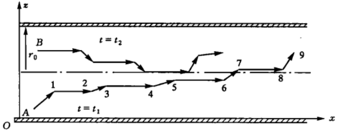


图 2-14　湍流管道流线示意图

由流体力学可知，流线是光滑曲线。光滑曲线出现的条件是流体微团都是连续的，这对层流运动是存在的，所以在层流运动中，可以绘出光滑的流线来；而对湍流运动而言，其流场中存在着相应数目的大小不同的涡旋微团运动，它们是分散相，而只有平移运动的微团是连续相，但由于它被分散相散布其中，无法绘出连续的光滑曲线的流线，而只能绘出连续不光滑的折线。

湍流脉动压力是否存在？在什么条件下存在？在什么条件下不存在？具体问题应当具体

分析。在有压湍流管道中充分发展管段上，管壁有脉动压力；而在其进口段管壁上，就没有脉动压力。认为湍流场均存在脉动压力 p'，脉动密度 ρ'是不符合实际的，任何湍流场均有速度脉动现象。

有压湍流充分发展管段壁面脉动压力 p'是怎么产生的呢？因为是圆形壁面，所以贴近壁面均为边层湍流。在某一断面平均速度 v_0 下，壁面上最高粗糙 Δ_1 临近边层流界面，开始产生横向运动涡旋微团。由于在一般情况下这批粗糙数量是较少的，断面速度分布产生的速度梯度也小，因此产生的涡旋微团直径大，沿径向走的距离也短，根本达不到管轴线位置，涡旋消失，壁面上压力平稳。

当平均速度 v_0 继续加大时，欠高粗糙 Δ_2 开始临近下移后的边层流界面，开始产生横向运动涡旋。由于 Δ_2 粗糙数量多于 Δ_1，产生的涡旋数目也多；加之在边层流界面处，速度梯度加大，产生的涡旋直径也变小；其沿径向运动距离也加大，可以冲过轴线，有的到达对面壁面，并对壁面形成微弱的冲击，壁面上压力开始产生波动。

当平均速度 v_0 连续加大时，壁面更小的粗糙 Δ_3 临近两次下移后的边层流界面，开始产生横向运动涡旋。由于 Δ_3 数量多于 Δ_2，产生的涡旋数量也多于前者；此外，速度梯度也加大，产生的涡旋直径变小，其径向速度转化为壁面上瞬时压力，而消失其旋转动能，剩下其平移速度。结论是：管壁上所谓的脉动压力，就是横向运动的涡旋对壁面的冲击力。

2.3.2 有压管道进口段

此处仅对有压管道进口段的特殊情况加以说明。进口段有理想的流体区，湍流运动不会扩展到理想流体的运动区；边层流界面输送的横向运动涡旋，其径向运动只能在实际流体区进行，不能进入中间理想流体区，当然也就无法冲到对面壁上，因此管壁上不会呈现脉动压力现象。由于边层流界面输送的涡旋只能在实际流体区运动，而实际流体区中间又是锥形理想流体区，所以涡旋微团消失只能在理想流体与实际流体分界面上或实际流体区内，沿管圆周内壁的边层流界面产生的涡旋运动不能穿过管轴线而达到对面实际流体区，所以其流线形状如图 2-15 所示。

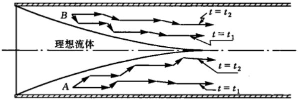
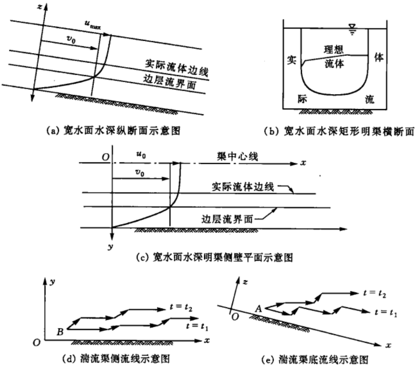


图 2-15 流线形状示意图

2.3.3 明 渠

这里不讨论明渠进口段，仅谈充分发展段湍流情况；只谈矩形明渠，分为水面宽水也深(如图 2-16 所示)与水面窄水也浅两种情况来讨论。

水面宽水也深情况如图 2-16 所示。其情况是，涡旋分别产生在渠底与侧壁边层流界面



(a) 宽水面水深纵断面示意图　(b) 宽水面水深矩形明渠横断面

(c) 宽水面水深明渠侧壁平面示意图

(d) 湍流渠侧流线示意图　(e) 湍流渠底流线示意图

图 2-16　矩形明渠充分发展段湍流情况

上；脉动速度现象分别发生在渠底与侧壁两个实际流体运动区，前者为 x 方向，后者为 y 方向；渠底边层流界面产生的涡旋斜向上运动，当升力与重力加速度引起的重力近似相等时消失，侧壁产生涡旋斜向下游运动，当达到理想流边线消失，即在没有速度梯度的地方消失；流线形状、湍流渠底如图 2-16(e)所示，渠侧壁如图 2-16(d)所示。

水面窄水也浅的矩形明渠如图 2-17 所示，没有压力脉动现象；在渠底及侧壁边层流界面上分别产生涡旋运动，引起速度在 x，y，z 三个方向有脉动。涡旋受重力的影响，当它受力与升力平衡时，则消失；不受重力影响，运动到中间速度梯度为零处，则消失；流线形状是多变的。

图 2-17(a)是示意立体分布，图 2-17(b)是断面上速度分布。

2.3.4　湍流平板边界层情况

流体运动属于二维二元问题，其速度分布 $u(x,\ y)$，$v(x,\ y)$如图 2-18 所示。其形成的涡旋强度为

$$\omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \tag{2-3-2}$$

式(2-3-2)可分为

$$\omega_z = \omega_z^+ + \omega_z^- \tag{2-3-3}$$

$$\omega_z^+ = \frac{1}{2}\frac{\partial v}{\partial x} \tag{2-3-4}$$

$$\omega_z^- = \frac{1}{2}\frac{\partial u}{\partial y} \tag{2-3-5}$$

在边层流区，流体微团受壁面摩擦力作用，将该断面平均速度一部分转为旋转运动，而且是顺时针方向，以 ω_1 表示。当涡旋微团 ω_1 经过边层流界面时，正好在粗糙最高处的顶部，这时该处 ω_z^- 也最大。在 ω_1 与 ω_z^- 同时作用下，微团产生 y 方向的升力；同时在 ω_z^+ 作用下，产生 x 方向的升力，克服阻力后，产生斜向上的运动，其值为

$$v_s = \sqrt{u_s^{*2} + v_s^{*2}} \tag{2-3-6}$$

速度脉动现象，当测 x 方向 $u(x, y)$ 的速度时，正好测点是涡旋微团，它在 x 方向的速度为

$$\tilde{u}_s = v_s^* + u_s \tag{2-3-7}$$

$\tilde{u}_s < u$，因此出现速度在 x 方向脉动；当测 y 方向速度 $v(x, y)$ 时，正好测点是涡旋微团，其在 y 方向的速度为

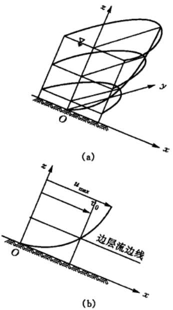


图 2-17　湍流矩形明渠水面窄水深浅速度分布示意图

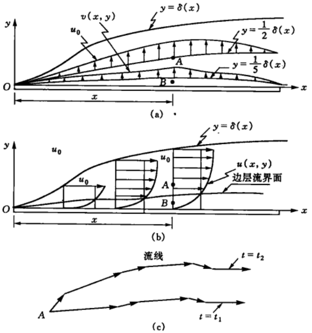


图 2-18　湍流平板边界层示意图

$$\tilde{v}_s = u_s^* + v_s \tag{2-3-8}$$

式中：u_s——涡旋在 x 方向的平移速度；

v_s——涡旋在 y 方向的平移速度；

u_s^*——涡旋在垂直于 x 方向的分速度；

v_s^*——涡旋在垂直于 y 方向的分速度。

涡旋运动到平板中间向下游方向后，重力与升力近似相等，同时又因为中间以后的 $v(x,\ y)$分布形成与前部速度梯度相反，成抵消作用，这样，涡旋失去 u_s^* 与 v_s^* 而消失，成为平移运动微团。流线特点如图 2-18(c)所示。在这里不存在压力脉动现象。

2.4　流态判别方法

2.4.1　雷诺数判别流态的商榷

以有压管道流动为例，当研究进口段流态时，其直径如何解决？其中间部分为理想流体时，不在流态判别范围；如果管很大，按计算为湍流，实际为层流；当非圆形管道，以水力半径代替直径，将流场形态特点变成圆形，失去其对流动的影响。当然，在没有弄清湍流形成的机理之前，利用它结合实际能解决实际问题，起到应有的作用。

2.4.2　判别流态新方法

当弄清湍流形成的机理后，结合具体问题的边界条件、壁面特点以及形成流场的状态，找出速度分布规律，确定出边层流的厚度，计算出横向运动涡旋到达的最远距离，以便确定湍流范围，计算出涡旋数量(φ)，确定湍流强度。这里仅谈原则，至于具体方法，必须结合具体流动问题才能说清楚。详见以后各章有关内容。

2.5　能量方程形式

研究湍流形成机理，须应用物理学中的运动能量守恒定律，它在物理学中的一般形式为

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgh_c \tag{2-5-1}$$

式中：E——刚体的总动能；

m——刚体的质量；

I——刚体的转动惯量；

ω——刚体的旋转速度；

g——重力加速度；

h_c——刚体的中心位置。

将式(2-5-1)变成适合球体的形式，将 I 用球体转动惯量代换，有

$$E_{球}=\frac{1}{2}mv^2+\frac{1}{5}mr^2\omega^2+mgh_c \tag{2-5-2}$$

将式(2-5-2)改为适合流体球形微团的形式，其质量为 m，则可写成

$$e_r=\frac{1}{2}v^2+\frac{1}{5}r^2\omega^2+gh_c \tag{2-5-3}$$

式(2-5-3)适合于明渠，若它适合有压管道流动，则可写成

$$e_r=\frac{1}{2}v^2+\frac{1}{5}r^2\omega^2+\frac{p}{\rho} \tag{2-5-4}$$

一个流体微团，在运动过程中要克服阻力，消耗能量，进行能量形式转化过程中，也要消耗能量，这一切都由微团本身的压能下降来完成。这样，作为一个流体微团，在运动过程中只考虑动能守恒就可以了，即

$$e_{rv}=\frac{1}{2}v_s^2+\frac{1}{5}r_s^2\omega^2=常数 \tag{2-5-5}$$

式(2-5-5)说明，当一个微团开始为只有平移速度而无旋转时，则其动能为

$$E_{rv}=\frac{1}{2}v_0^2 \tag{2-5-6}$$

当它在外力作用下形成涡旋后，则其总动能不变，即

$$\frac{1}{2}u^2=\frac{1}{2}u_s^2+\frac{1}{5}r_s^2\omega^2 \tag{2-5-7}$$

2.6 涡旋直径与转速的关系

设涡旋微团为球体，如图 2-19 所示。在球面上取一段 ds 微弧，其微环筒面积为

$$\mathrm{d}A=2\pi r\cdot r\cos\theta\mathrm{d}\theta \tag{2-6-1}$$

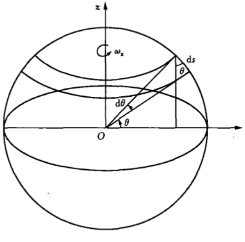


图 2-19 涡旋球体示意图

根据牛顿内摩擦定律，有

$$\Omega = \mu \cdot \frac{\mathrm{d}u}{\mathrm{d}\eta} \tag{2-6-2}$$

在球面上一点应表示为

$$\Omega = \mu \cos\theta\omega \tag{2-6-3}$$

则微环筒面上摩擦力为

$$\mathrm{d}T = \mathrm{d}A\Omega = 2\pi r^2 \cos^2\theta\mu\omega\,\mathrm{d}\theta \tag{2-6-4}$$

微环筒面上摩擦力对 z 轴的力矩为

$$\mathrm{d}M = 2\pi\mu\omega r^3 \cos^3\theta\,\mathrm{d}\theta \tag{2-6-5}$$

整个球面上摩擦力对 z 轴的力矩为

$$M = 4\pi\mu\omega r^r \int_0^{\frac{\pi}{2}} \cos^3\theta\,\mathrm{d}\theta \tag{2-6-6}$$

积分得

$$M = \frac{8}{3}\pi\mu\omega r^3 \tag{2-6-7}$$

由于问题具有特殊性，先有涡旋球体转动，其旋转动能为

$$E_\omega = \frac{1}{2} I\omega^2 \tag{2-6-8}$$

式中：I——涡旋球体的转动惯量；

ω——涡旋球体的转动速度。

当涡旋球体转动时，必然引起周围流体对它形成摩擦阻力。此阻力又对涡旋球体转轴形成阻力矩 M。当 M 与 E_ω 相等时，达到平衡，其转速为等速，即

$$M = E_\omega \tag{2-6-9}$$

从而得涡旋转速与其半径的关系为

$$r_s = \sqrt{\frac{10\nu}{\omega}} \tag{2-6-10}$$

式(2-6-10)说明，转速越大，涡旋半径越小，运动黏性系数越大，涡旋半径也越大。

2.7　本章小结

在第 1 章的基础之上，本章研究各种边界条件下的湍流形成的机理与湍流状态变化；同时探讨了流态判别方法的原则；结合流体微团的特点，引入了适合其运动的能量方程形式；推导出了涡旋直径与其转速之间的关系式。所有这些问题，不但有其本身应用的意义，而且是分析、研究、讨论以后各章的理论基础。

第 3 章　湍流运动基本方程组

所谓湍流运动基本方程组，是指湍流运动没有引入坐标系。又由前两章的分析可知，湍流运动的特点是存在着涡旋，它是引起湍流附加剪应力的根源。涡旋在湍流中是分散相，而只有平移运动的流体微团才是连续相，它在湍流运动中起主导作用。湍流运动实质是同介质的多相流。因此，依据多相流观点与力学定律，将拉格朗日与欧拉方法相结合，可建立不可压缩与可压缩流体输运公式，然后，建立湍流运动基本方程。

3.1　湍流系统积分方程组

在湍流流场中划出一有限大小的分离体，如图 3-1 所示，作为研究系统，它在运动过程中与外界没有质量交换。对它应用四大守恒定律，可建立湍流系统积分方程组。

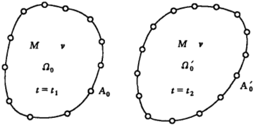


图 3-1　系统运动示意图

3.1.1　系统质量积分方程

如图 3-1 所示，系统中质量为 M。在运动过程中，系统服从质量守恒定律，对其运用拉格朗日方法，得其数学表达式为

$$\frac{\mathrm{D}M}{\mathrm{D}t}=\frac{\mathrm{D}}{\mathrm{D}t}\iiint_{\Omega_0}\rho\mathrm{d}\Omega_0=0 \tag{3-1-1}$$

式中：ρ——流体密度；

　　t——时间。

3.1.2　系统动量积分方程

如图 3-1 所示，在运动过程中，系统动量发生变化，但其变化服从动量守恒定律，即系统动量对时间的变化率等于外界作用在其上的合力，其数学表达式为

$$\frac{\mathrm{D}k}{\mathrm{D}t}=\frac{\mathrm{D}}{\mathrm{D}t}\iiint_{\Omega_0}\rho(\varphi\boldsymbol{v}_s+\varphi_1\boldsymbol{v})\mathrm{d}\Omega_0=\iiint_{\Omega_0}\boldsymbol{f}\rho\mathrm{d}\Omega_0+\oiint_{A_0}\boldsymbol{p}_n\mathrm{d}A \tag{3-1-2}$$

式中：k——系统中的动量；

φ——涡旋在系统中占的体积分数；

$\boldsymbol{f}$——单位质量所受的力；

$\boldsymbol{p}_n$——作用在系统表面上的应力矢量。

其中

$$\varphi_1=1-\varphi \tag{3-1-3}$$

3.1.3　系统动量矩积分方程

设系统对某点的动量矩为 M_0，它对时间的变化率等于外界作用在系统上的所有外力对该点力矩之和，其数学表达式为

$$\frac{\mathrm{D}M_0}{\mathrm{D}t}=\frac{\mathrm{D}}{\mathrm{D}t}\iiint_{\Omega_0}[\boldsymbol{r}\times(\varphi\boldsymbol{v}_s+\varphi_1\boldsymbol{v})]\mathrm{d}\Omega_0=\iiint_{\Omega_0}(\boldsymbol{r}\times\rho\boldsymbol{f})\mathrm{d}\Omega_0+\oiint_{A_0}(\boldsymbol{r}\times\boldsymbol{p}_n)\mathrm{d}A_0 \tag{3-1-4}$$

3.1.4　系统能量积分方程

外界传入系统中的热量与外界力对系统作用之和，等于系统总能量 E 对时间的变化率，其数学表达式为

$$\begin{aligned}\frac{\mathrm{D}E}{\mathrm{D}t}&=\frac{\mathrm{D}}{\mathrm{D}t}\iiint_{\Omega_0}\rho\left(e+\varphi\frac{v_s^2}{2}+\varphi_1\frac{v^2}{2}\right)\mathrm{d}\Omega_0\\&=\oiint_{A}\boldsymbol{n}\cdot q_\lambda\mathrm{d}A+\iiint_{\Omega_0}\rho q_R\mathrm{d}\Omega_0+\iiint_{\Omega_0}\boldsymbol{f}\cdot\rho(\varphi\boldsymbol{v}_s+\varphi_1\boldsymbol{v})\mathrm{d}\Omega_0\\&\quad+\oiint_{A_0}\boldsymbol{p}_n\cdot(\varphi^{\frac{2}{3}}\boldsymbol{v}_s+\varphi_2\boldsymbol{v})\mathrm{d}A_0\end{aligned} \tag{3-1-5}$$

式中：φ——涡旋在单位流体体积中占的体积分数；

$\varphi^{\frac{2}{3}}$——涡旋在单位面积流体上占的面积分数；

Ω_0——系统体积；

A_0——系统表面面积；

$\boldsymbol{v}_s$——涡旋微团速度矢量；

$\boldsymbol{v}$——无旋流体微团速度矢量；

v_s——涡旋微团标量速度；

v——无旋微团标量速度；

e——流体内能；

q_R——热辐射率；

q_λ——外界通过系统表面以导热方式的传热率。

其中

$$\varphi_2 = 1 - \varphi^{\frac{2}{3}} \tag{3-1-6}$$

3.2 不可压缩湍流输运公式

在流体力学中，用拉格朗日方法得到的积分公式，是对运动着的系统而言的。要弄清楚一个流场中流体运动参数变化的情况，必须跟踪所有的流体系统，这样工作量很大，以至无法完成。鉴于这种情况，流体力学研究史上，又出现了另一种研究流体运动的方法，称之为欧拉方法。欧拉方法是指，在流场中，划出分离体为固定不动空间(称之为控制体)，研究通过固定空间的流体运动参数变化的情况。至于通过该点的流体微团是哪一个，却不用管它。因为流场是固定的，其中空间点也是固定可数的，所以研究起来也就很方便。

欧拉方法虽然方便，但它划出的分离体是控制体，是不动的。不动的物理体是无法对它应用四大守恒定律的，也就得不到其对时间的变化率。而实际上，流场的流体总是运动的，于是将控制体内的流体取为系统，它的外表面与控制体表面吻合，其体积相同。因为系统是动的，对它可以应用四大守恒定律。经过一个 Δt 时间，系统运动到新的位置。当 $\Delta t \to 0$ 时，得到系统物理量对时间的变化率。而这时系统与控制体重合，从而也就是适合控制体物理量对时间的变化率。系统的积分方程转为控制体积分方程。联系这个转化过程的桥梁，就是不可压缩流体输运体导数公式，它是研究不可压缩流体运动基本方程组的出发点。

在湍流场中，画出系统与控制体关系示意图，如图 3-2 所示。Δt 时间内，系统运动到新的位置，当 $\Delta t \to 0$ 时，它们近似重合。由于限定研究对象为不可压缩流体，所以系统在运动过程中只有形状改变，而体积不变。

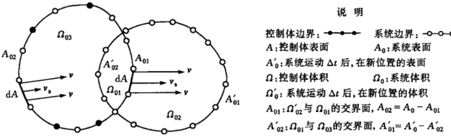


图 3-2 不可压缩流体系统与控制体关系示意图

利用图 3-2，首先研究系统导数。设系统中分散相物理量为 Φ_s，连续相物理量为 Φ，则其总量为

$$I_m = \iiint_{\Omega_0} [\varphi \Phi_s(r,t) + \varphi_1 \Phi(r,t)] \mathrm{d}\Omega_0 \tag{3-2-1}$$

当 $\Delta t \to 0$ 时，系统物理量对时间的变化率为

$$\lim_{\Delta t\to 0}\frac{\Delta I_m}{\Delta t}=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\iiint_{\Omega_{01}}[\varphi\Phi_s(r,t+\Delta t)+\varphi_1\Phi(r,t+\Delta t)-\varphi\Phi_s(r,t)+\varphi_1\Phi(r,t)]\mathrm{d}\Omega_0\right\}$$
$$+\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\iiint_{\Omega'_{02}}[\varphi\Phi_s(r,t+\Delta t)+\varphi_1\Phi(r,t+\Delta t)]\mathrm{d}\Omega_0-\iiint_{\Omega_{03}}[\varphi\Phi_s(r,t)+\varphi_1\Phi(r,t)]\mathrm{d}\Omega_0\right\}$$

(3-2-2)

式(3-2-2)中，对于第一个大括弧的积分，利用数学中的中值定理，可写成

$$\lim_{\Delta t\to 0}\frac{\Delta I_m}{\Delta t}=\lim_{\Delta t\to 0}\frac{\Delta t}{\Delta t}\left[\iiint_{\Omega_{01}}\left(\varphi\cdot\frac{\partial\Phi_s}{\partial t}+\varphi_1\cdot\frac{\partial\Phi}{\partial t}\right)\mathrm{d}\Omega_0\right]\tag{3-2-3}$$

第二个大括弧内，第一个与第二个积分，分别利用计算边界面流出的流量可得

$$\iiint_{\Omega'_{02}}[\varphi\Phi_s(r,t+\Delta t)+\varphi_1\Phi(r,t+\Delta t)]\mathrm{d}\Omega_0\approx\iint_{A_{01}}[\varphi^{\frac{2}{3}}\Phi_s(\boldsymbol{v}_s\cdot\boldsymbol{n})+\varphi_2\Phi(\boldsymbol{v}\cdot\boldsymbol{n})]\Delta t\mathrm{d}A_0$$

(3-2-4)

$$\iiint_{\Omega_{03}}[\varphi\Phi_s(r,t)+\varphi_1\Phi(r,t)]\mathrm{d}\Omega_0\approx-\iint_{A_{02}}[\varphi^{\frac{2}{3}}\Phi_s(\boldsymbol{v}_s\cdot\boldsymbol{n})+\varphi_2\Phi(\boldsymbol{v}\cdot\boldsymbol{n})]\Delta t\mathrm{d}A_0$$

(3-2-5)

将式(3-2-3)、式(3-2-4)、式(3-2-5)代入式(3-2-2)，并令 $\Delta t\to 0$，则 $\Omega_0\to\Omega$，$A_0\to A$，可得

$$\frac{\mathrm{D}I_m}{\mathrm{D}t}=\frac{\mathrm{D}}{\mathrm{D}t}\iiint_{\Omega_0}(\varphi\Phi_s+\varphi_1\Phi)\mathrm{d}\Omega_0$$
$$=\iiint_{\Omega}\left(\varphi\frac{\partial\Phi_s}{\partial t}+\varphi_1\frac{\partial\Phi}{\partial t}\right)\mathrm{d}\Omega+\oiint_A[\varphi^{\frac{2}{3}}\Phi_s(\boldsymbol{v}_s\cdot\boldsymbol{n})+\varphi_2\Phi(\boldsymbol{v}\cdot\boldsymbol{n})]\mathrm{d}A\tag{3-2-6}$$

式(3-2-6)就是不可压缩多相流输运公式，也称之为第一输运公式。它表明，控制体内物理量的导数，等于单位时间内控制体内物理量的增量与通过控制体表面流出的物理量之和。

3.3 不可压缩湍流积分方程组

3.3.1 质量守恒积分方程

将 $\Phi_s=\rho_s$，$\Phi=\rho$，$I_m=M$ 代入式(3-2-6)，然后代入式(3-1-1)，则有

$$\frac{\mathrm{D}M}{\mathrm{D}t}=-\iiint_{\Omega}\frac{\partial\rho}{\partial t}\mathrm{d}\Omega=\oiint_A[\varphi^{\frac{2}{3}}\rho_s(\boldsymbol{v}_s\cdot\boldsymbol{n})+\varphi_2\rho(\boldsymbol{v}\cdot\boldsymbol{n})]\mathrm{d}A\tag{3-3-1}$$

式(3-3-1)表明，单位时间内通过控制面 A 流出的湍流质量，等于同时间内控制体内质量的减少量。

3.3.2 动量守恒积分方程

令 $\Phi_s=\rho\boldsymbol{v}_s$，$\Phi=\rho\boldsymbol{v}$，$I_m=\boldsymbol{k}$，代入式(3-2-6)，然后代入式(3-1-2)，则有

$$\iiint_{\Omega}f\rho\mathrm{d}\Omega+\oiint_A\boldsymbol{p}_n\mathrm{d}A-\oiint_A[\varphi^{\frac{2}{3}}\rho\boldsymbol{v}_s(\boldsymbol{v}_s\cdot\boldsymbol{n})+\varphi_2\rho_n\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{n})]\mathrm{d}A$$

$$= \frac{\partial}{\partial t} \iiint_{\Omega} [\varphi(\rho \boldsymbol{v}_s) + \varphi_1(\rho \boldsymbol{v})] d\Omega \tag{3-3-2}$$

式(3-3-2)表明，作用在控制体内湍流上的合外力，加上单位时间内通过控制面流入湍流的动量，等于控制体内的湍流动量对时间的变化率。

3.3.3 动量矩守恒积分方程

令 $\boldsymbol{\Phi} = \rho\ (\boldsymbol{r} \times \boldsymbol{v})$，$\boldsymbol{\Phi}_s = \boldsymbol{r} \times \rho \boldsymbol{v}_s$，$I_m = M_0$，代入式(3-2-6)，然后代入式(3-1-4)，则有

$$\frac{\partial}{\partial t} \iiint_{\Omega} [\boldsymbol{r} \times (\varphi \rho \boldsymbol{v}_s + \varphi_1 \rho \boldsymbol{v})] d\Omega$$

$$= \iiint_{\Omega} (\boldsymbol{r} \times \boldsymbol{f}) \rho d\Omega + \oiint_A (\boldsymbol{r} \times \boldsymbol{p}_n) dA - \oiint_A \{\boldsymbol{r} \times [\varphi^{\frac{2}{3}} \rho \boldsymbol{v}_s (\boldsymbol{v}_s \cdot \boldsymbol{n}) + \varphi_2 \rho \boldsymbol{v} (\boldsymbol{v} \cdot \boldsymbol{n})]\} dA \tag{3-3-3}$$

式(3-3-3)表明，作用在控制体内湍流上的所有外力，对某点的合力矩，与单位时间内通过控制面流入的湍流对同一点动量矩之和，等于控制体内湍流对该点动量矩对时间的变化率。

3.3.4 能量守恒积分方程

令 $\Phi_c = \rho\left(e_s + \frac{v_s^2}{2}\right)$，$\Phi = \rho\left(e + \frac{v^2}{2}\right)$，$I_m = E$，代入式(3-2-6)，然后代入式(3-1-5)，则有

$$\iiint_{\Omega} \frac{\partial}{\partial t}\left[\varphi \rho_s \left(e_s + \frac{v_s^2}{2}\right) + \varphi_1 \rho \left(e + \frac{v^2}{2}\right)\right] d\Omega$$

$$= \oiint_A \boldsymbol{n} \cdot \boldsymbol{q}_\lambda dA - \oiint_A \left[\varphi^{\frac{2}{3}} \rho \left(e + \frac{v_s^2}{2}\right)(\boldsymbol{v}_s \cdot \boldsymbol{n}) + \varphi_2 \rho \left(e + \frac{v^2}{2}\right)(\boldsymbol{v} \cdot \boldsymbol{n})\right] dA$$

$$+ \iiint_{\Omega} \rho q_R d\Omega + \iiint_{\Omega} \boldsymbol{f} \cdot (\rho \varphi \boldsymbol{v}_s + \rho \varphi_1 \boldsymbol{v}) d\Omega + \oiint_A [\boldsymbol{p}_n \cdot (\varphi^{\frac{2}{3}} \boldsymbol{v}_s + \varphi_2 \boldsymbol{v})] dA \tag{3-3-4}$$

式(3-3-4)表明，单位时间内传给控制体内湍流的热量及外界力对控制体内湍流所做的功，与通过控制面流入的湍流能量之和，等于控制体内湍流能量对时间的变化率。

3.4 不可压缩湍流微分方程组

3.4.1 连续性微分方程

将式(3-3-1)中的面积分变成体积分，则有

$$\iiint_{\Omega} \left(\frac{\partial \rho}{\partial t} + \varphi^{\frac{2}{3}} \rho_s \nabla \cdot \boldsymbol{v}_s + \varphi_2 \rho \nabla \cdot \boldsymbol{v}\right) d\Omega = 0 \tag{3-4-1}$$

式(3-4-1)积分为 0，只能被积函数为 0，则有

$$\frac{\partial \rho}{\partial t} + \varphi^{\frac{2}{3}} \rho_s \nabla \cdot \boldsymbol{v}_s + \varphi_2 \rho \nabla \cdot \boldsymbol{v} = 0 \tag{3-4-1$'$}$$

$$\nabla \cdot \boldsymbol{v} + \varphi^{\frac{2}{3}} \nabla \cdot (\boldsymbol{v}_s - \boldsymbol{v}) = 0 \tag{3-4-2}$$

3.4.2 动量微分方程

将式(3-3-2)中的面积分变成体积分，则有

$$\iiint_{\Omega} \left[f\rho + \nabla \cdot \boldsymbol{p} - (\varphi^{\frac{2}{3}}\rho \nabla \cdot \boldsymbol{v}_s \boldsymbol{v}_s + \varphi_2 \rho \nabla \cdot \boldsymbol{v}\boldsymbol{v}) - \frac{\partial}{\partial t}(\varphi\rho \boldsymbol{v}_s + \varphi_1 \rho \boldsymbol{v}) \right] \mathrm{d}\Omega = 0 \tag{3-4-3}$$

式(3-4-3)积分为 0，则有

$$\frac{\partial}{\partial t}(\varphi \boldsymbol{v}_s + \varphi_1 \boldsymbol{v}) + \varphi^{\frac{2}{3}} \nabla \cdot \boldsymbol{v}_s \boldsymbol{v}_s + \varphi_2 \nabla \cdot \boldsymbol{v}\boldsymbol{v} = \boldsymbol{f} + \frac{1}{\rho} \nabla \cdot \boldsymbol{P} \tag{3-4-4}$$

3.4.3 动量矩微分方程

式(3-3-3)中

$$\oint_A \{ \boldsymbol{r} \times [\varphi^{\frac{2}{3}}\rho \boldsymbol{v}_s(\boldsymbol{v}_s \cdot \boldsymbol{n}) + \varphi_2 \rho \boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{n})] \} \mathrm{d}A$$

$$= \oint_A \boldsymbol{n} \cdot \boldsymbol{v}_s(\boldsymbol{r} \times \varphi^{\frac{2}{3}}\rho \boldsymbol{v}_s)\mathrm{d}A + \oint_A \boldsymbol{n} \cdot \boldsymbol{v}(\boldsymbol{r} \times \varphi_2 \rho \boldsymbol{v})\mathrm{d}A$$

$$= \iiint_{\Omega} \nabla \cdot \boldsymbol{v}_s(\boldsymbol{r} \times \varphi^{\frac{2}{3}}\rho \boldsymbol{v}_s)\mathrm{d}\Omega + \iiint_{\Omega} \nabla \cdot \boldsymbol{v}(\boldsymbol{r} \times \varphi_2 \rho \boldsymbol{v})\mathrm{d}\Omega \tag{3-4-5}$$

$$\oint_A (\boldsymbol{r} \times \boldsymbol{p}_n)\mathrm{d}A = \oint_A (\boldsymbol{r} \times \boldsymbol{n} \cdot \boldsymbol{P})\mathrm{d}A = \oint_A \boldsymbol{n}(\boldsymbol{r} \times \boldsymbol{P})\mathrm{d}A$$

$$= \iiint_{\Omega} \nabla \cdot (\boldsymbol{r} \times \boldsymbol{P})\mathrm{d}\Omega \tag{3-4-6}$$

式中：$\boldsymbol{P}$——应力张量；

$\boldsymbol{r}$——参变矢量。

将式(3-4-5)和式(3-4-6)代回式(3-3-3)，则有

$$\iiint_{\Omega} \left\{ \frac{\partial}{\partial t}[\boldsymbol{r} \times \rho(\varphi \boldsymbol{v}_s + \varphi_1 \boldsymbol{v})] + \nabla \cdot \boldsymbol{v}_s(\boldsymbol{r} \times \varphi^{\frac{2}{3}}\rho \boldsymbol{v}_s) + \nabla \cdot \boldsymbol{v}(\boldsymbol{r} \times \varphi_2 \rho \boldsymbol{v}) - \nabla \cdot (\boldsymbol{r} \times \boldsymbol{P}) - (\boldsymbol{r} \times \boldsymbol{f}) \cdot \rho \right\} \mathrm{d}\Omega = 0 \tag{3-4-7}$$

式(3-3-3)积分为 0，则一定是被积函数为 0，即

$$\frac{\partial}{\partial t}[\boldsymbol{r} \times \rho(\varphi \boldsymbol{v}_s + \varphi_1 \boldsymbol{v})] + \nabla \cdot \boldsymbol{v}_s(\boldsymbol{r} \times \varphi^{\frac{2}{3}}\rho \boldsymbol{v}_s) + \nabla \cdot \boldsymbol{v}(\boldsymbol{r} \times \varphi_2 \rho \boldsymbol{v})$$

$$= (\boldsymbol{r} \times \boldsymbol{f}) \cdot \rho + \nabla \cdot (\boldsymbol{r} \times \boldsymbol{P}) \tag{3-4-8}$$

3.4.4 能量微分方程

将式(3-3-4)中的面积分转变为体积分，则有

$$\oint_A \boldsymbol{n} \cdot \boldsymbol{q}_\lambda \mathrm{d}A = \iiint_{\Omega} \nabla \cdot \boldsymbol{q}_\lambda \mathrm{d}\Omega = \iiint_{\Omega} \nabla \cdot \nabla T \mathrm{d}\Omega \tag{3-4-9}$$

$$\oiint_A \varphi^{\frac{2}{3}}\rho\left(e_s+\frac{v_s^2}{2}\right)(\boldsymbol{v}_s\cdot\boldsymbol{n})+\varphi_2\rho\left(e+\frac{v^2}{2}\right)(\boldsymbol{v}\cdot\boldsymbol{n})\mathrm{d}A$$
$$=\oiint_A(\boldsymbol{v}_s\cdot\boldsymbol{n})\varphi^{\frac{2}{3}}\rho\left(e_s+\frac{v_s^2}{2}\right)\mathrm{d}A+\oiint_A(\boldsymbol{v}\cdot\boldsymbol{n})\varphi_2\rho\left(e+\frac{v^2}{2}\right)\mathrm{d}A$$
$$=\oiint_A(\boldsymbol{n}\cdot\boldsymbol{v}_s)\varphi^{\frac{2}{3}}\rho\left(e_s+\frac{v_s^2}{2}\right)\mathrm{d}A+\oiint_A(\boldsymbol{n}\cdot\boldsymbol{v})\varphi_2\rho\left(e+\frac{v^2}{2}\right)\mathrm{d}A$$
$$=\iiint_\Omega(\nabla\cdot\boldsymbol{v}_s)\varphi^{\frac{2}{3}}\rho\left(e_s+\frac{v_s^2}{2}\right)\mathrm{d}\Omega+\iiint_\Omega(\nabla\cdot\boldsymbol{v})\varphi_2\rho\left(e+\frac{v^2}{2}\right)\mathrm{d}\Omega \tag{3-4-10}$$

$$\oiint_A[\boldsymbol{p}_n\cdot(\varphi^{\frac{2}{3}}\boldsymbol{v}_s+\varphi_2\boldsymbol{v})]\mathrm{d}A=\oiint_A[\boldsymbol{n}\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{v}_s+\varphi_2\boldsymbol{v})]\mathrm{d}A$$
$$=\iiint_\Omega\nabla\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{v}_s+\varphi_2\boldsymbol{v})\mathrm{d}\Omega \tag{3-4-11}$$

将式(3-4-9)、式(3-4-10)、式(3-4-11)代回式(3-3-4)，则有

$$\iiint_\Omega\left\{\frac{\partial}{\partial t}\left[\rho\left(e+\frac{v_s^2}{2}+\frac{v^2}{2}\right)\right]+\nabla\cdot\left[\varphi^{\frac{2}{3}}\rho\left(e+\frac{v_s^2}{2}\right)\boldsymbol{v}_s+\varphi_2\rho\left(e+\frac{v^2}{2}\right)\boldsymbol{v}\right]\right.$$
$$\left.-\nabla\cdot\nabla T-\rho q_R-\boldsymbol{f}\cdot\rho(\varphi\boldsymbol{v}_s+\varphi_1\boldsymbol{v})-\nabla\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{v}_s+\varphi_2\boldsymbol{v})\right\}\mathrm{d}\Omega=0 \tag{3-4-12}$$

式(3-4-12)中，积分为 0，因是不可压缩流体，$\rho_s=\rho=$常数，$e_s=e$，全式除以 ρ，则有

$$\frac{\partial}{\partial t}\left(\varphi\cdot\frac{v_s^2}{2}+\varphi_1\cdot\frac{v^2}{2}\right)+\nabla\cdot\left(\varphi^{\frac{2}{3}}\cdot\frac{v_s^2}{2}\cdot\boldsymbol{v}_s+\varphi_2\cdot\frac{v^2}{2}\cdot\boldsymbol{v}\right)$$
$$=\frac{1}{\rho}\nabla\cdot\nabla T+q_R+\boldsymbol{f}\cdot(\varphi\boldsymbol{v}_s+\varphi_1\boldsymbol{v})+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{v}_s+\varphi_2\boldsymbol{v}) \tag{3-4-13}$$

3.5 可压缩湍流随体导数公式

当系统中为可压缩湍流时，它在运动过程中不仅形状改变，而且其体积也变化，相应的密度、温度也发生变化。这样，当 $\Delta t\to 0$ 时，系统导数与控制体导数相等，就可以得到可压缩流体输运公式。

在可压缩湍流场中，当 $t=t_1$ 时，划出一系统；当 $t=t_2$ 时，它运动到新的位置，如图 3-3 所示，此时，形状与体积均发生了变化。为研究其变化率，令 $t_2-t_1=\Delta t$ 趋于 0，计算其随体导数，以数学方式表示，则为

$$\frac{\mathrm{D}I_m}{\mathrm{D}t}=\frac{\mathrm{D}}{\mathrm{D}t}\iiint_{\Omega_0}(\varphi\Phi_s+\varphi_1\Phi)\mathrm{d}\Omega_0$$
$$=\iiint_{\Omega_0}\left[\frac{\mathrm{D}}{\mathrm{D}t}(\varphi\Phi_s\mathrm{d}\Omega_0)+\frac{\mathrm{D}}{\mathrm{D}t}(\varphi_1\Phi\mathrm{d}\Omega_0)\right]$$
$$=\iiint_{\Omega_0}\left[\varphi\mathrm{d}\Omega_0\cdot\frac{\mathrm{D}\Phi_s}{\mathrm{D}t}+\varphi\Phi_s\cdot\frac{\mathrm{D}(\mathrm{d}\Omega_0)}{\mathrm{D}t}+\varphi_1\mathrm{d}\Omega_0\cdot\frac{\mathrm{D}\Phi}{\mathrm{D}t}+\Phi\varphi_1\cdot\frac{\mathrm{D}(\mathrm{d}\Omega_0)}{\mathrm{D}t}\right] \tag{3-5-1}$$

根据散度的定义，结合湍流特点，分别定义分散相、连续相的散度为

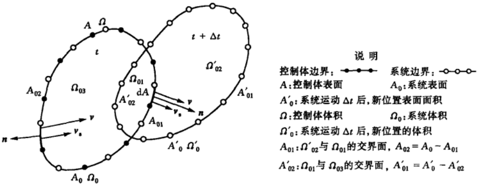


图 3-3 可压缩湍流系统运动示意图

$$\nabla\cdot\boldsymbol{v}_s=\frac{\varphi\mathrm{D}\ (\mathrm{d}\Omega_0)}{\mathrm{d}\Omega_0\mathrm{D}t} \tag{3-5-2}$$

$$\nabla\cdot\boldsymbol{v}=\frac{\varphi_1\mathrm{D}\ (\mathrm{d}\Omega_0)}{\mathrm{d}\Omega_0\mathrm{D}t} \tag{3-5-3}$$

将式(3-5-2)和式(3-5-3)代入式(3-5-1)，则有

$$\begin{aligned}\frac{\mathrm{D}I_m}{\mathrm{D}t} &= \iiint_{\Omega_0}\left(\varphi\frac{\mathrm{D}\Phi_s}{\mathrm{D}t}+\Phi_s\nabla\cdot\boldsymbol{v}_s+\varphi_1\frac{\mathrm{D}\Phi}{\mathrm{D}t}+\Phi\nabla\cdot\boldsymbol{v}\right)\mathrm{d}\Omega_0\\ &= \iiint_{\Omega}\left[\left(\varphi\frac{\mathrm{D}\Phi_s}{\mathrm{D}t}+\Phi_s\nabla\cdot\boldsymbol{v}_s\right)+\left(\varphi_1\frac{\mathrm{D}\Phi}{\mathrm{D}t}+\Phi\nabla\cdot\boldsymbol{v}\right)\right]\mathrm{d}\Omega\end{aligned} \tag{3-5-4}$$

式(3-5-4)就是可压缩湍流输运公式。也称之为第二输运公式。

3.6 可压缩湍流积分方程组

3.6.1 质量积分方程

令 $\Phi_s=\rho_s$，$\Phi=\rho$，$I_m=M$，代入式(3-5-4)，然后代入式(3-1-1)，则有

$$\frac{\mathrm{D}M}{\mathrm{D}t} = \iiint_{\Omega}\left[\varphi\frac{\mathrm{D}\rho_s}{\mathrm{D}t}+(1-\varphi)\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho_s\nabla\cdot\boldsymbol{v}_s+\rho\nabla\cdot\boldsymbol{v}\right]\mathrm{d}\Omega = 0 \tag{3-6-1}$$

3.6.2 动量积分方程

令 $I_m=k$，$\Phi_s=\rho_s\boldsymbol{v}_s$，$\Phi=\rho\boldsymbol{v}$，代入式(3-5-4)，然后代入式(3-1-2)，有

$$\begin{aligned}\frac{\mathrm{D}k}{\mathrm{D}t} &= \iiint_{\Omega}\left[\varphi\frac{\mathrm{D}(\rho_s\boldsymbol{v}_s)}{\mathrm{D}t}+\varphi_1\frac{\mathrm{D}(\rho\boldsymbol{v})}{\mathrm{D}t}+\rho_s\boldsymbol{v}_s(\nabla\cdot\boldsymbol{v}_s)+\rho\boldsymbol{v}(\nabla\cdot\boldsymbol{v})\right]\mathrm{d}\Omega\\ &= \iiint_{\Omega}f[\varphi\rho_s+(1-\varphi)\rho]\mathrm{d}\Omega+\oiint_A\boldsymbol{p}_n\mathrm{d}A\end{aligned} \tag{3-6-2}$$

3.6.3 动量矩积分方程

令 $I_m = M_0$，$\Phi_s = \boldsymbol{r}\times\rho_s\boldsymbol{v}_s$，$\Phi = \boldsymbol{r}\times\rho\boldsymbol{v}$，代入式(3-5-4)，然后代入式(3-1-4)，则有

$$\frac{\mathrm{D}M_0}{\mathrm{D}t} = \iiint_\Omega \left[\varphi\frac{\mathrm{D}(\boldsymbol{r}\times\rho_s\boldsymbol{v}_s)}{\mathrm{D}t} + (\boldsymbol{r}\times\rho_s\boldsymbol{v}_s)\nabla\cdot\boldsymbol{v}_s + \varphi_1\frac{\mathrm{D}(\boldsymbol{r}\times\rho\boldsymbol{v})}{\mathrm{D}t} + (\boldsymbol{r}\times\rho\boldsymbol{v})\nabla\cdot\boldsymbol{v}\right]\mathrm{d}\Omega$$
$$= \iiint_\Omega \boldsymbol{r}\times[\varphi\rho_s + (1-\varphi)\rho]\boldsymbol{f} + \oiint_A (\boldsymbol{r}\times\boldsymbol{p}_n)\mathrm{d}A \tag{3-6-3}$$

3.6.4 能量积分方程

令 $I_m = E$，$\Phi_s = \rho_s\left(e_s + \dfrac{v_s^2}{2}\right)$，$\Phi = \rho\left(e + \dfrac{v^2}{2}\right)$，代入式(3-5-4)，然后代入式(3-1-5)，则有

$$\frac{\mathrm{D}E}{\mathrm{D}t} =$$
$$\iiint_\Omega \left\{\varphi\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho_s\left(e_s + \frac{v_s^2}{2}\right)\right] + \rho_s\left(e_s + \frac{v_s^2}{2}\right)\nabla\cdot\boldsymbol{v}_s + \varphi_1\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho\left(e + \frac{v^2}{2}\right)\right] + \rho\left(e + \frac{v^2}{2}\right)\nabla\cdot\boldsymbol{v}\right\}\mathrm{d}\Omega$$
$$= \oiint_A \boldsymbol{n}\cdot\boldsymbol{q}_\lambda\mathrm{d}A + \iiint_\Omega \rho q_R\mathrm{d}\Omega + \iiint \boldsymbol{f}\cdot(\varphi\rho_s\boldsymbol{v}_s + \varphi_1\rho\boldsymbol{v}) + \oiint_A \boldsymbol{p}_n\cdot(\varphi^{\frac{2}{3}}\boldsymbol{v}_s + \varphi_2\boldsymbol{v})\mathrm{d}A \tag{3-6-4}$$

3.7 可压缩湍流微分方程组

3.7.1 连续性微分方程

由式(3-6-1)可知，积分式为0，只有被积函数为0，则

$$\varphi\frac{\mathrm{D}\rho_s}{\mathrm{D}t} + (1-\varphi)\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho_s\nabla\cdot\boldsymbol{v}_s + \rho\nabla\cdot\boldsymbol{v} = 0 \tag{3-7-1}$$

$$\rho_s\nabla\cdot\boldsymbol{v}_s = -\left[\varphi\frac{\mathrm{D}\rho_s}{\mathrm{D}t} + (1-\varphi)\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\boldsymbol{v}\right] \tag{3-7-2}$$

3.7.2 动量微分方程

将式(3-6-2)中的面积分变成体积分，则有

$$\oiint_A \boldsymbol{p}_n\mathrm{d}A = \oiint_A \boldsymbol{n}\cdot\boldsymbol{p}_n\mathrm{d}A = \iiint_\Omega \nabla\cdot\boldsymbol{p}_n\mathrm{d}\Omega = \iiint_\Omega \nabla\cdot\boldsymbol{P}\mathrm{d}\Omega \tag{3-7-3}$$

将式(3-7-3)代入式(3-6-2)，有

$$\iiint_\Omega \left[\varphi\frac{\mathrm{D}(\rho_s\boldsymbol{v}_s)}{\mathrm{D}t} + \varphi_1\frac{\mathrm{D}(\rho\boldsymbol{v})}{\mathrm{D}t} + \rho_s\boldsymbol{v}_s(\nabla\cdot\boldsymbol{v}_s) + \rho\boldsymbol{v}(\nabla\cdot\boldsymbol{v})\right]\mathrm{d}\Omega$$
$$= \iiint_\Omega \boldsymbol{f}[\varphi\rho_s + (1-\varphi)\rho]\mathrm{d}\Omega + \iiint_\Omega \nabla\cdot\boldsymbol{P}\mathrm{d}\Omega \tag{3-7-4}$$

$$\iiint_{\Omega}\left\{\left[\varphi\frac{\mathrm{D}(\rho_s\boldsymbol{v}_s)}{\mathrm{D}t}+\varphi_1\frac{\mathrm{D}(\rho\boldsymbol{v})}{\mathrm{D}t}+\rho_s\boldsymbol{v}_s(\nabla\cdot\boldsymbol{v}_s)+\rho\boldsymbol{v}(\nabla\cdot\boldsymbol{v})\right]-\boldsymbol{f}[\varphi\rho_s+(1-\varphi)\rho]-\nabla\cdot\boldsymbol{P}\right\}\mathrm{d}\Omega$$
$$=0 \tag{3-7-5}$$

式(3-7-5)积分为 0，而 $\mathrm{d}\Omega$ 不为 0，只有被积函数为 0，即

$$\varphi\frac{\mathrm{D}(\rho_s\boldsymbol{v}_s)}{\mathrm{D}t}+\varphi_1\frac{\mathrm{D}(\rho\boldsymbol{v})}{\mathrm{D}t}+\rho_s\boldsymbol{v}_s(\nabla\cdot\boldsymbol{v}_s)+\rho\boldsymbol{v}(\nabla\cdot\boldsymbol{v})=\boldsymbol{f}[\varphi\rho_s+(1-\varphi)\rho]+\nabla\cdot\boldsymbol{P} \tag{3-7-6}$$

3.7.3 动量矩微分方程

将式(3-6-3)中的面积分变成体积分，有

$$\oiint_A(\boldsymbol{r}\times\boldsymbol{p}_n)\mathrm{d}A=\oiint_A\boldsymbol{n}\cdot(\boldsymbol{r}\times\boldsymbol{p}_n)\mathrm{d}A=\iiint_{\Omega}\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P})\mathrm{d}\Omega \tag{3-7-7}$$

将式(3-7-7)代入式(3-6-3)，全式为体积分，积分为 0，则有

$$\varphi\frac{\mathrm{D}(\boldsymbol{r}\times\rho\boldsymbol{v}_s)}{\mathrm{D}t}+(\boldsymbol{r}\times\rho\boldsymbol{v}_s)\nabla\cdot\boldsymbol{v}_s+\varphi_1\frac{\mathrm{D}(\boldsymbol{r}\times\rho\boldsymbol{v})}{\mathrm{D}t}+(\boldsymbol{r}\times\rho\boldsymbol{v})\nabla\cdot\boldsymbol{v}=\boldsymbol{r}\times\rho\boldsymbol{f}+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{3-7-8}$$

3.7.4 能量微分方程

同理，将式(3-6-4)变为

$$\oiint_A\boldsymbol{n}\cdot q_\lambda\mathrm{d}A=\oiint_A\boldsymbol{n}\cdot(\lambda\nabla T)\mathrm{d}A=\iiint_{\Omega}\nabla\cdot(\lambda\nabla T)\mathrm{d}\Omega \tag{3-7-9}$$

$$\begin{aligned}\oiint_A\boldsymbol{p}_n\cdot(\varphi^{\frac{2}{3}}\boldsymbol{v}_s+\varphi_2\boldsymbol{v})\mathrm{d}A&=\oiint_A\boldsymbol{n}\cdot\boldsymbol{p}_n\cdot(\varphi^{\frac{2}{3}}\boldsymbol{v}_s+\varphi_2\boldsymbol{v})\mathrm{d}A\\&=\iiint_{\Omega}\nabla\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{v}_s+\varphi_2\boldsymbol{v})\mathrm{d}\Omega\end{aligned} \tag{3-7-10}$$

将式(3-7-9)和式(3-7-10)代入式(3-6-4)，成为单一的体积分，由于其积分为 0，则得

$$\begin{aligned}&\varphi\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho_s\left(e_s+\frac{v_s^2}{2}\right)\right]+\rho_s\left(e_s+\frac{v_s^2}{2}\right)\nabla\cdot\boldsymbol{v}_s+\varphi_1\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho\left(e+\frac{v^2}{2}\right)\right]+\rho\left(e+\frac{v^2}{2}\right)\nabla\cdot\boldsymbol{v}\\&=\nabla\cdot(\lambda\nabla T)+\rho q_\lambda+\boldsymbol{f}(\varphi\rho_s\boldsymbol{v}_s+\varphi_1\rho\boldsymbol{v})+\nabla\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{v}_s+\varphi_2\boldsymbol{v})\end{aligned} \tag{3-7-11}$$

湍流微分方程，无论是可压缩还是不可压缩，均是对单位体积而言的。

第 4 章　直角坐标系流体运动微分方程组

第 3 章导出了无坐标系微分方程，要想得到在实际工程中的应用，必须将其转化为有坐标系的微分方程。本章先讨论直角坐标系，然后讨论柱坐标系和球坐标系。

下面讨论层流与湍流，每种情况还分可压缩与不可压缩流体运动。

无坐标系微分方程，是对单位体积而言的，下面所讨论的方程均是对 1kg 流体而言的。

4.1　不可压缩层流微分方程组

根据定义，层流运动没有涡旋体积分数，$\varphi=0$，涡旋垂直于流向的速度 $\boldsymbol{v}_s=\mathbf{0}$。层流运动，也分层外区与边层流区。在层外区，流体微团只有平移运动，而无旋转，各层流微团均在所在流层平移向前运动。在边层流区，有旋与无旋流体层相间存在，有旋流体层也有平移运动，即边旋转边平移向前运动，而其旋转强度较弱，不能配合层间速度差构成升力作用，所以，只能边旋转边平移向前运动。总之，层流场没有垂直于主流向的旋转涡旋运动，无论有旋还是无旋流体微团，均沿所在流层向前平移运动。

4.1.1　不可压缩层流连续性微分方程

将层流条件 $v_s=0$，$\varphi=0$，$\varphi_2=1-\varphi^{\frac{2}{3}}$代入式(3-4-1)，得

$$\frac{\partial\rho}{\partial t}+\rho\nabla\cdot\boldsymbol{v}=0 \tag{4-1-1}$$

将不可压缩条件 $\rho=$常数代入式(4-1-1)，则有

$$\nabla\cdot\boldsymbol{v}=0 \tag{4-1-2}$$

4.1.2　不可压缩层流动量微分方程

将层流条件和不可压缩条件代入式(3-4-2)，再除以 ρ，变成对单位质量即 1kg 流体而言的方程，即

$$\frac{\partial\boldsymbol{v}}{\partial t}+\nabla\cdot\boldsymbol{v}\boldsymbol{v}=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{4-1-3}$$

根据数学运算，式(4-1-3)可以写成

$$\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{4-1-4}$$

式(4-1-4)仍然是无坐标系下不可压缩层流运动的动量微分方程。现在将它转变为直角坐标系下对应的方程。

计算张量散度$\nabla\cdot\boldsymbol{P}$：

$$
\begin{aligned}
\nabla\cdot\boldsymbol{P} &= \nabla\cdot(\boldsymbol{p}_x\boldsymbol{i}+\boldsymbol{p}_y\boldsymbol{j}+\boldsymbol{p}_z\boldsymbol{k})\\
&= \nabla\cdot[(p_{xx}\boldsymbol{ii}+p_{xy}\boldsymbol{ji}+p_{xz}\boldsymbol{ki})+(p_{yx}\boldsymbol{ij}+p_{yy}\boldsymbol{jj}+p_{yz}\boldsymbol{kj})+(p_{zx}\boldsymbol{ik}+p_{zy}\boldsymbol{jk}+p_{zz}\boldsymbol{kk})]\\
&= \nabla\cdot[\boldsymbol{i}(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\boldsymbol{k})+\boldsymbol{j}(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})+\boldsymbol{k}(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})]\\
&= \frac{\partial}{\partial x}(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\boldsymbol{k})+\frac{\partial}{\partial y}(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})+\frac{\partial}{\partial z}(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})\\
&= \left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{xy}}{\partial y}+\frac{\partial p_{xz}}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial p_{yx}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{yz}}{\partial z}\right)\boldsymbol{j}+\left(\frac{\partial p_{zx}}{\partial x}+\frac{\partial p_{zy}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)\boldsymbol{k}\\
&= \left[-\frac{\partial p}{\partial x}+2\mu\frac{\partial^2 u}{\partial x^2}+\mu\left(\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 u}{\partial y^2}\right)+\mu\left(\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 u}{\partial z^2}\right)\right]\boldsymbol{i}\\
&\quad+\left[\mu\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial x^2}\right)-\frac{\partial p}{\partial y}+2\mu\frac{\partial^2 v}{\partial y^2}+\mu\left(\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial^2 v}{\partial z^2}\right)\right]\boldsymbol{j}\\
&\quad+\left[\mu\left(\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+\mu\left(\frac{\partial^2 v}{\partial z\partial y}+\frac{\partial^2 w}{\partial y^2}\right)-\frac{\partial p}{\partial z}+2\mu\frac{\partial^2 w}{\partial z^2}\right]\boldsymbol{k}
\end{aligned}
\tag{4-1-5}
$$

因为是不可压缩层流，所以$\nabla\cdot\boldsymbol{v}=0$，则有

$$\frac{\partial^2 w}{\partial z\partial x}=-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}\right)\tag{4-1-6}$$

$$\frac{\partial^2 u}{\partial x\partial y}=-\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial z\partial y}\right)\tag{4-1-7}$$

$$\frac{\partial^2 u}{\partial x\partial z}=-\left(\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial z^2}\right)\tag{4-1-8}$$

将式(4-1-6)、式(4-1-7)、式(4-1-8)代入式(4-1-5)，则有

$$
\begin{aligned}
\nabla\cdot\boldsymbol{P}&=\left[-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right]\boldsymbol{i}+\left[-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\right]\boldsymbol{j}\\
&\quad+\left[-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)\right]\boldsymbol{k}
\end{aligned}
\tag{4-1-9}
$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=f_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\tag{4-1-10}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=f_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\tag{4-1-11}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}-w\frac{\partial w}{\partial z}=f_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)\tag{4-1-12}$$

这就是流体力学中不可压缩层流运动动量微分方程。说明以这种方法推导并建立流体力学的方程是可行的。

4.1.3 不可压缩层流动量矩微分方程

利用$\varphi=0$，$v_s=0$，$\rho=$常数，$\boldsymbol{r}$为参变量，将式(3-4-3)改写为

$$\frac{\partial}{\partial t}(\boldsymbol{r}\times\boldsymbol{v})+\boldsymbol{v}\cdot\nabla(\boldsymbol{r}\times\boldsymbol{v})=\boldsymbol{r}\times\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot(\boldsymbol{v}\times\boldsymbol{P})\tag{4-1-13}$$

它仍然是无坐标系的方程。现将它转化为直角坐标系下的不可压缩层流动量矩微分方程。其

他各项简单，将 $\boldsymbol{v}\cdot\nabla(\boldsymbol{r}\times\boldsymbol{v})$项与$\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P})$项分别推演如下：

$$
\begin{aligned}
\boldsymbol{v}\cdot\nabla(\boldsymbol{r}\times\boldsymbol{v}) &= \left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)[(r_y w-r_z v)\boldsymbol{i}+(r_z u-r_x w)\boldsymbol{j}+(r_x v-r_y u)\boldsymbol{k}] \\
&= \left[u\frac{\partial}{\partial x}(r_y w-r_z v)+v\frac{\partial}{\partial y}(r_y w-r_z v)+w\frac{\partial}{\partial z}(r_y w-r_z v)\right]\boldsymbol{i} \\
&\quad+\left[u\frac{\partial}{\partial x}(r_z u-r_x w)+v\frac{\partial}{\partial y}(r_z u-r_x w)+w\frac{\partial}{\partial z}(r_z u-r_x w)\right]\boldsymbol{j} \\
&\quad+\left[u\frac{\partial}{\partial x}(r_x v-r_y u)+v\frac{\partial}{\partial y}(r_x v-r_y u)+w\frac{\partial}{\partial z}(r_x v-r_y u)\right]\boldsymbol{k}
\end{aligned}
\tag{4-1-14}
$$

式(4-1-14)是数学运算结果，它不能作为处理力学问题的结果。根据是否形成动量矩的原则，对式(4-1-14)最后确定应保留项。因为 $\boldsymbol{r}$ 是参变量，不参加微积分运算，所以

$$
\boldsymbol{v}\cdot\nabla(\boldsymbol{r}\times\boldsymbol{v})=\left(r_y w\frac{\partial w}{\partial z}-r_z v\frac{\partial v}{\partial y}\right)\boldsymbol{i}+\left(r_z u\frac{\partial u}{\partial x}-r_x w\frac{\partial w}{\partial z}\right)\boldsymbol{j}+\left(r_x v\frac{\partial v}{\partial y}-r_y u\frac{\partial u}{\partial x}\right)\boldsymbol{k}
\tag{4-1-15}
$$

$$
\begin{aligned}
\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) &= \nabla\cdot[(r_y\boldsymbol{p}_z-r_z\boldsymbol{p}_y)\boldsymbol{i}+(r_z\boldsymbol{p}_x-r_x\boldsymbol{p}_z)\boldsymbol{j}+(r_x\boldsymbol{p}_y-r_y\boldsymbol{p}_x)\boldsymbol{k}] \\
&= \nabla\cdot\{[r_y(p_{zx}\boldsymbol{i}+p_{zy}\boldsymbol{j}+p_{zz}\boldsymbol{k})-r_z(p_{yx}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{yz}\boldsymbol{k})]\boldsymbol{i} \\
&\quad+[r_z(p_{xx}\boldsymbol{i}+p_{xy}\boldsymbol{j}+p_{xz}\boldsymbol{k})-r_x(p_{zx}\boldsymbol{i}+p_{zy}\boldsymbol{j}+p_{zz}\boldsymbol{k})]\boldsymbol{j} \\
&\quad+[r_x(p_{yx}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{yz}\boldsymbol{k})-r_y(p_{xx}\boldsymbol{i}+p_{xy}\boldsymbol{j}+p_{xz}\boldsymbol{k})]\boldsymbol{k}\} \\
&= \left[r_y\left(\frac{\partial p_{zx}}{\partial x}+\frac{\partial p_{zy}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)-r_z\left(\frac{\partial p_{yx}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{yz}}{\partial z}\right)\right]\boldsymbol{i} \\
&\quad+\left[r_z\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{xy}}{\partial y}+\frac{\partial p_{xz}}{\partial z}\right)-r_x\left(\frac{\partial p_{zx}}{\partial x}+\frac{\partial p_{zy}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)\right]\boldsymbol{j} \\
&\quad+\left[r_x\left(\frac{\partial p_{yx}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{yz}}{\partial z}\right)-r_y\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{xy}}{\partial y}+\frac{\partial p_{xz}}{\partial z}\right)\right]\boldsymbol{k} \\
&= \left\{r_y\left[\frac{\partial}{\partial x}\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(-p+2\mu\frac{\partial w}{\partial z}\right)\right]\right. \\
&\quad\left.-r_z\left[\frac{\partial}{\partial x}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(-p+2\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]\right\}\boldsymbol{i} \\
&\quad+\left\{r_z\left[\frac{\partial}{\partial x}\left(-p+2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]\right. \\
&\quad\left.-r_x\left[\frac{\partial}{\partial x}\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(-p+2\mu\frac{\partial w}{\partial z}\right)\right]\right\}\boldsymbol{j} \\
&\quad+\left\{r_x\left[\frac{\partial}{\partial x}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(-p+2\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]\right. \\
&\quad\left.-r_y\left[\frac{\partial}{\partial x}\left(-p+2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)++\frac{\partial}{\partial z}\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]\right\}\boldsymbol{k}
\end{aligned}
\tag{4-1-16}
$$

引入条件$\nabla\cdot\boldsymbol{v}=0$，则式(4-1-16)可以简化为

$$
\begin{aligned}
\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) &= \left\{\left(r_z\frac{\partial p}{\partial y}-r_y\frac{\partial p}{\partial z}\right)+\mu\left[r_y\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-r_z\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\right]\right\}\boldsymbol{i} \\
&\quad+\left\{\left(r_x\frac{\partial p}{\partial z}-r_z\frac{\partial p}{\partial x}\right)+\mu\left[r_z\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-r_x\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)\right]\right\}\boldsymbol{j}
\end{aligned}
$$

$$+\left\{\left(r_y\frac{\partial p}{\partial x}-r_x\frac{\partial p}{\partial y}\right)+\mu\left[r_x\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-r_y\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right]\right\}\boldsymbol{k} \tag{4-1-17}$$

将式(4-1-15)和式(4-1-17)代回式(4-1-13)，然后将其他各项展开，接三个坐标系，列出：

$$r_y\frac{\partial w}{\partial t}-r_z\frac{\partial v}{\partial t}+r_y w\frac{\partial w}{\partial z}-r_z v\frac{\partial v}{\partial y}$$
$$=r_y f_z-r_z f_y+\frac{1}{\rho}\left(r_z\frac{\partial p}{\partial y}-r_y\frac{\partial p}{\partial z}\right)+\nu\left[r_y\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-r_z\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\right] \tag{4-1-18}$$

$$r_z\frac{\partial u}{\partial t}-r_x\frac{\partial w}{\partial t}+r_z u\frac{\partial u}{\partial x}-r_x w\frac{\partial w}{\partial z}$$
$$=r_z f_x-r_x f_z+\frac{1}{\rho}\left(r_x\frac{\partial p}{\partial z}-r_z\frac{\partial p}{\partial x}\right)+\nu\left[r_z\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-r_x\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)\right] \tag{4-1-19}$$

$$r_x\frac{\partial v}{\partial t}-r_y\frac{\partial u}{\partial t}+r_x v\frac{\partial v}{\partial y}-r_y u\frac{\partial u}{\partial x}$$
$$=r_x f_y-r_y f_x+\frac{1}{\rho}\left(r_y\frac{\partial p}{\partial x}-r_x\frac{\partial p}{\partial y}\right)+\nu\left[r_x\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-r_y\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right] \tag{4-1-20}$$

4.1.4　不可压缩层流能量微分方程

将层流条件 $\varphi=0$，$v_s=0$ 以及不可压缩条件 ρ 为常数代入式(3-4-13)，并除以 ρ，依数学规则，它可写成

$$\frac{\partial}{\partial t}\left(e+\frac{v^2}{2}\right)+(\boldsymbol{v}\cdot\nabla)\left(e+\frac{v^2}{2}\right)=\frac{1}{\rho}\nabla\cdot\nabla\boldsymbol{T}+q_{\mathrm{R}}+\boldsymbol{f}\cdot\boldsymbol{v}+\frac{1}{\rho}(\nabla\cdot\boldsymbol{P}\cdot\boldsymbol{v}) \tag{4-1-21}$$

式(4-1-21)是层流不可压缩能量无坐标系微分方程。现将其转化为直角坐标系相应方程。

首先研究 $\nabla\cdot\boldsymbol{P}\cdot\boldsymbol{v}$ 项：

$$\begin{aligned}\nabla\cdot\boldsymbol{P}\cdot\boldsymbol{v}&=\nabla\cdot(\boldsymbol{p}_x u+\boldsymbol{p}_y v+\boldsymbol{p}_z w)=\nabla\cdot\boldsymbol{p}_x u+\nabla\cdot\boldsymbol{p}_y v+\nabla\cdot\boldsymbol{p}_z w\\
&=\nabla\cdot(p_{xx}\boldsymbol{i}+p_{xy}\boldsymbol{j}+p_{xz}\boldsymbol{k})u+\nabla\cdot(p_{yx}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{yz}\boldsymbol{k})v+\nabla\cdot(p_{zx}\boldsymbol{i}+p_{zy}\boldsymbol{j}+p_{zz}\boldsymbol{k})w\\
&=u\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{xy}}{\partial y}+\frac{\partial p_{xz}}{\partial z}\right)+v\left(\frac{\partial p_{yx}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{yz}}{\partial z}\right)+w\left(\frac{\partial p_{zx}}{\partial x}+\frac{\partial p_{zy}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)\\
&=u\left[\frac{\partial}{\partial x}\left(-p+2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]\\
&\quad+v\left[\frac{\partial}{\partial x}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(-p+2\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]\\
&\quad+w\left[\frac{\partial}{\partial x}\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(-p+2\mu\frac{\partial w}{\partial z}\right)\right]\end{aligned} \tag{4-1-22}$$

利用不可压缩条件 $\nabla\cdot\boldsymbol{v}=0$，将式(4-1-22)化为

$$\nabla\cdot\boldsymbol{P}\cdot\boldsymbol{v}=u\left[-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right]+v\left[-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\right]$$
$$+w\left[-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)\right] \tag{4-1-23}$$

将式(4-1-23)代回式(4-1-21)，并将其他项展开，则有

$$\frac{\partial}{\partial t}\left(e+\frac{v^2}{2}\right)+u\frac{\partial}{\partial x}\left(e+\frac{v^2}{2}\right)+v\frac{\partial}{\partial y}\left(e+\frac{v^2}{2}\right)+w\frac{\partial}{\partial z}\left(e+\frac{v^2}{2}\right)$$
$$=\frac{\lambda}{\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+q_R+\boldsymbol{f}\cdot\boldsymbol{v}+u\left[-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right]$$
$$+v\left[-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\right]+w\left[-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)\right] \tag{4-1-24}$$

式(4-1-24)就是层流不可压缩能量直角坐标系微分方程；定常流时，去掉$\frac{\partial}{\partial t}=0$项即可。

4.2 可压缩层流微分方程组

4.2.1 层流可压缩连续性微分方程

将 $\rho_s=0$，$\varphi=0$，$v_s=0$ 代入式(3-7-1)，得

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\boldsymbol{v}=0 \tag{4-2-1}$$

式(4-2-1)是层流可压缩无坐标系下连续性微分方程。现将其转化为直角坐标系下的对应方程：

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0 \tag{4-2-2}$$

除以 ρ，变成对 1kg 流体而言的方程：

$$\frac{1}{\rho}\left(\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}\right)+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{4-2-3}$$

当定常流时，去掉$\frac{\partial\rho}{\partial t}=0$项即可。

4.2.2 层流可压缩动量微分方程

将 $\rho_s=0$，$\boldsymbol{v}_s=\boldsymbol{0}$，$\varphi=0$ 代入式(3-7-2)，得

$$\frac{D(\rho\boldsymbol{v})}{Dt}+\rho\boldsymbol{v}(\nabla\cdot\boldsymbol{v})=\boldsymbol{f}\rho+\nabla\cdot\boldsymbol{P} \tag{4-2-4}$$

将式(4-2-4)展开，除以 ρ，则有

$$\frac{D\boldsymbol{v}}{Dt}+\frac{\boldsymbol{v}}{\rho}\frac{D\rho}{Dt}+\boldsymbol{v}(\nabla\cdot\boldsymbol{v})=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{4-2-5}$$

将式(4-2-1)除以 ρ 后代入式(4-2-5)，则有

$$\frac{D\boldsymbol{v}}{Dt}=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{4-2-6}$$

展开，得

$$\nabla\cdot\boldsymbol{P}=\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{xy}}{\partial y}+\frac{\partial p_{xz}}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial p_{yx}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{yz}}{\partial z}\right)\boldsymbol{j}+\left(\frac{\partial p_{zx}}{\partial x}+\frac{\partial p_{zy}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)\boldsymbol{k} \tag{4-2-7}$$

令

$$p_{xx}=-p+2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu(\nabla\cdot\boldsymbol{v}) \tag{4-2-8}$$

$$p_{yy}=-p+2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu(\nabla\cdot\boldsymbol{v}) \tag{4-2-9}$$

$$p_{zz}=-p+2\mu\frac{\partial w}{\partial z}-\frac{2}{3}\mu(\nabla\cdot\boldsymbol{v}) \tag{4-2-10}$$

将式(4-2-8)、式(4-2-9)、式(4-2-10)代入式(4-2-7)，然后代回式(4-2-6)，分开 x，y，z 三个方向，则有

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=f_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{4}{3}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\frac{1}{3}\nu\left(\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 w}{\partial z\partial x}\right) \tag{4-2-11}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=f_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{4}{3}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\frac{1}{3}\nu\left(\frac{\partial^2 w}{\partial z\partial y}+\frac{\partial^2 u}{\partial x\partial y}\right) \tag{4-2-12}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=f_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\frac{1}{3}\nu\left(\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 u}{\partial y\partial z}\right) \tag{4-2-13}$$

4.2.3 层流可压缩动量矩微分方程

将 $\varphi=0$，$\rho_s=0$，$\boldsymbol{v}_s=\boldsymbol{0}$代入式(3-7-8)，得

$$\frac{D(\boldsymbol{r}\times\rho\boldsymbol{v})}{Dt}+(\boldsymbol{r}\times\rho\boldsymbol{v})\nabla\cdot\boldsymbol{v}=\boldsymbol{r}\times\rho\boldsymbol{f}+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{4-2-14}$$

$\boldsymbol{r}$ 是参变量，故式(4-2-14)可写成

$$\boldsymbol{r}\times\frac{D(\rho\boldsymbol{v})}{Dt}+(\boldsymbol{r}\times\rho\boldsymbol{v})\nabla\cdot\boldsymbol{v}=\boldsymbol{r}\times\rho\boldsymbol{f}+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{4-2-15}$$

$$\boldsymbol{r}\times\left(\rho\frac{D\boldsymbol{v}}{Dt}+\boldsymbol{v}\frac{D\rho}{Dt}\right)+(\boldsymbol{r}\times\rho\boldsymbol{v})\nabla\cdot\boldsymbol{v}=\boldsymbol{r}\times\rho\boldsymbol{f}+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{4-2-16}$$

式(4-2-16)除以 ρ，得

$$\boldsymbol{r}\times\frac{D\boldsymbol{v}}{Dt}+\boldsymbol{r}\times\frac{\boldsymbol{v}}{\rho}\frac{D\rho}{Dt}+(\boldsymbol{r}\times\boldsymbol{v})\nabla\cdot\boldsymbol{v}=\boldsymbol{r}\times\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{4-2-17}$$

将式(4-2-1)代入式(4-2-17)，得

$$\boldsymbol{r}\times\frac{D\boldsymbol{v}}{Dt}=\boldsymbol{r}\times\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{4-2-18}$$

对式(4-2-18)中的$\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P})$进行如下推演：

$$
\begin{aligned}
\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) &= \nabla\cdot[\boldsymbol{r}\times(\boldsymbol{p}_x\boldsymbol{i}+\boldsymbol{p}_y\boldsymbol{j}+\boldsymbol{p}_z\boldsymbol{k})] \\
&= \nabla\cdot\{\boldsymbol{r}\times[(p_{xx}\boldsymbol{i}+p_{xy}\boldsymbol{j}+p_{xz}\boldsymbol{k})\boldsymbol{i}+(p_{yx}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{yz}\boldsymbol{k})\boldsymbol{j}+(p_{zx}\boldsymbol{i}+p_{zy}\boldsymbol{j}+p_{zz}\boldsymbol{k})\boldsymbol{k}]\} \\
&= \nabla\cdot\{(r_x\boldsymbol{i}+r_y\boldsymbol{j}+r_z\boldsymbol{k})\times[\boldsymbol{i}(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\boldsymbol{k}) \\
&\quad +\boldsymbol{j}(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})+\boldsymbol{k}(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})]\} \\
&= \nabla\cdot[\boldsymbol{k}r_x(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})-\boldsymbol{j}r_x(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})-\boldsymbol{k}r_y(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\boldsymbol{k})+ \\
&\quad \boldsymbol{i}r_y(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})+\boldsymbol{j}r_z(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\boldsymbol{k})-\boldsymbol{i}r_z(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})] \\
&= \nabla\cdot\{\boldsymbol{i}[r_y(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})-r_z(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})] \\
&\quad +\boldsymbol{j}[r_z(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\boldsymbol{k})-r_x(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})] \\
&\quad +\boldsymbol{k}[r_x(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})-r_y(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\boldsymbol{k})]\} \\
&= r_y\left(\frac{\partial p_{xz}}{\partial x}\boldsymbol{i}+\frac{\partial p_{yz}}{\partial x}\boldsymbol{j}+\frac{\partial p_{zz}}{\partial x}\boldsymbol{k}\right)-r_z\left(\frac{\partial p_{xy}}{\partial x}\boldsymbol{i}+\frac{\partial p_{yy}}{\partial x}\boldsymbol{j}+\frac{\partial p_{zy}}{\partial x}\boldsymbol{k}\right) \\
&\quad +r_z\left(\frac{\partial p_{xx}}{\partial y}\boldsymbol{i}+\frac{\partial p_{yx}}{\partial y}\boldsymbol{j}+\frac{\partial p_{zx}}{\partial y}\boldsymbol{k}\right)-r_x\left(\frac{\partial p_{xz}}{\partial y}\boldsymbol{i}+\frac{\partial p_{yz}}{\partial y}\boldsymbol{j}+\frac{\partial p_{zz}}{\partial y}\boldsymbol{k}\right) \\
&\quad +r_x\left(\frac{\partial p_{xy}}{\partial z}\boldsymbol{i}+\frac{\partial p_{yy}}{\partial z}\boldsymbol{j}+\frac{\partial p_{zy}}{\partial z}\boldsymbol{k}\right)-r_y\left(\frac{\partial p_{xx}}{\partial z}\boldsymbol{i}+\frac{\partial p_{yx}}{\partial z}\boldsymbol{j}+\frac{\partial p_{zx}}{\partial z}\boldsymbol{k}\right) \\
&= r_y\left\{\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial z\partial x}\right)\boldsymbol{i}+\mu\left(\frac{\partial^2 w}{\partial y\partial x}+\frac{\partial^2 v}{\partial z\partial x}\right)\boldsymbol{j}\right. \\
&\quad \left.+\left[-\frac{\partial p}{\partial x}+2\mu\frac{\partial^2 w}{\partial z\partial x}-\frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 w}{\partial z\partial x}\right)\right]\boldsymbol{k}\right\} \\
&\quad -r_z\left\{\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial y\partial x}\right)\boldsymbol{i}+\left[-\frac{\partial p}{\partial x}+2\mu\frac{\partial^2 v}{\partial y\partial x}-\frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 w}{\partial z\partial x}\right)\right]\boldsymbol{j}\right. \\
&\quad \left.+\mu\left(\frac{\partial^2 v}{\partial z\partial x}+\frac{\partial^2 w}{\partial y\partial x}\right)\boldsymbol{k}\right\}+r_z\left\{\left[-\frac{\partial p}{\partial y}+2\mu\frac{\partial^2 u}{\partial x\partial y}-\frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial z\partial y}\right)\right]\boldsymbol{i}\right. \\
&\quad \left.+\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)\boldsymbol{j}+\mu\left(\frac{\partial^2 u}{\partial z\partial y}+\frac{\partial^2 w}{\partial x\partial y}\right)\boldsymbol{k}\right\} \\
&\quad -r_x\left\{\mu\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 u}{\partial z\partial y}\right)\boldsymbol{i}+\mu\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 v}{\partial z\partial y}\right)\boldsymbol{j}\right. \\
&\quad \left.+\left[-\frac{\partial p}{\partial y}+2\mu\frac{\partial^2 w}{\partial z\partial y}-\frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial z\partial y}\right)\right]\boldsymbol{k}\right\} \\
&\quad +r_x\left\{\mu\left(\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 u}{\partial y\partial z}\right)\boldsymbol{i}+\left[-\frac{\partial p}{\partial z}+2\mu\frac{\partial^2 v}{\partial y\partial z}-\frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial z^2}\right)\right]\boldsymbol{j}\right. \\
&\quad \left.+\mu\left(\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}\right)\boldsymbol{k}\right\}-r_y\left\{\left[-\frac{\partial p}{\partial z}+2\mu\frac{\partial^2 u}{\partial x\partial z}-\frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial z^2}\right)\right]\boldsymbol{i}\right. \\
&\quad \left.+\mu\left(\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 v}{\partial x\partial z}\right)\boldsymbol{j}++\mu\left(\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}\right)\boldsymbol{k}\right\} \\
&= r_y\frac{\partial p}{\partial z}-r_z\frac{\partial p}{\partial y}+\mu\left[r_y\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y\partial x}+\frac{2}{3}\frac{\partial^2 w}{\partial z^2}+\frac{1}{3}\frac{\partial^2 w}{\partial x\partial z}\right)-r_z\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial x}\right.\right.
\end{aligned}
$$

$$+\frac{2}{3}\frac{\partial^2 v}{\partial y^2}+\frac{1}{3}\frac{\partial^2 v}{\partial x\partial y}\Big)\Big]+r_z\frac{\partial p}{\partial x}-r_x\frac{\partial p}{\partial z}+\mu\Big[r_z\Big(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z\partial y}+\frac{2}{3}\frac{\partial^2 u}{\partial x^2}+\frac{1}{3}\frac{\partial^2 u}{\partial x\partial y}\Big)$$

$$-r_x\Big(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial x\partial y}+\frac{2}{3}\frac{\partial^2 w}{\partial y^2}+\frac{1}{3}\frac{\partial^2 w}{\partial y\partial z}\Big)\Big]+r_x\frac{\partial p}{\partial y}-r_y\frac{\partial p}{\partial x}+\mu\Big[r_x\Big(\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial x\partial z}$$

$$+\frac{2}{3}\frac{\partial^2 v}{\partial y^2}+\frac{1}{3}\frac{\partial^2 v}{\partial z\partial y}\Big)-r_y\Big(\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial y\partial z}+\frac{2}{3}\frac{\partial^2 u}{\partial x^2}+\frac{1}{3}\frac{\partial^2 u}{\partial x\partial z}\Big)\Big] \tag{4-2-19}$$

将式(4-2-19)代入式(4-2-18)，展开其他项，按三个转轴列出：

$$r_y\frac{\partial w}{\partial t}-r_z\frac{\partial v}{\partial t}+r_y w\frac{\partial w}{\partial z}-v r_z\frac{\partial v}{\partial y}=r_y f_z-r_z f_y+\frac{1}{\rho}\Big(r_y\frac{\partial p}{\partial z}-r_z\frac{\partial p}{\partial y}\Big)$$

$$+\nu\Big[r_y\Big(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y\partial x}+\frac{2}{3}\frac{\partial^2 w}{\partial z^2}+\frac{1}{3}\frac{\partial^2 w}{\partial x\partial z}\Big)-r_z\Big(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial x}+\frac{2}{3}\frac{\partial^2 v}{\partial y^2}+\frac{1}{3}\frac{\partial^2 v}{\partial x\partial y}\Big)\Big] \tag{4-2-20}$$

$$r_z\frac{\partial u}{\partial t}-r_x\frac{\partial w}{\partial t}+r_z u\frac{\partial u}{\partial x}-r_x w\frac{\partial w}{\partial z}=r_z f_x-r_x f_z+\frac{1}{\rho}\Big(r_z\frac{\partial p}{\partial x}-r_x\frac{\partial p}{\partial z}\Big)$$

$$+\nu\Big[r_z\Big(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z\partial y}+\frac{2}{3}\frac{\partial^2 u}{\partial x^2}+\frac{1}{3}\frac{\partial^2 u}{\partial x\partial y}\Big)-r_x\Big(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial x\partial y}+\frac{2}{3}\frac{\partial^2 w}{\partial y^2}+\frac{1}{3}\frac{\partial^2 w}{\partial y\partial z}\Big)\Big] \tag{4-2-21}$$

$$r_x\frac{\partial v}{\partial t}-r_y\frac{\partial u}{\partial t}+r_x v\frac{\partial v}{\partial y}-r_y u\frac{\partial u}{\partial x}=r_x f_y-r_y f_x+\frac{1}{\rho}\Big(r_x\frac{\partial p}{\partial y}-r_y\frac{\partial p}{\partial x}\Big)$$

$$+\nu\Big[r_x\Big(\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial x\partial z}+\frac{2}{3}\frac{\partial^2 v}{\partial y^2}+\frac{1}{3}\frac{\partial^2 v}{\partial z\partial y}\Big)-r_y\Big(\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial y\partial z}+\frac{2}{3}\frac{\partial^2 u}{\partial x^2}+\frac{1}{3}\frac{\partial^2 u}{\partial x\partial z}\Big)\Big] \tag{4-2-22}$$

4.2.4　可压缩层流能量微分方程

将 $\varphi=0$，$\boldsymbol{v}_s=\boldsymbol{0}$，$e_s=0$，$\rho_s=0$ 代入式(3-7-11)，并除以 ρ，得

$$\frac{1}{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\rho\Big(e+\frac{v^2}{2}\Big)+\Big(e+\frac{v^2}{2}\Big)\nabla\cdot\boldsymbol{v}=\frac{1}{\rho}\nabla\cdot(\lambda\nabla T)+q_{\mathrm{R}}+\boldsymbol{f}\cdot\boldsymbol{v}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot\boldsymbol{v} \tag{4-2-23}$$

将式(4-2-1)代入式(4-2-23)，得

$$\frac{\mathrm{D}}{\mathrm{D}t}\Big(e+\frac{v^2}{2}\Big)=\frac{1}{\rho}\nabla\cdot(\lambda\nabla T)+q_{\mathrm{R}}+\boldsymbol{f}\cdot\boldsymbol{v}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot\boldsymbol{v} \tag{4-2-24}$$

展开 $\nabla\cdot\boldsymbol{P}\cdot\boldsymbol{v}$，其他均与式(4-1-21)形式一样，有

$$\nabla\cdot\boldsymbol{P}\cdot\boldsymbol{v}=u\Big(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{xy}}{\partial y}+\frac{\partial p_{xz}}{\partial z}\Big)+v\Big(\frac{\partial p_{yx}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{yz}}{\partial z}\Big)+w\Big(\frac{\partial p_{zx}}{\partial x}+\frac{\partial p_{zy}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\Big)$$

$$=u\Big\{\frac{\partial}{\partial x}\Big[-p+2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big]+\frac{\partial}{\partial y}\mu\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big)+\frac{\partial}{\partial z}\mu\Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big)\Big\}$$

$$+v\Big\{\frac{\partial}{\partial x}\mu\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)+\frac{\partial}{\partial y}\Big[-p+2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big]+\frac{\partial}{\partial z}\mu\Big(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\Big)\Big\}$$

$$+w\Big\{\frac{\partial}{\partial x}\mu\Big(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\Big)+\frac{\partial}{\partial y}\mu\Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big)+\frac{\partial}{\partial z}\Big[-p+2\mu\frac{\partial w}{\partial z}-\frac{2}{3}\mu\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big]\Big\}$$

$$=u\Big[-\frac{\partial p}{\partial x}+\mu\Big(\frac{4}{3}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{3}\frac{\partial^2 v}{\partial y\partial x}+\frac{1}{3}\frac{\partial^2 w}{\partial z\partial x}\Big)\Big]$$

$$+v\left[-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{4}{3}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+\frac{1}{3}\frac{\partial^2 u}{\partial y\partial x}+\frac{1}{3}\frac{\partial^2 w}{\partial y\partial z}\right)\right]$$

$$+w\left[-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{4}{3}\frac{\partial^2 w}{\partial z^2}+\frac{1}{3}\frac{\partial^2 u}{\partial z\partial x}+\frac{1}{3}\frac{\partial^2 v}{\partial z\partial y}\right)\right] \tag{4-2-25}$$

将式(4-2-25)代入式(4-2-24)后，展开其他项，则有

$$\frac{\partial}{\partial t}\left(e+\frac{v^2}{2}\right)+u\frac{\partial}{\partial x}\left(e+\frac{v^2}{2}\right)+v\frac{\partial}{\partial y}\left(e+\frac{v^2}{2}\right)+w\frac{\partial}{\partial z}\left(e+\frac{v^2}{2}\right)$$

$$=\frac{\lambda}{\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+q_{\mathrm{R}}+f_x u+f_y v+f_z w$$

$$+u\left[-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{4}{3}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{3}\frac{\partial^2 v}{\partial y\partial x}+\frac{1}{3}\frac{\partial^2 w}{\partial z\partial x}\right)\right]$$

$$+v\left[-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{4}{3}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+\frac{1}{3}\frac{\partial^2 u}{\partial y\partial x}+\frac{1}{3}\frac{\partial^2 w}{\partial y\partial z}\right)\right]$$

$$+w\left[-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{4}{3}\frac{\partial^2 w}{\partial z^2}+\frac{1}{3}\frac{\partial^2 u}{\partial z\partial x}+\frac{1}{3}\frac{\partial^2 v}{\partial z\partial y}\right)\right] \tag{4-2-26}$$

式(4-2-26)就是层流可压缩能量微分方程式。定常流时，去掉$\frac{\partial}{\partial t}$项即可。

4.3 湍流方程封闭相关公式

湍流是同介质的多相流，因此，按照研究多相流的方法，建立起其运动微分方程，而最终目的是建立只有连续相为变量的微分方程。为此，必须建立分散相与连续相变量之间的关系。

4.3.1 湍流剪应力分解

湍流中存在着涡旋运动，由它引起附加剪应力；由连续相速度差引起的是黏性剪应力。湍流中涡旋体积分数为 φ，则其面积分数为 $\varphi^{\frac{2}{3}}$，则湍流时，附加剪应力为

$$x\text{ 方向附加剪应力}=\varphi^{\frac{2}{3}}(p'_{yx}+p'_{zx}) \tag{4-3-1}$$

$$y\text{ 方向附加剪应力}=\varphi^{\frac{2}{3}}(p'_{xy}+p'_{zy}) \tag{4-3-2}$$

$$z\text{ 方向附加剪应力}=\varphi^{\frac{2}{3}}(p'_{xz}+p'_{yz}) \tag{4-3-3}$$

黏性剪应力为

$$x\text{ 方向黏性剪应力}=(1-\varphi^{\frac{2}{3}})(p_{yx}+p_{zx}) \tag{4-3-4}$$

$$y\text{ 方向黏性剪应力}=(1-\varphi^{\frac{2}{3}})(p_{xy}+p_{zy}) \tag{4-3-5}$$

$$z\text{ 方向黏性剪应力}=(1-\varphi^{\frac{2}{3}})(p_{xz}+p_{yz}) \tag{4-3-6}$$

4.3.2　涡旋公式分解

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \omega_{x^+} + \omega_{x^-} \tag{4-3-7}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \omega_{y^+} + \omega_{y^-} \tag{4-3-8}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \omega_{z^+} + \omega_{z^-} \tag{4-3-9}$$

4.3.3　附加剪应力以涡旋平移速度表示

$$\varphi^{\frac{2}{3}}(p'_{yx} + p'_{zx}) = \varphi^{\frac{2}{3}}\rho(u^*_{yz^-}u_{z^-} + u^*_{zy^+}u_{y^+}) \tag{4-3-10}$$

$$\varphi^{\frac{2}{3}}(p'_{xy} + p'_{zy}) = \varphi^{\frac{2}{3}}\rho(v^*_{xz^+}v_{z^+} + v^*_{zx^-}v_{x^-}) \tag{4-3-11}$$

$$\varphi^{\frac{2}{3}}(p'_{xz} + p'_{yz}) = \varphi^{\frac{2}{3}}\rho(w^*_{xy^-}w_{y^-} + w^*_{yx^+}w_{x^+}) \tag{4-3-12}$$

4.3.4　涡旋平移速度图解说明

涡旋平移速度见图 4-1～图 4-6。

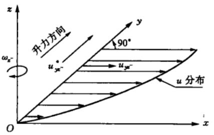


图 4-1　ω_{z^-}具有速度 $u^*_{yz^-}$，u_{z^-}

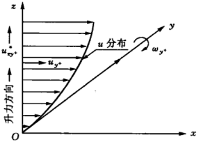


图 4-2　ω_{y^+}具有速度 $u^*_{zy^+}$，u_{y^+}

图 4-1 与图 4-2 中符号说明：

$u^*_{yz^-}$——由涡旋 ω_{z^-}引起的升力，使其产生垂直穿过 y 平面的速度；

u_{z^-}——具有 ω_{z^-}的涡旋沿 x 方向的速度；

$u^*_{zy^+}$——由涡旋 ω_{y^+}引起的升力，使其产生垂直穿过 z 平面的速度；

u_{y^+}——具有 ω_{y^+}的涡旋沿 x 方向运动的速度。

图 4-3 与图 4-4 中符号说明：

$v^*_{xz^+}$——由涡旋 ω_{z^+}引起的升力，使其产生垂直穿过 x 平面的速度；

v_{z^+}——具有 ω_{z^+}的涡旋沿 y 方向的速度；

$v^*_{zx^-}$——由 ω_{x^-}涡旋引起的升力，使其产生垂直穿过 z 平面的速度；

v_{x^-}——具有 ω_{x^-} 的涡旋沿 y 方向运动的速度。

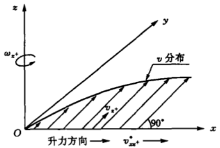


图 4-3　ω_{z^+} 具有速度 $v^*_{xz^+}$，v_{z^+}

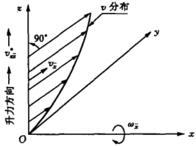


图 4-4　ω_{x^-} 具有速度 $v^*_{zx^-}$，v_{x^-}

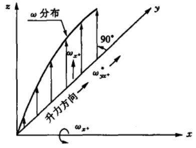


图 4-5　ω_{z^+} 具有速度 $w^*_{yz^+}$，w_{z^+}

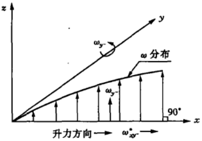


图 4-6　ω_{y^-} 具有速度 $w^*_{xy^-}$，w_{y^-}

图 4-5 与图 4-6 中符号说明：

$w^*_{yz^+}$——由涡旋 ω_{z^+} 引起的升力，使其产生垂直穿过 y 平面的速度；

w_{z^+}——涡旋 ω_{z^+} 沿 z 方向运动的速度；

$w^*_{xy^-}$——由涡旋 ω_{y^-} 引起的升力，使其产生垂直穿过 x 平面的速度；

w_{y^-}——涡旋 ω_{y^-} 沿 z 方向运动的速度。

4.3.5　涡旋平移速度以连续相速度表示

要使湍流运动微分方程数目与未知函数的数目相等，必须将分散相函数与连续相函数联系起来。这样，方程组才能达到封闭。为此，应用式(2-5-7)，现将它写成如下物理方程：

$$u^2 = u_s^2 + \frac{2}{5} r_s^2 \omega^2 \tag{4-3-13}$$

式中，r_s 是涡旋半径。将它用式(2-6-10)置换，则有

$$u_s = \sqrt{u^2 - 4\nu\omega} \tag{4-3-14}$$

流体旋转微团，在边层流界面上产生涡旋运动，即一边旋转，一边以垂直于连续相速度

u^* 平移运动，而且还保留部分能量，使它仍然具有平行于连续相速度 u_s。当它进入层外流区后，它到什么地方，就自动调节能量与该点平行流的能量相等。

在直角坐标系下，涡旋在湍流场中平行于连续相速度表达式。

x 方向：由图 4-1 与图 4-2 可知

$$u_{z^-} = \sqrt{u^2 - 4\nu\omega_{z^-}} = \sqrt{u^2 - 2\nu\frac{\partial u}{\partial y}} \tag{4-3-15}$$

$$u_{y^+} = \sqrt{u^2 - 4\nu\omega_{y^+}} = \sqrt{u^2 - 2\nu\frac{\partial u}{\partial z}} \tag{4-3-16}$$

y 方向：由图 4-3 与图 4-4 可知

$$v_{x^-} = \sqrt{v^2 - 4\nu\omega_{x^-}} = \sqrt{v^2 - 2\nu\frac{\partial v}{\partial z}} \tag{4-3-17}$$

$$v_{z^+} = \sqrt{v^2 - 4\nu\omega_{z^+}} = \sqrt{v^2 - 2\nu\frac{\partial v}{\partial x}} \tag{4-3-18}$$

z 方向：由图 4-5 与图 4-6 可知

$$w_{y^-} = \sqrt{w^2 - 4\nu\omega_{y^-}} = \sqrt{w^2 - 2\nu\frac{\partial w}{\partial x}} \tag{4-3-19}$$

$$w_{x^+} = \sqrt{w^2 - 4\nu\omega_{x^+}} = \sqrt{w^2 - 2\nu\frac{\partial w}{\partial y}} \tag{4-3-20}$$

4.3.6 涡旋垂直无旋流速度公式

涡旋在其升力作用下，使其具有垂直于无旋平行流的速度 u^*。其计算公式应用式(2-2-12)。现将它写成物理方程形式：

$$u^* = \sqrt{\frac{k_1(e^{2t\sqrt{k_1k_2}} - 1)}{k_2(e^{2t\sqrt{k_1k_2}} + 1)}} \tag{4-3-21}$$

作近似处理后，为

$$u^* = \frac{tk_1}{1 + t\sqrt{k_1k_2}} \tag{4-3-22}$$

式中，k_2 由式(2-2-11)计算，并用式(2-6-10)置换后为

$$k_2 = 0.052\nu^{-\frac{1}{2}}\omega^{\frac{1}{2}} \tag{4-3-23}$$

$$k_1 = 6(u - u_s)\omega = 6(U - \sqrt{u^2 - 4\nu\omega})\omega \tag{4-3-24}$$

将 k_1 作近似处理后，为

$$k_1 = \frac{12\nu\omega^2}{u} \tag{4-3-25}$$

$$k_1k_2 = 0.63\nu^{\frac{1}{2}}\omega^{\frac{5}{2}}u^{-1} \tag{4-3-26}$$

将式(4-3-25)与式(4-3-26)代入式(4-3-22)，有

$$u^{*}=\frac{12t\nu\omega^{2}u^{-1}}{1+t\sqrt{0.63\nu^{\frac{1}{2}}\omega^{\frac{5}{2}}u^{-1}}} \tag{4-3-27}$$

在直角坐标系中，u^{*}的计算公式如下。

垂直于 x 轴：

$$u_{yz}^{*}{}^{-}=\frac{12t\nu\omega_{z}^{2}{}^{-}u^{-1}}{1+t\sqrt{0.63\nu^{\frac{1}{2}}\omega_{z}^{\frac{5}{2}}{}^{-}u^{-1}}}=\frac{3t\nu\left(\frac{\partial u}{\partial y}\right)^{2}u^{-1}}{1+t\sqrt{0.11\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)^{2.5}u^{-1}}} \tag{4-3-28}$$

$$u_{zy}^{*}{}^{+}=\frac{12t\nu\omega_{y}^{2}{}^{+}u^{-1}}{1+t\sqrt{0.63\nu^{\frac{1}{2}}\omega_{y}^{2.5}{}^{+}u^{-1}}}=\frac{3t\nu\left(\frac{\partial u}{\partial z}\right)^{2}u^{-1}}{1+t\sqrt{0.11\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial z}\right)^{2.5}u^{-1}}} \tag{4-3-29}$$

垂直于 y 轴：

$$v_{xz}^{*}{}^{+}=\frac{12t\nu\omega_{z}^{2}{}^{+}v^{-1}}{1+t\sqrt{0.63\nu^{\frac{1}{2}}\omega_{z}^{2.5}{}^{+}v^{-1}}}=\frac{3t\nu\left(\frac{\partial v}{\partial x}\right)^{2}v^{-1}}{1+t\sqrt{0.11\nu^{\frac{1}{2}}\left(\frac{\partial v}{\partial x}\right)^{2.5}v^{-1}}} \tag{4-3-30}$$

$$v_{zx}^{*}{}^{-}=\frac{12t\nu\omega_{x}^{2}{}^{-}v^{-1}}{1+t\sqrt{0.63\nu^{\frac{1}{2}}\omega_{x}^{2.5}{}^{-}v^{-1}}}=\frac{3t\nu\left(\frac{\partial v}{\partial z}\right)^{2}v^{-1}}{1+t\sqrt{0.11\nu^{\frac{1}{2}}\left(\frac{\partial v}{\partial z}\right)^{2.5}v^{-1}}} \tag{4-3-31}$$

垂直于 z 轴：

$$w_{xy}^{*}{}^{-}=\frac{12t\nu\omega_{y}^{2}{}^{-}w^{-1}}{1+t\sqrt{0.63\nu^{\frac{1}{2}}\omega_{y}^{2.5}{}^{-}w^{-1}}}=\frac{3t\nu\left(\frac{\partial w}{\partial x}\right)^{2}w^{-1}}{1+t\sqrt{0.11\nu^{\frac{1}{2}}\left(\frac{\partial w}{\partial x}\right)^{2.5}w^{-1}}} \tag{4-3-32}$$

$$w_{yx}^{*}{}^{+}=\frac{12t\nu\omega_{x}^{2}{}^{+}w^{-1}}{1+t\sqrt{0.63\nu^{\frac{1}{2}}\omega_{x}^{2.5}{}^{+}w^{-1}}}=\frac{3t\nu\left(\frac{\partial w}{\partial y}\right)^{2}w^{-1}}{1+t\sqrt{0.11\nu^{\frac{1}{2}}\left(\frac{\partial w}{\partial y}\right)^{2.5}w^{-1}}} \tag{4-3-33}$$

4.3.7 涡旋平移速度自积

在 x 方向，分别有 y 面与 z 面上涡旋平移速度自积，为 $u_{yz}^{*}{}^{-}u_{z}{}^{-}$，$u_{zy}^{*}{}^{+}u_{y}{}^{+}$。
由式(4-3-15)与式(4-3-28)有

$$u_{yz}^{*}{}^{-}u_{z}{}^{-}=\frac{3t\nu\left(\frac{\partial u}{\partial y}\right)^{2}u^{-1}\left(u^{2}-2\nu\frac{\partial u}{\partial y}\right)^{\frac{1}{2}}}{1+t\sqrt{0.11\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)^{2.5}u^{-1}}} \tag{4-3-34}$$

近似处理后，为

$$u_{yz}^{*-}u_z^- = 3\nu t\left(\frac{\partial u}{\partial y}\right)^2 \tag{4-3-35}$$

由式(4-3-16)与式(4-3-29)，有

$$u_{zy}^{*+}u_y^+ = \frac{3t\nu\left(\frac{\partial u}{\partial z}\right)^2 u^{-1}\left(u^2 - 2\nu\frac{\partial u}{\partial z}\right)^{\frac{1}{2}}}{1 + t\sqrt{0.11\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial z}\right)^{2.5} u^{-1}}} \tag{4-3-36}$$

作近似处理后，为

$$u_{zy}^{*+}u_y^+ = 3\nu t\left(\frac{\partial u}{\partial z}\right)^2 \tag{4-3-37}$$

在 y 方向，分别有 x 面与 z 面上涡旋自积，为 $v_{xz}^{*+}v_z^+$，$v_{zx}^{*-}v_x^-$。

由式(4-3-17)与式(4-3-30)，有

$$v_{zx}^{*-}v_x^- = \frac{3t\nu\left(\frac{\partial v}{\partial z}\right)^2 v^{-1}\left(v^2 - 2\nu\frac{\partial v}{\partial z}\right)^{\frac{1}{2}}}{1 + t\sqrt{0.11\nu^{\frac{1}{2}}\left(\frac{\partial v}{\partial z}\right)^{2.5} v^{-1}}} \tag{4-3-38}$$

作近似处理后，为

$$v_{zx}^{*-}v_x^- = 3\nu t\left(\frac{\partial v}{\partial z}\right)^2 \tag{4-3-39}$$

由式(4-3-18)与式(4-3-31)，有

$$v_{xz}^{*+}v_z^+ = \frac{3t\nu\left(\frac{\partial v}{\partial x}\right)^2 v^{-1}\left(v^2 - 2\nu\frac{\partial v}{\partial x}\right)^{\frac{1}{2}}}{1 + t\sqrt{0.11\nu^{\frac{1}{2}}\left(\frac{\partial v}{\partial x}\right)^{2.5} v^{-1}}} \tag{4-3-40}$$

作近似处理后，为

$$v_{xz}^{*+}v_z^+ = 3\nu t\left(\frac{\partial v}{\partial x}\right)^2 \tag{4-3-41}$$

在 z 方向，分别有 x 面与 y 面上涡旋自积，为 $w_{xy}^{*-}w_y^-$，$w_{yx}^{*+}w_x^+$。

由式(4-3-19)与式(4-3-32)，有

$$w_{xy}^{*-}w_y^- = \frac{3t\nu\left(\frac{\partial w}{\partial x}\right)^2 w^{-1}\left(w^2 - 2\nu\frac{\partial w}{\partial x}\right)^{\frac{1}{2}}}{1 + t\sqrt{0.11\nu^{\frac{1}{2}}\left(\frac{\partial w}{\partial x}\right)^{2.5} w^{-1}}} \tag{4-3-42}$$

作近似处理后，为

$$w_{xy}^{*-}w_y^- = 3t\nu\left(\frac{\partial w}{\partial x}\right)^2 \tag{4-3-43}$$

由式(4-3-20)与式(4-3-33)，有

$$w^{*}_{yx^+}w_{x^+}=\frac{3\nu t\left(\frac{\partial w}{\partial y}\right)^2 w^{-1}\left(w^2-2\nu\frac{\partial w}{\partial y}\right)^{\frac{1}{2}}}{1+t\sqrt{0.11\nu^{\frac{1}{2}}\left(\frac{\partial w}{\partial y}\right)^{2.5}w^{-1}}}\tag{4-3-44}$$

作近似处理后，有

$$w^{*}_{yx^+}w_{x^+}=3\nu t\left(\frac{\partial w}{\partial y}\right)^2\tag{4-3-45}$$

4.3.8 附加剪应力以连续相速度表示

要使湍流微分方程组封闭，必须将分散相运动要素转化为用连续相运动要素表示。

首先，必须将湍流附加剪应力转化为以连续相速度表示，x 方向、y 方向、z 方向分别将式(4-3-25)、式(4-3-27)、式(4-3-29)、式(4-3-41)、式(4-3-43)、(4-3-45)代入式(4-3-10)、式(4-3-11)、式(4-3-12)，则有

$$\varphi^{\frac{2}{3}}(p'_{yx}+p'_{zx})=\varphi^{\frac{2}{3}}\rho\nu t\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]\tag{4-3-46}$$

$$\varphi^{\frac{2}{3}}(p'_{zy}+p'_{xy})=\varphi^{\frac{2}{3}}\rho\nu t\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]\tag{4-3-47}$$

$$\varphi^{\frac{2}{3}}(p'_{xz}+p'_{yz})=\varphi^{\frac{2}{3}}\rho\nu t\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\tag{4-3-48}$$

4.3.9 涡旋平移速度表达式

涡旋形成条件不同，其平行于坐标轴的平移速度也各不相同。沿同一个坐标轴有两个涡旋平移速度，取其平均值，作为涡旋平移运动代表速度。

由式(4-3-15)与式(4-3-16)，取

$$u_s=\frac{1}{2}(u_{z^-}+u_{y^+})=u-\frac{\nu}{2u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\tag{4-3-49}$$

由式(4-3-17)与式(4-3-18)，取

$$v_s=\frac{1}{2}(v_{x^-}+v_{z^+})=v-\frac{\nu}{2v}\left(\frac{\partial v}{\partial z}+\frac{\partial v}{\partial x}\right)\tag{4-3-50}$$

由式(4-3-19)与式(4-3-20)，取

$$w_s=\frac{1}{2}(w_{y^-}+w_{x^+})=w-\frac{\nu}{2w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\tag{4-3-51}$$

4.4 不可压缩湍流微分方程组

4.4.1 连续性微分方程

将 $\rho_s=\rho=$常数，代入式(3-4-1)′，得

$$\varphi^{\frac{2}{3}}\nabla\cdot\boldsymbol{v}_s+\varphi_2\nabla\cdot\boldsymbol{v}=0\tag{4-4-1}$$

$$\nabla\cdot\boldsymbol{v}_s=-\frac{\varphi_2}{\varphi^{\frac{2}{3}}}\nabla\cdot\boldsymbol{v} \tag{4-4-2}$$

4.4.2 动量微分方程

将 $\rho_s=\rho=$常数代入式(3-4-4)，并除以 ρ，得

$$\frac{\partial}{\partial t}(\varphi\boldsymbol{v}_s+\varphi_1\boldsymbol{v})+\varphi^{\frac{2}{3}}(\nabla\cdot\boldsymbol{v}_s)\boldsymbol{v}_s+\varphi_2(\nabla\cdot\boldsymbol{v})\boldsymbol{v}=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{4-4-3}$$

将式(4-4-2)代入式(4-4-3)，可化为

$$\frac{\partial\boldsymbol{v}}{\partial t}+\frac{\partial}{\partial t}\varphi(\boldsymbol{v}_s-\boldsymbol{v})+\varphi_2(\boldsymbol{v}\cdot\nabla)(\boldsymbol{v}-\boldsymbol{v}_s)=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{4-4-4}$$

式(4-4-4)仍为无坐标系的不可压缩湍流微分方程。现将其转化为直角坐标系下的对应方程。

分项推导：

$$\frac{\partial}{\partial t}\varphi(\boldsymbol{v}_s-\boldsymbol{v})=\varphi\frac{\partial}{\partial t}(\boldsymbol{v}_s-\boldsymbol{v})=\varphi\frac{\partial}{\partial t}[(u_s-u)\boldsymbol{i}+(v_s-v)\boldsymbol{j}+(w_s-w)\boldsymbol{k}] \tag{4-4-5}$$

将式(4-3-49)至式(4-3-51)代入式(4-4-5)中，得

$$\begin{aligned}\frac{\partial}{\partial t}\varphi(\boldsymbol{v}_s-\boldsymbol{v})&=-\frac{\nu}{2}\varphi\frac{\partial}{\partial t}\left[\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\boldsymbol{i}+\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\boldsymbol{j}+\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\boldsymbol{k}\right]\\&=-\frac{\nu}{2}\varphi\left\{\left[\frac{1}{u}\left(\frac{\partial^2 u}{\partial y\partial t}+\frac{\partial^2 u}{\partial z\partial t}\right)-\frac{1}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial t}\right]\boldsymbol{i}\right.\\&\quad+\left[\frac{1}{v}\left(\frac{\partial^2 v}{\partial x\partial t}+\frac{\partial^2 v}{\partial z\partial t}\right)-\frac{1}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial t}\right]\boldsymbol{j}\\&\quad\left.+\left[\frac{1}{w}\left(\frac{\partial^2 w}{\partial x\partial t}+\frac{\partial^2 w}{\partial y\partial t}\right)-\frac{1}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial t}\right]\boldsymbol{k}\right\}\end{aligned} \tag{4-4-6}$$

$$\begin{aligned}\boldsymbol{v}\cdot\nabla(\boldsymbol{v}-\boldsymbol{v}_s)&=u\frac{\partial(\boldsymbol{v}-\boldsymbol{v}_s)}{\partial x}+v\frac{\partial(\boldsymbol{v}-\boldsymbol{v}_s)}{\partial y}+w\frac{\partial(\boldsymbol{v}-\boldsymbol{v}_s)}{\partial z}\\&=u\frac{\partial}{\partial x}[(u-u_s)\boldsymbol{i}+(v-v_s)\boldsymbol{j}+(w-w_s)\boldsymbol{k}]\\&\quad+v\frac{\partial}{\partial y}[(u-u_s)\boldsymbol{i}+(v-v_s)\boldsymbol{j}+(w-w_s)\boldsymbol{k}]\\&\quad+w\frac{\partial}{\partial z}[(u-u_s)\boldsymbol{i}+(v-v_s)\boldsymbol{j}+(w-w_s)\boldsymbol{k}]\\&=\frac{\nu}{2}\left\{u\frac{\partial}{\partial x}\left[\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\boldsymbol{i}+\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\boldsymbol{j}+\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\boldsymbol{k}\right]\right.\\&\quad\left.+v\frac{\partial}{\partial y}\left[\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\boldsymbol{i}+\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\boldsymbol{j}+\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\boldsymbol{k}\right]\right\}\\&\quad+w\frac{\partial}{\partial z}\left[\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\boldsymbol{i}+\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\boldsymbol{j}+\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\boldsymbol{k}\right]\\&=\frac{\nu}{2}\left\{\left[\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial z\partial x}-\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial x}\right]\boldsymbol{i}+\left[\frac{u}{v}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial x}\right)\right.\right.\\&\quad\left.-\frac{u}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\right]\frac{\partial v}{\partial x}\boldsymbol{j}+\left[\frac{u}{w}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y\partial x}\right)-\frac{u}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial x}\right]\boldsymbol{k}\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{v}{u}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z\partial y}\right)-\frac{v}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial y}\right]\boldsymbol{i}+\left[\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 v}{\partial z\partial y}\right.\\
&\left.-\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial y}\right]\boldsymbol{j}+\left[\frac{v}{w}\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\right)-\frac{u}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial y}\right]\boldsymbol{k}\\
&+\left[\frac{w}{u}\left(\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{w}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial z}\right]\boldsymbol{i}+\left[\frac{w}{v}\left(\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 v}{\partial z^2}\right)\right.\\
&\left.-\frac{w}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial z}+\frac{\partial^2 w}{\partial x\partial z}\right]\boldsymbol{j}+\left[\frac{\partial^2 w}{\partial y\partial z}-\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial z}\right]\boldsymbol{k}\Big\}
\end{aligned}
\tag{4-4-7}
$$

式(4-4-7)中，$\boldsymbol{i}$，$\boldsymbol{j}$，$\boldsymbol{k}$ 是代表 x 平面，y 平面，z 平面按速度决定的方向。

$$
\begin{aligned}
\boldsymbol{v}\cdot\nabla(\boldsymbol{v}-\boldsymbol{v}_s)=\frac{\nu}{2}\Big\{&\left[\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial z\partial x}-\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial x}+\frac{v}{u}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z\partial y}\right)\right.\\
&\left.-\frac{v}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial y}+\frac{w}{u}\left(\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{w}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial z}\right]\boldsymbol{i}\\
&+\left[\frac{u}{v}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial x}\right)-\frac{u}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 v}{\partial z\partial y}\right.\\
&\left.-\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial y}+\frac{w}{v}\left(\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{w}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial z}\right]\boldsymbol{j}\\
&+\left[\frac{u}{w}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y\partial x}\right)-\frac{u}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial x}+\frac{v}{w}\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\right.\\
&\left.-\frac{v}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial y}+\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}-\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial z}\right]\boldsymbol{k}\Big\}
\end{aligned}
\tag{4-4-8}
$$

$$
\begin{aligned}
\nabla\cdot\boldsymbol{P}&=\nabla\cdot(\boldsymbol{p}_x\boldsymbol{i}+\boldsymbol{p}_y\boldsymbol{j}+\boldsymbol{p}_z\boldsymbol{k})\\
&=\nabla\cdot[(p_{xx}\boldsymbol{i}+p_{xy}\boldsymbol{j}+p_{xz}\boldsymbol{k})\boldsymbol{i}+(p_{yx}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{yz}\boldsymbol{k})\boldsymbol{j}+(p_{zx}\boldsymbol{i}+p_{zy}\boldsymbol{j}+p_{zz}\boldsymbol{k})\boldsymbol{k}]\\
&=\nabla\cdot[\boldsymbol{i}(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\boldsymbol{k})+\boldsymbol{j}(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})+\boldsymbol{k}(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})]\\
&=\frac{\partial}{\partial x}(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\boldsymbol{k})+\frac{\partial}{\partial y}(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})+\frac{\partial}{\partial z}(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})\\
&=\left(\frac{\partial}{\partial x}p_{xx}+\frac{\partial}{\partial y}p_{xy}+\frac{\partial}{\partial z}p_{xz}\right)\boldsymbol{i}+\left(\frac{\partial}{\partial x}p_{yx}+\frac{\partial}{\partial y}p_{yy}+\frac{\partial}{\partial z}p_{yz}\right)\boldsymbol{j}+\left(\frac{\partial}{\partial x}p_{zx}+\frac{\partial}{\partial y}p_{zy}+\frac{\partial}{\partial z}p_{zz}\right)\boldsymbol{k}\\
&=\left[\frac{\partial}{\partial x}p_{xx}+\frac{\partial}{\partial y}p_{xy}+\frac{\partial}{\partial z}p_{xz}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{xy}}{\partial y}+\frac{\partial p'_{xz}}{\partial z}-\frac{\partial p_{xy}}{\partial y}-\frac{\partial p_{xz}}{\partial z}\right)\right]\boldsymbol{i}\\
&\quad\left[\frac{\partial}{\partial y}p_{yy}+\frac{\partial}{\partial x}p_{yx}+\frac{\partial}{\partial z}p_{yz}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{yx}}{\partial x}+\frac{\partial p'_{yz}}{\partial z}-\frac{\partial p_{yx}}{\partial x}-\frac{\partial p_{yz}}{\partial z}\right)\right]\boldsymbol{j}\\
&\quad\left[\frac{\partial}{\partial z}p_{zz}+\frac{\partial}{\partial x}p_{zx}+\frac{\partial}{\partial y}p_{zy}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{zx}}{\partial x}+\frac{\partial p'_{zy}}{\partial y}-\frac{\partial p_{zx}}{\partial x}-\frac{\partial p_{zy}}{\partial y}\right)\right]\boldsymbol{k}
\end{aligned}
\tag{4-4-9}
$$

将湍流附加剪应力式(4-3-46)至式(4-3-48)代入式(4-4-9)，得

$$
\begin{aligned}
\nabla\cdot\boldsymbol{P}=\Big\{&-\frac{\partial p}{\partial x}+2\mu\frac{\partial^2 u}{\partial x^2}+\mu\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 u}{\partial z^2}\right)\\
&+\varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial z}\right)-\mu\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 u}{\partial z^2}\right)\right]\Big\}\boldsymbol{i}
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{-\frac{\partial p}{\partial y}+\mu\left(2\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial^2 v}{\partial z^2}\right)+\varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y\partial z}\right)\right.\right.\\
&\left.\left.-\mu\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial^2 u}{\partial z^2}\right)\right]\right\}\boldsymbol{j}+\left\{-\frac{\partial p}{\partial z}+\mu\left(2\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial y}\right.\right.\\
&\left.\left.+\frac{\partial^2 w}{\partial y^2}\right)+\varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial u}{\partial z}-\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial v}{\partial z}-\frac{\partial^2 v}{\partial z\partial y}\right)-\mu\left(\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^w}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\right]\right\}\boldsymbol{k}\\
&=\left\{-\frac{\partial p}{\partial x}+\mu\left(2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial z\partial x}\right)+\varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial x}\right)\right.\right.\\
&\left.\left.-\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial z\partial x}\right)\right]\right\}\boldsymbol{i}+\left\{-\frac{\partial p}{\partial y}+\mu\left(2\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial x\partial y}\right.\right.\\
&\left.\left.+\frac{\partial^2 v}{\partial z\partial y}\right)+\varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial y}\right)-\mu\left(\frac{\partial v^2}{\partial x\partial y}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial z\partial y}\right)\right]\right\}\boldsymbol{j}\\
&+\left\{-\frac{\partial p}{\partial z}+\mu\left(2\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}\right)+\varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y\partial z}\right)\right.\right.\\
&\left.\left.-\mu\left(\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\right]\right\}\boldsymbol{k}
\end{aligned}
\tag{4-4-10}
$$

$$
\begin{aligned}
&\frac{\partial u}{\partial t}-\frac{\nu}{2}\varphi\left[\frac{1}{u}\left(\frac{\partial^2 u}{\partial y\partial t}+\frac{\partial^2 u}{\partial z\partial t}\right)-\frac{1}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial t}\right]+\frac{\nu}{2}\varphi_2\left[\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial z\partial x}-\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\right.\\
&\left.+\frac{v}{u}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z\partial y}\right)-\frac{v}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial y}+\frac{w}{u}\left(\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{w}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial z}\right]\\
&=f_x-\frac{\partial p}{\rho\partial x}+\nu\left(2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial z\partial x}\right)+\varphi^{\frac{2}{3}}\left[2\nu t'\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial x}\right)\right.\\
&\left.-\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial z\partial x}\right)\right]
\end{aligned}
\tag{4-4-11}
$$

$$
\begin{aligned}
&\frac{\partial v}{\partial t}-\frac{\nu}{2}\varphi\left[\frac{1}{v}\left(\frac{\partial^2 v}{\partial x\partial t}+\frac{\partial^2 u}{\partial z\partial t}\right)-\frac{1}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial t}\right]+\frac{\nu}{2}\varphi_2\left[\frac{u}{v}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial x}\right)\right.\\
&\left.-\frac{u}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 v}{\partial z\partial y}-\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial y}+\frac{w}{v}\left(\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{w}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial z}\right]\\
&=f_y-\frac{\partial p}{\rho\partial y}+\nu\left(2\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 v}{\partial z\partial y}\right)+\varphi^{\frac{2}{3}}\left[2\nu t'\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial y}\right)\right.\\
&\left.-\nu\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial z\partial y}\right)\right]
\end{aligned}
\tag{4-4-12}
$$

$$
\begin{aligned}
&\frac{\partial w}{\partial t}-\frac{\nu}{2}\varphi\left[\frac{1}{w}\left(\frac{\partial^2 w}{\partial x\partial t}+\frac{\partial^2 w}{\partial y\partial t}\right)-\frac{1}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial t}\right]+\frac{\nu}{2}\varphi_2\left[\frac{u}{w}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y\partial x}\right)-\frac{u}{w^2}\right.\\
&\left.\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial x}+\frac{v}{w}\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\right)-\frac{v}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial y}+\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}-\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial z}\right]\\
&=f_z-\frac{\partial p}{\rho\partial z}+\nu\left(2\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}\right)+\varphi^{\frac{2}{3}}\left[2\nu t'\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y\partial z}\right)\right.
\end{aligned}
$$

$$-\nu\left(\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\Big] \tag{4-4-13}$$

式(4-4-11)至式(4-4-13)就是不可压缩湍流动量在直角坐标下的微分方程，当定常流时去掉含$\frac{\partial}{\partial t}$项。

4.4.3 动量矩微分方程

将 $\rho_s=\rho=$常数，代入(3-4-8)式，得

$$\frac{\partial}{\partial t}[\boldsymbol{r}\times(\varphi\boldsymbol{v}_s+\varphi_1\boldsymbol{v})]+\nabla\cdot\boldsymbol{v}_s(\boldsymbol{r}\times\varphi^{\frac{2}{3}}\boldsymbol{v}_s)+\nabla\cdot\boldsymbol{v}(\boldsymbol{r}\times\varphi_2\boldsymbol{v})=\boldsymbol{r}\times\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{4-4-14}$$

利用式(4-4-2)化简上式，注意到 $\boldsymbol{r}$ 是参变量，有

$$\boldsymbol{r}\times\frac{\partial}{\partial t}[\varphi(\boldsymbol{v}_s-\boldsymbol{v})]+\boldsymbol{r}\times\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{r}\times[\boldsymbol{v}\cdot\nabla\varphi_2(\boldsymbol{v}-\boldsymbol{v}_s)]=\boldsymbol{r}\times\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{4-4-15}$$

首先推导式(4-4-15)的第一项。

$$\boldsymbol{r}\times[\boldsymbol{v}\cdot\nabla\varphi_2(\boldsymbol{v}-\boldsymbol{v}_s)]$$

$$=\boldsymbol{r}\times\varphi_2\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\left[-\frac{\nu}{2u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\boldsymbol{i}-\frac{\nu}{2v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\boldsymbol{j}-\frac{\nu}{2w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\boldsymbol{k}\right]$$

$$=\frac{\nu}{2}\varphi_2\boldsymbol{v}\cdot\nabla\left\{\boldsymbol{r}\times\left[\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\boldsymbol{i}+\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\boldsymbol{j}+\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\boldsymbol{k}\right]\right\}$$

$$=\frac{\nu}{2}\varphi_2\boldsymbol{v}\cdot\nabla\left\{\boldsymbol{i}\left[r_y\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)-\frac{r_z}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\right]+\boldsymbol{j}\left[\frac{r_z}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)-\frac{r_x}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]\right.$$
$$\left.+\boldsymbol{k}\left[\frac{r_x}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)-\frac{r_y}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\right]\right\}$$

$$=-\frac{\nu}{2}\varphi_2\left\{u\left\{r_y\left[\frac{1}{w}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y\partial x}\right)-\frac{1}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial x}\right]-r_z\left[\frac{1}{v}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial x}\right)\right.\right.\right.$$
$$\left.\left.-\frac{1}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial x}\right]\right\}+v\left\{r_y\left[\frac{1}{w}\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\right)-\frac{1}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial y}\right]\right.$$
$$\left.-r_z\left[\frac{1}{v}\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 v}{\partial z\partial y}\right)-\frac{1}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial y}\right]\right\}+w\left\{r_y\left[\frac{1}{w}\left(\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}\right)\right.\right.$$
$$\left.\left.\left.-\frac{1}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial z}\right]-r_z\left[\frac{1}{v}\left(\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{1}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial z}\right]\right\}\right\}\boldsymbol{i}$$

$$+\frac{\nu}{2}\varphi_2\left\{u\left\{r_z\left[\frac{1}{u}\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial z\partial x}\right)-\frac{1}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial x}\right]-r_x\left[\frac{1}{w}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y\partial x}\right)\right.\right.\right.$$
$$\left.\left.-\frac{1}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial x}\right]\right\}+v\left\{r_z\left[\frac{1}{u}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z\partial y}\right)-\frac{1}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial y}\right]\right.$$
$$\left.-r_x\left[\frac{1}{w}\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\right)-\frac{1}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial y}\right]\right\}+w\left\{r_z\left[\frac{1}{u}\left(\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial z^2}\right)\right.\right.$$
$$\left.\left.\left.-\frac{1}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial z}\right]-r_x\left[\frac{1}{w}\left(\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}\right)-\frac{1}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial z}\right]\right\}\right\}\boldsymbol{j}$$

$$
\begin{aligned}
&+\frac{\nu}{2}\varphi_2\left\{u\left\{r_x\left[\frac{1}{v}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial x}\right)-\frac{1}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial x}\right]-r_y\left[\frac{1}{u}\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial z\partial x}\right)\right.\right.\right.\\
&\left.\left.-\frac{1}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial x}\right]\right\}+v\left\{r_x\left[\frac{1}{v}\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 v}{\partial z\partial y}\right)-\frac{1}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial y}\right]\right.\\
&\left.-r_y\left[\frac{1}{u}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z\partial y}\right)-\frac{1}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial y}\right]\right\}+w\left\{r_x\left[\frac{1}{v}\left(\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 v}{\partial z^2}\right)\right.\right.\\
&\left.\left.\left.-\frac{1}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial z}\right]-r_y\left[\frac{1}{u}\left(\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{1}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial z}\right]\right\}\right\}\boldsymbol{k}
\end{aligned}
\tag{4-4-16}
$$

推导

$$
\begin{aligned}
\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P})&=\nabla\cdot[\boldsymbol{r}\times(\boldsymbol{p}_x\boldsymbol{i}+\boldsymbol{p}_y\boldsymbol{j}+\boldsymbol{p}_z\boldsymbol{k})]\\
&=\nabla\cdot\{\boldsymbol{r}\times[(p_{xx}\boldsymbol{i}+p_{xy}\boldsymbol{j}+p_{xz}\boldsymbol{k})\boldsymbol{i}+(p_{yx}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{yz}\boldsymbol{k})\boldsymbol{j}+(p_{zx}\boldsymbol{i}+p_{zy}\boldsymbol{j}+p_{zz}\boldsymbol{k})\boldsymbol{k}]\}\\
&=\nabla\cdot\{\boldsymbol{r}\times[\boldsymbol{i}(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\boldsymbol{k})+\boldsymbol{j}(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})+\boldsymbol{k}(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})]\}\\
&=\nabla\cdot[\boldsymbol{k}\times r_x(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})-\boldsymbol{j}r_x(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})-\boldsymbol{k}r_y(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\\
&\quad\boldsymbol{k})+\boldsymbol{i}r_y(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})+\boldsymbol{j}r_z(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\boldsymbol{k})-\boldsymbol{i}r_z(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})]\\
&=\nabla\cdot\{\boldsymbol{i}[r_y(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})-r_z(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})]+\boldsymbol{j}[r_z(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\\
&\quad\boldsymbol{k})-r_x(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})]+\boldsymbol{k}[r_x(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})-r_y(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\\
&\quad\boldsymbol{k})]\}\\
&=\frac{\partial}{\partial x}[r_y(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})-r_z(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})]\\
&\quad+\frac{\partial}{\partial y}[r_z(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\boldsymbol{k})-r_x(p_{xz}\boldsymbol{i}+p_{yz}\boldsymbol{j}+p_{zz}\boldsymbol{k})]\\
&\quad+\frac{\partial}{\partial z}[r_x(p_{xy}\boldsymbol{i}+p_{yy}\boldsymbol{j}+p_{zy}\boldsymbol{k})-r_y(p_{xx}\boldsymbol{i}+p_{yx}\boldsymbol{j}+p_{zx}\boldsymbol{k})]\\
&=r_y\left[\frac{\partial}{\partial x}(p_{xz}+p'_{xz})\boldsymbol{i}+\frac{\partial}{\partial x}(p_{yz}+p'_{yz})\boldsymbol{j}+\frac{\partial}{\partial x}p_{zz}\boldsymbol{k}\right]\\
&\quad-r_z\left[\frac{\partial}{\partial y}p_{xx}\boldsymbol{i}+\frac{\partial}{\partial y}(p_{yx}+p'_{yx})\boldsymbol{j}+\frac{\partial}{\partial y}(p_{zx}+p'_{zx})\boldsymbol{k}\right]\\
&\quad-r_x\left[\frac{\partial}{\partial y}(p_{xz}+p'_{xz})\boldsymbol{i}+\frac{\partial}{\partial y}(p_{yz}+p'_{yz})\boldsymbol{j}+\frac{\partial}{\partial y}p_{zz}\boldsymbol{k}\right]\\
&\quad+r_x\left[\frac{\partial}{\partial z}(p_{xy}+p'_{xy})\boldsymbol{i}+\frac{\partial}{\partial z}p_{yy}\boldsymbol{j}+\frac{\partial}{\partial z}(p_{zy}+p'_{zy})\boldsymbol{k}\right]\\
&\quad-r_y\left[\frac{\partial}{\partial z}p_{xx}\boldsymbol{i}+\frac{\partial}{\partial z}(p_{yx}+p'_{yx})\boldsymbol{j}+\frac{\partial}{\partial z}(p_{zx}+p'_{zx})\boldsymbol{k}\right]\\
&=r_y\left\{\left[\frac{\partial}{\partial x}p_{xz}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{xz}}{\partial x}-\frac{\partial p_{xz}}{\partial x}\right)\right]\boldsymbol{i}+\left[\frac{\partial}{\partial x}p_{yz}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{yz}}{\partial x}-\frac{\partial p_{yz}}{\partial x}\right)\right]\boldsymbol{j}+\frac{\partial}{\partial x}p_{zz}\boldsymbol{k}\right\}\\
&\quad-r_z\left\{\left[\frac{\partial}{\partial x}p_{xy}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{xy}}{\partial x}-\frac{\partial p_{xy}}{\partial x}\right)\right]\boldsymbol{i}+\frac{\partial}{\partial x}p_{yy}\boldsymbol{j}+\left[\frac{\partial p_{zy}}{\partial x}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{zy}}{\partial x}-\frac{\partial p_{zy}}{\partial x}\right)\right]\boldsymbol{k}\right\}\\
&\quad+r_z\left\{\frac{\partial}{\partial y}p_{xx}\boldsymbol{i}+\left[\frac{\partial p_{yx}}{\partial y}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{yx}}{\partial y}-\frac{\partial p_{yx}}{\partial y}\right)\right]\boldsymbol{j}+\left[\frac{\partial p_{zx}}{\partial y}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{zx}}{\partial y}-\frac{\partial p_{zx}}{\partial y}\right)\right]\boldsymbol{k}\right\}\\
&\quad-r_x\left\{\left[\frac{\partial p_{xz}}{\partial y}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{xz}}{\partial y}-\frac{\partial p_{xz}}{\partial y}\right)\right]\boldsymbol{i}+\left[\frac{\partial p_{yz}}{\partial y}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{yz}}{\partial y}-\frac{\partial p_{yz}}{\partial y}\right)\right]\boldsymbol{j}+\frac{\partial}{\partial y}p_{zz}\boldsymbol{k}\right\}
\end{aligned}
$$

$$+ r_x\left\{\left[\frac{\partial p_{xz}}{\partial z}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{xz}}{\partial z}-\frac{\partial p_{xz}}{\partial z}\right)\right]\boldsymbol{i}+\frac{\partial}{\partial z}p_{yy}\boldsymbol{j}+\left[\frac{\partial p_{zy}}{\partial z}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{zy}}{\partial z}-\frac{\partial p_{zy}}{\partial z}\right)\right]\boldsymbol{k}\right\}$$

$$- r_y\left\{\frac{\partial}{\partial z}p_{xx}\boldsymbol{i}+\left[\frac{\partial p_{xy}}{\partial z}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{xy}}{\partial z}-\frac{\partial p_{xy}}{\partial z}\right)\right]\boldsymbol{j}+\left[\frac{\partial p_{zx}}{\partial z}+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{zx}}{\partial z}-\frac{\partial p_{zx}}{\partial z}\right)\right]\boldsymbol{k}\right\}$$

$$= r_y\left\{\left\{\frac{\partial}{\partial x}\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial x}\rho\nu t'\left(\frac{\partial w}{\partial x}\right)^2-\frac{\partial}{\partial x}\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]\right\}\boldsymbol{i}\right.$$

$$\left.+\left\{\frac{\partial}{\partial x}\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial x}\rho\nu t'\left(\frac{\partial w}{\partial y}\right)^2-\frac{\partial \mu}{\partial x}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]\right\}\boldsymbol{j}+\frac{\partial}{\partial x}\left(-p+2\mu\frac{\partial w}{\partial z}\right)\boldsymbol{k}\right\}$$

$$- r_z\left\{\left\{\frac{\partial}{\partial x}\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial x}\rho\nu t'\left(\frac{\partial v}{\partial x}\right)^2-\frac{\partial}{\partial x}\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]\right\}\boldsymbol{i}\right.$$

$$\left.+\frac{\partial}{\partial x}\left(-p+2\mu\frac{\partial v}{\partial y}\right)\boldsymbol{j}+\left\{\frac{\partial}{\partial x}\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial x}\rho\nu t'\left(\frac{\partial v}{\partial z}\right)^2-\frac{\partial \mu}{\partial x}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]\right\}\boldsymbol{k}\right\}$$

$$+ r_z\left\{\frac{\partial}{\partial y}\left(-p+2\mu\frac{\partial u}{\partial x}\right)\boldsymbol{i}+\left\{\frac{\partial}{\partial y}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial y}\rho\nu t'\left(\frac{\partial u}{\partial y}\right)^2-\frac{\partial}{\partial y}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\right\}\boldsymbol{j}\right.$$

$$\left.+\left\{\frac{\partial}{\partial y}\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial y}\rho\nu t'\left(\frac{\partial u}{\partial z}\right)^2-\frac{\partial}{\partial y}\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]\right\}\boldsymbol{k}\right\}$$

$$- r_x\left\{\left\{\frac{\partial \mu}{\partial y}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial y}\rho\nu t'\left(\frac{\partial w}{\partial x}\right)^2-\frac{\partial}{\partial y}\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]\right\}\boldsymbol{i}\right.$$

$$\left.+\left\{\frac{\partial}{\partial y}\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial y}\rho\nu t'\left(\frac{\partial w}{\partial y}\right)^2-\frac{\partial}{\partial y}\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]\right\}\boldsymbol{j}+\frac{\partial}{\partial y}\left(-p+2\mu\frac{\partial w}{\partial z}\right)\boldsymbol{k}\right\}$$

$$+ r_x\left\{\left\{\frac{\partial}{\partial z}\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial z}\rho\nu t'\left(\frac{\partial w}{\partial x}\right)^2-\frac{\partial}{\partial z}\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]\right\}\boldsymbol{i}\right.$$

$$\left.+\frac{\partial}{\partial z}\left(-p+2\mu\frac{\partial v}{\partial y}\right)\boldsymbol{j}+\left\{\frac{\partial}{\partial z}\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial z}\rho\nu t'\left(\frac{\partial v}{\partial z}\right)^2-\frac{\partial}{\partial z}\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]\right\}\boldsymbol{k}\right\}$$

$$- r_y\left\{\frac{\partial}{\partial z}\left(-p+2\mu\frac{\partial u}{\partial x}\right)\boldsymbol{i}+\left\{\frac{\partial}{\partial z}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial z}\rho\nu t'\left(\frac{\partial u}{\partial y}\right)^2-\frac{\partial}{\partial z}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\right\}\boldsymbol{j}\right.$$

$$\left.+\left\{\frac{\partial}{\partial z}\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial z}\rho\nu t'\left(\frac{\partial u}{\partial z}\right)^2-\frac{\partial}{\partial z}\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]\right\}\boldsymbol{k}\right\}$$

$$= r_y\frac{\partial p}{\partial z}-r_z\frac{\partial p}{\partial y}+r_y\left\{\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z\partial x}\right)+\varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y\partial x}\right)\right.\right.$$

$$\left.\left.-\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y\partial x}\right)\right]\right\}-r_z\left\{\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial x}+\frac{\partial^2 v}{\partial y\partial x}\right)\right.$$

$$\left.+\varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial x}\right)-\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial x}\right)\right]\right\}\boldsymbol{i}$$

$$+ r_z\frac{\partial p}{\partial x}-r_x\frac{\partial p}{\partial z}+r_z\left\{\mu\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z\partial y}\right)+\varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial y}\right)\right.\right.$$

$$\left.\left.-\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z\partial y}\right)\right]\right\}-r_x\left\{\mu\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z\partial y}-\frac{\partial^2 w}{\partial x\partial z}\right)\right.$$

$$\left.+\varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial z}\right)-\mu\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w}{\partial x\partial z}-\frac{\partial^2 w}{\partial y\partial z}\right)\right]\right\}\boldsymbol{j}$$

$$+ r_x \frac{\partial p}{\partial y} - r_y \frac{\partial p}{\partial x} + r_x \left\{ \mu \left(\frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 v}{\partial z^2} - \frac{\partial^2 v}{\partial z \partial y} \right) + \varphi^{\frac{2}{3}} \left[2\rho\nu t' \left(\frac{\partial v}{\partial z} \frac{\partial^2 v}{\partial z^2} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y \partial z} \right) \right. \right.$$

$$\left. \left. - \mu \left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial z \partial y} \right) \right] \right\} - r_y \left\{ \mu \left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z^2} \right) \right.$$

$$\left. + \varphi^{\frac{2}{3}} \left[2\rho\nu t' \left(\frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial z^2} \right) - \mu \left(\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z^2} \right) \right] \right\} \boldsymbol{k} \tag{4-4-17}$$

将式(4-4-16)、式(4-4-17)代回式(4-4-15)，并按三个转轴分开列出动量矩微分方程。

$$r_y \frac{\partial w}{\partial t} - r_z \frac{\partial v}{\partial t} - \frac{\nu}{2} \varphi \left[\frac{1}{u} \left(\frac{\partial^2 u}{\partial y \partial t} + \frac{\partial^2 u}{\partial z \partial t} \right) - \frac{1}{u^2} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) \frac{\partial u}{\partial t} \right] + \frac{\nu}{2} \varphi_2$$

$$\left\{ u \left\{ r_y \left[\frac{1}{w} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x} \right) - \frac{1}{w^2} \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial w}{\partial x} \right] - r_z \left[\frac{1}{v} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z \partial x} \right) - \frac{1}{v^2} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \right) \frac{\partial v}{\partial x} \right] \right\} \right.$$

$$+ v \left\{ r_y \left[\frac{1}{w} \left(\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} \right) - \frac{1}{w^2} \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial w}{\partial y} \right] - r_z \left[\frac{1}{v} \left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 v}{\partial z \partial y} \right) - \frac{1}{v^2} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \right) \frac{\partial v}{\partial y} \right] \right\}$$

$$\left. + w \left\{ r_y \left[\frac{1}{w} \left(\frac{\partial^2 w}{\partial x \partial z} + \frac{\partial^2 w}{\partial y \partial z} \right) - \frac{1}{w^2} \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial w}{\partial z} \right] - r_z \left[\frac{1}{v} \left(\frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 v}{\partial z^2} \right) - \frac{1}{v^2} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \right) \frac{\partial v}{\partial z} \right] \right\} \right\}$$

$$= r_y f_z - r_z f_y + r_y \frac{\rho \partial p}{\partial z} - r_z \frac{\partial p}{\rho \partial y}$$

$$+ r_y \left\{ \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z \partial x} \right) + \varphi^{\frac{2}{3}} \left[2\nu t' \left(\frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y \partial x} \right) - \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x} \right) \right] \right\}$$

$$- r_z \left\{ \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z \partial x} + \frac{\partial^2 v}{\partial y \partial x} \right) + \varphi^{\frac{2}{3}} \left[2\nu t' \left(\frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial z} \frac{\partial^2 v}{\partial z \partial x} \right) - \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z \partial x} \right) \right] \right\} \tag{4-4-18}$$

$$r_z \frac{\partial u}{\partial t} - r_x \frac{\partial w}{\partial t} - \frac{\nu}{2} \varphi \left[\frac{1}{v} \left(\frac{\partial^2 v}{\partial x \partial t} + \frac{\partial^2 v}{\partial z \partial t} \right) - \frac{1}{v^2} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \right) \frac{\partial v}{\partial t} \right] + \frac{\nu}{2} \varphi_2$$

$$\left\{ u \left\{ r_z \left[\frac{1}{u} \left(\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial z \partial x} \right) - \frac{1}{u^2} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) \frac{\partial u}{\partial x} \right] - r_x \left[\frac{1}{w} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x} \right) - \frac{1}{w^2} \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial w}{\partial x} \right] \right\} \right.$$

$$+ v \left\{ r_z \left[\frac{1}{u} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z \partial y} \right) - \frac{1}{u^2} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) \frac{\partial u}{\partial y} \right] - r_x \left[\frac{1}{w} \left(\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} \right) - \frac{1}{w^2} \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial w}{\partial y} \right] \right\}$$

$$\left. + w \left\{ r_z \left[\frac{1}{u} \left(\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{1}{u^2} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) \frac{\partial u}{\partial z} \right] - r_x \left[\frac{1}{w} \left(\frac{\partial^2 w}{\partial x \partial z} + \frac{\partial^2 w}{\partial y \partial z} \right) - \frac{1}{w^2} \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial w}{\partial z} \right] \right\} \right\}$$

$$= r_z f_x - r_x f_z + \frac{1}{\rho} \left(r_z \frac{\partial p}{\partial x} - r_x \frac{\partial p}{\partial z} \right)$$

$$+ r_z \left\{ \nu \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial z \partial y} \right) + \varphi^{\frac{2}{3}} \left[2\nu t' \left(\frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial z \partial y} \right) - \nu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z \partial y} - \frac{\partial^2 u}{\partial y \partial x} \right) \right] \right\}$$

$$- r_x \left\{ \nu \left(\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z \partial y} - \frac{\partial^2 w}{\partial x \partial z} \right) + \varphi^{\frac{2}{3}} \left[2\nu t' \left(\frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x \partial z} \right) \right. \right.$$

$$\left. \left. - \nu \left(\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x \partial z} - \frac{\partial^2 w}{\partial y \partial z} \right) \right] \right\} \tag{4-4-19}$$

$$r_x\frac{\partial v}{\partial t}-r_y\frac{\partial u}{\partial t}-\frac{\nu}{2}\varphi\left[\frac{1}{w}\left(\frac{\partial^2 w}{\partial x\partial t}+\frac{\partial^2 w}{\partial y\partial t}\right)-\frac{1}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial t}\right]+\frac{\nu}{2}\varphi_2$$

$$\left\{u\left\{r_x\left[\frac{1}{v}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial x}\right)-\frac{1}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial z}\right]-r_y\left[\frac{1}{u}\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial z\partial x}\right)-\frac{1}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial x}\right]\right\}\right.$$

$$+v\left\{r_x\left[\frac{1}{v}\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 v}{\partial z\partial y}\right)-\frac{1}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial y}\right]-r_y\left[\frac{1}{u}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z\partial y}\right)-\frac{1}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial y}\right]\right\}$$

$$\left.+w\left\{r_x\left[\frac{1}{v}\left(\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{1}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial z}\right]-r_y\left[\frac{1}{u}\left(\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{1}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial z}\right]\right\}\right\}$$

$$=r_xf_y-r_yf_x+\frac{1}{\rho}\left(r_x\frac{\partial p}{\partial y}-r_y\frac{\partial p}{\partial x}\right)$$

$$+r_x\left\{\nu\left(\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial z\partial y}\right)+\varphi^{\frac{2}{3}}\left[2\nu t'\left(\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial z}\right)-\nu\left(\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial z\partial y}\right)\right]\right\}$$

$$-r_y\left\{\nu\left(\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial z^2}\right)+\varphi^{\frac{2}{3}}\left[2\nu t'\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right)-\nu\left(\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial z^2}\right)\right]\right\}$$

(4-4-20)

4.4.4 能量微分方程

将式(3-4-13)转化为直角坐标系的对应方程，将 φ 作为常量，有

$$\frac{\partial}{\partial t}\left[\varphi\frac{v_s^2}{2}+(1-\varphi)\frac{v^2}{2}\right]+\varphi^{\frac{2}{3}}\frac{v_s^2}{2}\nabla\cdot\boldsymbol{v}_s(1-\varphi^{\frac{2}{3}})\frac{v^2}{2}\nabla\cdot\boldsymbol{v}$$

$$=\frac{1}{\rho}\nabla\cdot\nabla\boldsymbol{T}+q_R+\boldsymbol{f}\cdot[\varphi\boldsymbol{v}_s+(1-\varphi)\boldsymbol{v}]+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot[\varphi^{\frac{2}{3}}\boldsymbol{v}_s+(1-\varphi^{\frac{2}{3}})\boldsymbol{v}]\tag{4-4-21}$$

将式(4-4-2)代入式(4-4-21)，进一步整理为

$$\frac{\partial}{\partial t}\left(\frac{v^2}{2}\right)+\varphi\frac{\partial}{\partial t}\left(\frac{v_s^2}{2}-\frac{v^2}{2}\right)+(\boldsymbol{v}\cdot\nabla)\left(\frac{v^2}{2}+\frac{v_s^2}{2}\right)+\varphi^{\frac{2}{3}}(\boldsymbol{v}\cdot\nabla)\left(\frac{v_s^2}{2}-\frac{v^2}{2}\right)$$

$$=\frac{1}{\rho}\nabla\cdot\nabla\boldsymbol{T}+q_R+\boldsymbol{f}\cdot\boldsymbol{v}+\boldsymbol{f}\cdot\varphi(\boldsymbol{v}_s-\boldsymbol{v})+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}[\boldsymbol{v}+\varphi^{\frac{2}{3}}(\boldsymbol{v}_s-\boldsymbol{v})]\tag{4-4-22}$$

式中，$\boldsymbol{v}_s$，$\boldsymbol{v}$ 是矢量；v，v_s 是标量。

$$v^2=u^2+v^2+w^2\tag{4-4-23}$$

$$v_s^2=u_s^2+v_s^2+w_s^2\tag{4-4-24}$$

依据式(4-3-49)至式(4-3-51)可得

$$u_s^2=u^2-\nu\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{\nu^2}{4u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)^2$$

因为 $\nu^2\leqslant 1$，故

$$u_s^2=u^2-\nu\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\tag{4-4-25}$$

同理：

$$v_s^2=v^2-\nu\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\tag{4-4-26}$$

$$w_s^2 = w^2 - \nu\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right) \tag{4-4-27}$$

单独推导$\nabla \cdot \boldsymbol{P}[\boldsymbol{v} + \varphi^{\frac{2}{3}}(\boldsymbol{v}_s - \boldsymbol{v})]$项如下：

$$\nabla \cdot \boldsymbol{P}[\boldsymbol{v} + \varphi^{\frac{2}{3}}(\boldsymbol{v}_s - \boldsymbol{v})] = \nabla \cdot \boldsymbol{P} \cdot \boldsymbol{v} + \varphi^{\frac{2}{3}} \nabla \cdot \boldsymbol{P}(\boldsymbol{v}_s - \boldsymbol{v}) \tag{4-4-28}$$

式(4-4-28)中第一项推导：

$$\begin{aligned}
\nabla \cdot \boldsymbol{P} \cdot \boldsymbol{v} &= \nabla \cdot [(p_{xx}\boldsymbol{i} + p_{yx}\boldsymbol{j} + p_{zx}\boldsymbol{k})\boldsymbol{i} + (p_{xy}\boldsymbol{j} + p_{yy}\boldsymbol{j} + p_{zy}\boldsymbol{k})\boldsymbol{j} + (p_{xz}\boldsymbol{i} + p_{yz}\boldsymbol{j} + p_{zz}\boldsymbol{k})\boldsymbol{k}] \cdot (u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}) \\
&= \nabla \cdot [(p_{xx}\boldsymbol{i} + p_{yx}\boldsymbol{j} + p_{zx}\boldsymbol{k})u + (p_{xy}\boldsymbol{j} + p_{yy}\boldsymbol{j} + p_{zy}\boldsymbol{k})v + (p_{xz}\boldsymbol{i} + p_{yz}\boldsymbol{j} + p_{zz}\boldsymbol{k})w] \\
&= \nabla \cdot [(p_{xx}u + p_{xy}v + p_{xz}w)\boldsymbol{i} + (p_{yx}u + p_{yy}v + p_{yz}w)\boldsymbol{j} + (p_{zx}u + p_{zy}v + p_{zz}w)\boldsymbol{k}] \\
&= \frac{\partial}{\partial x}(p_{xx}u + p_{xy}v + p_{xz}w) + \frac{\partial}{\partial y}(p_{yx}u + p_{yy}v + p_{yz}w) + \frac{\partial}{\partial z}(p_{zx}u + p_{zy}v + p_{zz}w) \\
&= p_{xx}\frac{\partial u}{\partial x} + u\,\frac{\partial p_{xx}}{\partial x} + p_{xy}\frac{\partial v}{\partial x} + v\,\frac{\partial p_{xy}}{\partial x} + p_{xz}\frac{\partial w}{\partial x} + w\,\frac{\partial p_{xz}}{\partial x} + p_{yx}\frac{\partial u}{\partial y} + u\,\frac{\partial p_{yx}}{\partial y} + p_{yy}\frac{\partial v}{\partial y} \\
&\quad + v\,\frac{\partial p_{yy}}{\partial y} + p_{yz}\frac{\partial w}{\partial y} + w\,\frac{\partial p_{yz}}{\partial y} + p_{zx}\frac{\partial u}{\partial z} + u\,\frac{\partial p_{zx}}{\partial z} + p_{zy}\frac{\partial v}{\partial z} + v\,\frac{\partial p_{zy}}{\partial z} + p_{zz}\frac{\partial w}{\partial z} + w\,\frac{\partial p_{zz}}{\partial z} \\
&= \left(-p + 2\mu\,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x} + u\left(-\frac{\partial p}{\partial x} + 2\mu\,\frac{\partial^2 u}{\partial x^2}\right) + \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial x} + v\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y \partial x}\right) \\
&\quad + \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\frac{\partial w}{\partial x} + w\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial z \partial x}\right) + \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial y} + u\mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) \\
&\quad + \left(-p + 2\mu\,\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial y} + v\left(-\frac{\partial p}{\partial y} + 2\mu\,\frac{\partial^2 v}{\partial y^2}\right) + \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\frac{\partial w}{\partial y} \\
&\quad + w\mu\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial z \partial y}\right) + \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\frac{\partial u}{\partial z} + u\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\frac{\partial v}{\partial z} \\
&\quad + v\mu\left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial y \partial z}\right) + \left(-p + 2\mu\,\frac{\partial w}{\partial z}\right)\frac{\partial w}{\partial z} + w\left(-\frac{\partial p}{\partial z} + 2\mu\,\frac{\partial^2 w}{\partial z^2}\right)
\end{aligned} \tag{4-4-29}$$

式(4-4-28)中第二项推导：

$$\begin{aligned}
\varphi^{\frac{2}{3}} \nabla \cdot \boldsymbol{P} \cdot (\boldsymbol{v}_s - \boldsymbol{v}) &= \varphi^{\frac{2}{3}} \nabla \cdot \boldsymbol{P} \cdot \left(-\frac{\nu}{2}\right)\left[\frac{1}{u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\boldsymbol{i} + \frac{1}{v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\right)\boldsymbol{j} + \frac{1}{w}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\boldsymbol{k}\right] \\
&= -\frac{\nu}{2}\varphi^{\frac{2}{3}} \nabla \cdot (\boldsymbol{p}_x\boldsymbol{i} + \boldsymbol{p}_y\boldsymbol{j} + \boldsymbol{p}_z\boldsymbol{k}) \\
&\quad \cdot \left[\frac{1}{u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\boldsymbol{i} + \frac{1}{v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\right)\boldsymbol{j} + \frac{1}{w}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\boldsymbol{k}\right] \\
&= -\frac{\nu}{2}\varphi^{\frac{2}{3}} \nabla \cdot [(p_{xx}\boldsymbol{i} + p_{yx}\boldsymbol{j} + p_{zx}\boldsymbol{k})\boldsymbol{i} + (p_{xy}\boldsymbol{j} + p_{yy}\boldsymbol{j} + p_{zy}\boldsymbol{k})\boldsymbol{j} + (p_{xz}\boldsymbol{i} + p_{yz}\boldsymbol{j} + p_{zz}\boldsymbol{k})\boldsymbol{k}] \\
&\quad \cdot \left[\frac{1}{u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\boldsymbol{i} + \frac{1}{v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\right)\boldsymbol{j} + \frac{1}{w}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\boldsymbol{k}\right] \\
&= -\frac{\nu}{2}\varphi^{\frac{2}{3}} \nabla \cdot \left[(p_{xx}\boldsymbol{i} + p_{yx}\boldsymbol{j} + p_{zx}\boldsymbol{k})\frac{1}{u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) + (p_{xy}\boldsymbol{j} + p_{yy}\boldsymbol{j} + p_{zy}\boldsymbol{k})\right. \\
&\quad \left.\frac{1}{v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\right) + (p_{xz}\boldsymbol{i} + p_{yz}\boldsymbol{j} + p_{zz}\boldsymbol{k})\frac{1}{w}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\right] \\
&= -\frac{\nu}{2}\varphi^{\frac{2}{3}}\left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right)
\end{aligned}$$

$$\cdot\left\{\left[p_{xx}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+p_{xy}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+p_{xz}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]i\right.$$

$$+\left[p_{yx}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+p_{yy}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+p_{yz}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]j$$

$$\left.+\left[p_{zx}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+p_{zy}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+p_{zz}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]k\right\}$$

$$=-\frac{\nu}{2}\varphi^{\frac{2}{3}}\left\{\frac{\partial}{\partial x}\left[\frac{p_{xx}}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{p_{xy}}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+\frac{p_{xz}}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]\right.$$

$$+\frac{\partial}{\partial y}\left[\frac{p_{yx}}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{p_{yy}}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+\frac{p_{yz}}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]$$

$$\left.+\frac{\partial}{\partial z}\left[\frac{p_{zx}}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{p_{zy}}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+\frac{p_{zz}}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]\right\}$$

$$=-\frac{\nu}{2}\varphi^{\frac{2}{3}}\left[\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\left(\frac{\partial}{\partial x}p_{xx}+\frac{\partial}{\partial y}p_{yx}+\frac{\partial}{\partial z}p_{zx}\right)+(p_{xx}+p_{yx}+p_{zx})\right.$$

$$\frac{\partial}{\partial x}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\left(\frac{\partial}{\partial x}p_{xy}+\frac{\partial}{\partial y}p_{yy}+\frac{\partial}{\partial z}p_{zy}\right)+(p_{xy}+p_{yy}+p_{zy})$$

$$\frac{\partial}{\partial x}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\left(\frac{\partial}{\partial x}p_{xz}+\frac{\partial}{\partial y}p_{yz}+\frac{\partial}{\partial z}p_{zz}\right)+(p_{xz}+p_{yz}+p_{zz})$$

$$\left.\frac{\partial}{\partial x}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]$$

$$=-\frac{\nu}{2}\varphi^{\frac{2}{3}}\left\{\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\left\{-\frac{\partial p}{\partial x}+2\mu\frac{\partial^2 u}{\partial x^2}+\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}\right)\right.\right.$$

$$\left.+\varphi^{\frac{2}{3}}\left[2\rho vt'\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right)-\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}\right)\right]\right\}+\left(-p+2\mu\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\left\{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\varphi^{\frac{2}{3}}\left[\rho vt'\left(\frac{\partial u}{\partial y}\right)^2-\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\right\}$$

$$\frac{\partial}{\partial y}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\left\{\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\varphi^{\frac{2}{3}}\left[\rho vt'\left(\frac{\partial u}{\partial z}\right)^2-\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]\right\}$$

$$+\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\left\{-\frac{\partial p}{\partial y}+2\mu\frac{\partial v}{\partial y}+\mu\left(\frac{\partial v^2}{\partial x^2}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial v^2}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}\right)\right.$$

$$\left.+\varphi^{\frac{2}{3}}\left[2\rho vt'\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}\right)-\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}\right)\right]\right\}$$

$$+\left\{\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\varphi^{\frac{2}{3}}\left[\rho vt'\left(\frac{\partial v}{\partial x}\right)^2-\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]\right\}$$

$$\frac{\partial}{\partial x}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+\left(-p+2\mu\frac{\partial v}{\partial y}\right)\frac{\partial}{\partial y}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)$$

$$+\left\{\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+\varphi^{\frac{2}{3}}\left[\rho vt'\left(\frac{\partial v}{\partial z}\right)^2-\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]\right\}$$

$$\frac{\partial}{\partial z}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\left\{\mu\left(\frac{\partial w^2}{\partial x^2}+\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial w^2}{\partial y^2}+\frac{\partial^2 v}{\partial z\partial y}\right)\right.$$

$$+\varphi^{\frac{2}{3}}\left[2\rho vt'\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}\right)-\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 v}{\partial z\partial y}\right)\right]$$

$$+\left(-\frac{\partial p}{\partial z}+2\mu\frac{\partial w}{\partial z}\right)\Big\}+\left\{\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+\varphi^{\frac{2}{3}}\left[\rho\nu t'\left(\frac{\partial w}{\partial x}\right)^2-\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]\right\}$$

$$\frac{\partial}{\partial x}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)+\left\{\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\varphi^{\frac{2}{3}}\left[\rho\nu t'\left(\frac{\partial w}{\partial y}\right)^2-\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]\right\}$$

$$\frac{\partial}{\partial y}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)+\left(-p+2\mu\frac{\partial w}{\partial z}\right)\frac{\partial}{\partial z}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\Big\} \tag{4-4-30}$$

将式(4-4-29)与式(4-4-30)代入式(4-4-28)，然后将(4-4-25)至式(4-4-28)代入式(4-4-22)，则有

$$\frac{\partial}{\partial t}\left(\frac{v^2}{2}\right)-\frac{\nu\varphi}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)$$

$$+\boldsymbol{v}\cdot\nabla\left[u^2+v^2+w^2-\frac{\nu}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]$$

$$+\frac{\nu}{2}\varphi^{\frac{2}{3}}\boldsymbol{v}\cdot\nabla\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)$$

$$=\frac{\lambda}{\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+f_x u+f_y v+f_z w$$

$$-\frac{\varphi}{2}\nu\left[f_x\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{f_y}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+\frac{f_z}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]$$

$$+\left(-\frac{p}{\rho}+2\nu\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x}+u\left(-\frac{\partial p}{\rho\partial x}+2\nu\frac{\partial^2 u}{\partial x^2}\right)+\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial x}+\nu v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial y\partial x}\right)$$

$$+\nu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\frac{\partial w}{\partial x}+\nu w\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial z\partial x}\right)+\nu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\nu u\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)$$

$$+\left(-\frac{p}{\rho}+2\nu\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial y}+v\left(-\frac{\partial p}{\rho\partial y}+2\nu\frac{\partial^2 v}{\partial y^2}\right)+\nu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\frac{\partial w}{\partial y}+\nu w\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 v}{\partial z\partial y}\right)$$

$$+\nu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\frac{\partial u}{\partial z}+\nu u\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\nu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\frac{\partial v}{\partial z}+\nu v\left(\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}\right)$$

$$+\left(-\frac{p}{\rho}+2\nu\frac{\partial w}{\partial z}\right)+w\left(-\frac{\partial p}{\rho\partial z}+2\nu\frac{\partial^2 w}{\partial z^2}\right)-\frac{\nu}{2}\varphi^{\frac{2}{3}}\left\{\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\right\}\left\{-\frac{\partial p}{\partial x}+2\mu\frac{\partial^2 u}{\partial x^2}\right.$$

$$+\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}\right)+\varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right)\right.$$

$$\left.\left.-\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}\right)\right]\right\}+\left(-p+2\mu\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial x}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$+\left\{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\varphi^{\frac{2}{3}}\left[\rho\nu t'\left(\frac{\partial u}{\partial y}\right)^2-\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\right\}\frac{\partial}{\partial y}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$+\left\{\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\varphi^{\frac{2}{3}}\left[\rho\nu t'\left(\frac{\partial u}{\partial z}\right)^2-\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]\right\}$$

$$+\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\left\{-\frac{\partial p}{\partial y}+2\mu\frac{\partial v}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}\right)\right.$$

$$\left.+\varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}\right)-\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}\right)\right]\right\}$$

$$
\begin{aligned}
&+\Big\{\mu\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big)+\varphi^{\frac{2}{3}}\Big[\rho\nu t'\Big(\frac{\partial v}{\partial x}\Big)^2-\mu\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big)\Big]\Big\}\frac{\partial}{\partial x}\frac{1}{v}\Big(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\Big)\\
&+\Big(-p+2\mu\frac{\partial v}{\partial y}\Big)\frac{\partial}{\partial y}\frac{1}{v}\Big(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\Big)+\Big\{\mu\Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big)+\varphi^{\frac{2}{3}}\Big[\rho\nu t'\Big(\frac{\partial v}{\partial z}\Big)^2-\mu\Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big)\Big]\Big\}\\
&\frac{\partial}{\partial z}\frac{1}{v}\Big(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\Big)+\frac{1}{w}\Big(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\Big)\Big\{\mu\Big(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 v}{\partial z\partial y}\Big)\\
&+\varphi^{\frac{2}{3}}\Big[2\rho\nu t'\Big(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}\Big)-\mu\Big(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 v}{\partial z\partial y}\Big)\Big]\\
&+\Big(-\frac{\partial p}{\partial z}+2\mu\frac{\partial w}{\partial z}\Big)\Big\}+\Big\{\mu\Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big)+\varphi^{\frac{2}{3}}\Big[\rho\nu t'\Big(\frac{\partial w}{\partial x}\Big)^2-\mu\Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big)\Big]\Big\}\\
&\frac{\partial}{\partial x}\frac{1}{w}\Big(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\Big)+\Big\{\mu\Big(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\Big)+\varphi^{\frac{2}{3}}\Big[\rho\nu t'\Big(\frac{\partial w}{\partial y}\Big)^2-\mu\Big(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\Big)\Big]\Big\}\\
&\frac{\partial}{\partial y}\frac{1}{w}\Big(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\Big)+\Big(-p+2\mu\frac{\partial w}{\partial z}\Big)\frac{\partial}{\partial z}\frac{1}{w}\Big(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\Big)\Big\}\Big\}
\end{aligned}
\tag{4-4-31}
$$

式(4-4-31)是直角坐标系不可压缩湍流运动能量微分方程；当定常流时，去掉含$\frac{\partial}{\partial t}$项。

4.5 可压缩湍流微分方程组

可压缩湍流的特点是涡旋微团温度 T_s，密度 ρ_s 与周围流体不同，但其压力与其一样。据此，有

$$\rho_s=\frac{\rho T}{T_s} \tag{4-5-1}$$

涡旋温度变化为

$$T_s=T+(T_{s0}-T)e^{-B_iF_i} \tag{4-5-2}$$

式中，设 $T_{s0}>T$，且

T——连续相温度；

T_s——涡旋温度；

T_{s0}——涡旋初始温度；

B_i——毕渥准数；

F_i——傅里叶准数。

$$B_i=\frac{hd_s}{6\lambda} \tag{4-5-3}$$

$$F_i=\frac{36at'}{d_s^2} \tag{4-5-4}$$

$$a=\frac{\lambda}{\rho_s c_p} \tag{4-5-5}$$

h——表面传热系数，单位：$W/(m^2\cdot k)$；

d_s——涡旋直径；

t'——涡旋产生到消失的时间；

a——热扩散系数；

λ——导热系数；

c_p——涡旋微团定压比值；

ρ_s——涡旋密度。

4.5.1 连续性微分方程

依式(3-7-2)体现了质量守恒，运动为可压缩流体，则它变为连续性微分方程：

$$\nabla\cdot(\alpha \boldsymbol{v}_s+\boldsymbol{v})+\ [\varphi(\alpha-1)+1]\ \frac{1}{\rho}\frac{D\rho}{Dt}=0 \tag{4-5-6}$$

$$(\alpha+1)\ \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)-\alpha\ \frac{\nu}{2}\left[\frac{\partial}{\partial x}\frac{1}{u}\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial y}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\right.$$
$$\left.+\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]+\ [\varphi(\alpha-1)+1]\ \frac{1}{\rho}\frac{D\rho}{D}=0 \tag{4-5-7}$$

4.5.2 动量微分方程

将式(3-7-6)转化为直角坐标系下对应微分方程，首先将连续性微分方程(3-7-2)代入式(3-7-6)：

$$\frac{D}{Dt}(\rho\boldsymbol{v})+\varphi\frac{D}{Dt}(\rho_s\boldsymbol{v}_s-\rho\boldsymbol{v})+\boldsymbol{v}_s\varphi\frac{D}{Dt}(\rho-\rho_s)-\boldsymbol{v}_s\frac{D\rho}{Dt}-\boldsymbol{v}_s\rho\nabla\cdot\boldsymbol{v}=\boldsymbol{f}[\varphi\rho_s+(1-\varphi)\rho]+\nabla\cdot\boldsymbol{P} \tag{4-5-8}$$

用式(4-5-1)、式(4-5-2)消去 ρ_s，使方程含连续相温度变化。

$$\frac{D}{Dt}(\rho\boldsymbol{v})+\varphi\frac{D}{Dt}\left(\frac{\rho T}{T_s}\boldsymbol{v}_s-\rho\boldsymbol{v}\right)+\boldsymbol{v}_s\varphi\frac{D}{Dt}\left(\rho-\frac{\rho T}{T_s}\right)-\boldsymbol{v}_s\frac{D\rho}{Dt}-(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}_s\rho$$
$$=\boldsymbol{f}\left[\varphi\frac{\rho T}{T_s}+(1-\varphi)\rho\right]+\nabla\cdot\boldsymbol{P} \tag{4-5-9}$$

$$\rho\frac{D\boldsymbol{v}}{Dt}+\boldsymbol{v}\frac{D\rho}{Dt}+\varphi\rho\frac{D}{Dt}\left[\frac{T\boldsymbol{v}_s}{T-(T_{s0}-T)e^{B_iF_i}}-\boldsymbol{v}\right]+\varphi\left[\frac{T\boldsymbol{v}_s}{1-(T_{s0}-T)e^{B_iF_i}}-\boldsymbol{v}\right]\frac{D\rho}{Dt}$$
$$+\boldsymbol{v}_s\varphi\rho\frac{D}{Dt}\left(1-\frac{T}{T_s}\right)+\boldsymbol{v}_s\varphi\left(1-\frac{T}{T_s}\right)\frac{D\rho}{Dt}-\boldsymbol{v}_s\frac{D\rho}{Dt}-(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}_s\rho$$
$$=f\rho\left[1+\left(\varphi\frac{T}{T_s}-1\right)\right]+\nabla\cdot\boldsymbol{P} \tag{4-5-10}$$

全式除以 ρ，近似取 $T_{s0}=0$，则令

$$\eta=1\pm e^{-B_iF_i} \tag{4-5-11}$$

当 $T_{s0}>T$，取(+)号，$T_{s0}<T$ 取(−)号。

$$\frac{D\boldsymbol{v}}{Dt}+\frac{\boldsymbol{v}}{\rho}\frac{D\rho}{Dt}+\varphi\frac{D}{Dt}\left(\frac{1}{\eta}\boldsymbol{v}_s-\boldsymbol{v}\right)+\varphi\left(\frac{1}{\eta}\boldsymbol{v}_s-\boldsymbol{v}\right)\frac{1}{\rho}\frac{D\rho}{Dt}+\boldsymbol{v}_s\varphi\frac{D}{Dt}\left(1-\frac{1}{\eta}\right)+\boldsymbol{v}_s\varphi\left(1-\frac{1}{\eta}\right)\frac{D\rho}{\rho Dt}$$

$$- \boldsymbol{v}_s \frac{\mathrm{D}\rho}{\rho \mathrm{D}t} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}_s = f\rho\left[1 + \left(\frac{1}{\eta} - 1\right)\right] + \frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{4-5-12}$$

当 η 为常数(具体问题)，则$\frac{\mathrm{D}}{\mathrm{D}t}\left(1 - \frac{1}{\eta}\right) = 0$，将式(4-5-12)进一步整理为

$$\frac{\mathrm{D}}{\mathrm{D}t}\boldsymbol{v} + \frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}(\boldsymbol{v} - \boldsymbol{v}_s) + \varphi\left(1 - \frac{1}{\eta}\right)\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}\boldsymbol{v}_s + \varphi\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{\eta}\boldsymbol{v}_s - \boldsymbol{v}\right) + \varphi\left(\frac{1}{\eta}\boldsymbol{v}_s - \boldsymbol{v}\right)\frac{\mathrm{D}\rho}{\rho\mathrm{D}t} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}_s$$

$$= f\rho\left[1 + \left(\frac{1}{\eta} - \boldsymbol{v}\right)\right] + \frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{4-5-13}$$

整理为

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + (\varphi - 1)(\boldsymbol{v}_s - \boldsymbol{v})\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \varphi\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{\eta}\boldsymbol{v}_s - \boldsymbol{v}\right) - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}_s = \boldsymbol{f}\left[1 + \left(\frac{1}{\eta} - 1\right)\right] + \frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{4-5-14}$$

转变为只有连续相变量方程，则为

$$\begin{aligned}
&\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} - (\varphi - 1)\frac{\nu}{2}\left[\frac{1}{u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\boldsymbol{i} + \frac{1}{v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\right)\boldsymbol{j} + \frac{1}{w}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\boldsymbol{k}\right]\frac{\mathrm{D}\rho}{\rho\mathrm{D}t} \\
&+ \varphi\frac{\mathrm{D}}{\mathrm{D}t}\left[\left(\frac{1}{\eta} - 1\right)(u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k})\right] - \frac{1}{\eta}\frac{\nu}{2}\left[\frac{1}{u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\boldsymbol{i} + \frac{1}{v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\right)\boldsymbol{j} + \frac{1}{w}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\boldsymbol{k}\right] \\
&- (\boldsymbol{v}\cdot\nabla)\left\{(u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}) - \frac{\nu}{2}\left[\frac{1}{u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\boldsymbol{i} + \frac{1}{v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\right)\boldsymbol{j} + \frac{1}{w}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\boldsymbol{k}\right]\right\} \\
&= \boldsymbol{f}\left[1 + \left(\frac{1}{\eta} - 1\right)\right] + \frac{1}{\rho}\left\{\left\{-\frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\mu\frac{\partial}{\partial x}(\nabla\cdot\boldsymbol{v}) + \mu\left(\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial x\partial z} + \frac{\partial^2 u}{\partial z^2}\right)\right.\right. \\
&\quad + \varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x\partial z}\right) - \mu\left(\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial x\partial z} + \frac{\partial^2 u}{\partial z^2}\right)\right]\Bigg\}\boldsymbol{i} \\
&\quad + \left\{-\frac{\partial p}{\partial y} + 2\mu\frac{\partial^2 v}{\partial y^2} - \frac{2}{3}\mu\frac{\partial}{\partial y}(\nabla\cdot\boldsymbol{v}) + \mu\left(\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial y\partial z} + \frac{\partial^2 v}{\partial z^2}\right)\right. \\
&\quad + \varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y\partial z}\right) - \mu\left(\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial y\partial z} + \frac{\partial^2 v}{\partial z^2}\right)\right]\Bigg\}\boldsymbol{j} \\
&\quad \left\{-\frac{\partial p}{\partial z} + 2\mu\frac{\partial^2 w}{\partial z^2} - \frac{2}{3}\mu\frac{\partial}{\partial z}(\nabla\cdot\boldsymbol{v}) + \mu\left(\frac{\partial^2 u}{\partial z\partial x} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial z\partial y} + \frac{\partial^2 w}{\partial y^2}\right)\right. \\
&\quad + \varphi^{\frac{2}{3}}\left[2\rho\nu t'\left(\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial x} + \frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial z\partial y}\right) - \mu\left(\frac{\partial^2 u}{\partial z\partial x} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial z\partial y} + \frac{\partial^2 w}{\partial y^2}\right)\right]\Bigg\}\boldsymbol{k}\Bigg\}
\end{aligned} \tag{4-5-15}$$

按三个坐标轴列出：

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - (\varphi - 1)\frac{\nu}{2}\frac{1}{u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \varphi\left(\frac{1}{\eta} - 1\right) \\
&\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - \frac{\varphi}{\eta}\frac{\nu}{2}\frac{\partial}{\partial t}\left[\frac{1}{u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\right] - (\boldsymbol{v}\cdot\nabla)\left[u - \frac{\nu}{2}\frac{1}{u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\right] \\
&= f_x\left[1 + \left(\frac{1}{\eta} - 1\right)\right] - \frac{\partial p}{\rho\partial x} + 2\nu\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\nu\frac{\partial}{\partial x}(\nabla\cdot\boldsymbol{v}) + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y\partial x} + \frac{\partial^2 u}{\partial z\partial x}\right) \\
&\quad + \varphi^{\frac{2}{3}}\left[2\nu t'\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial x}\right) - \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y\partial x} + \frac{\partial^2 u}{\partial z\partial x}\right)\right]
\end{aligned} \tag{4-5-16}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} - (\varphi - 1)\frac{\nu}{2}\frac{1}{v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\right)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \varphi\left(\frac{1}{\eta} - 1\right)$$

$$\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)-\frac{\varphi}{\eta}\frac{\nu}{2}\frac{\partial}{\partial t}\left[\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\right]-(\boldsymbol{v}\cdot\nabla)\left[v-\frac{\nu}{2}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\right]$$
$$=f_y\left[1+\left(\frac{1}{\eta}-1\right)\right]-\frac{\partial p}{\rho\partial y}+2\nu\frac{\partial^2 v}{\partial y^2}-\frac{2}{3}\nu\frac{\partial}{\partial y}(\nabla\cdot\boldsymbol{v})+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 v}{\partial z\partial y}\right)$$
$$+\varphi^{\frac{2}{3}}\left[2\nu t'\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial y}\right)-\nu\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial z\partial y}\right)\right] \tag{4-5-17}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}-(\varphi-1)\frac{\nu}{2}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\varphi\left(\frac{1}{\eta}-1\right)$$
$$\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)-\frac{\varphi}{\eta}\frac{\nu}{2}\frac{\partial}{\partial t}\left[\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]-(\boldsymbol{v}\cdot\nabla)\left[w-\frac{\nu}{2}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]$$
$$=f_z\left[1+\left(\frac{1}{\eta}-1\right)\right]-\frac{\partial p}{\rho\partial z}+2\nu\frac{\partial^2 w}{\partial z^2}-\frac{2}{3}\nu\frac{\partial}{\partial z}(\nabla\cdot\boldsymbol{v})+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}\right)$$
$$+\varphi^{\frac{2}{3}}\left[2\nu t'\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y\partial z}\right)-\nu\left(\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\right] \tag{4-5-18}$$

式(4-5-16)至式(4-5-18)是可压缩湍流在直角坐标系下的动量微分方程。

4.5.3 动量矩微分方程

将式(3-7-2)代入式(3-7-8)，则为

$$\varphi\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{r}\times\rho_s\boldsymbol{v}_s)-(\boldsymbol{r}\times\rho_s\boldsymbol{v}_s)\frac{1}{\rho_s}\left[\varphi\frac{\mathrm{D}\rho_s}{\mathrm{D}t}+(1-\varphi)\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\nabla\cdot\boldsymbol{v}\right]+(1-\varphi)\frac{\mathrm{D}(\boldsymbol{r}\times\rho\boldsymbol{v})}{\mathrm{D}t}+(\boldsymbol{r}\times\rho\boldsymbol{v})\nabla\cdot\boldsymbol{v}$$
$$=\boldsymbol{r}\times[\varphi\rho_s(1-\varphi)\rho]\boldsymbol{f}+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{4-5-19}$$

$\boldsymbol{r}$ 是参变量，化简它为

$$\boldsymbol{r}\times\frac{\mathrm{D}}{\mathrm{D}t}(\rho\boldsymbol{v})+\varphi\left[\boldsymbol{r}\times\frac{\mathrm{D}}{\mathrm{D}t}(\rho_s\boldsymbol{v}_s-\rho\boldsymbol{v})\right]-(\boldsymbol{r}\times\boldsymbol{v}_s)\left[\varphi\frac{\mathrm{D}}{\mathrm{D}t}(\rho_s-\rho)\right]-(\boldsymbol{r}\times\boldsymbol{v}_s)\frac{\mathrm{D}\rho}{\mathrm{D}t}-[\boldsymbol{r}\times\rho(\boldsymbol{v}-\boldsymbol{v}_s)]\nabla\cdot\boldsymbol{v}$$
$$=\boldsymbol{r}\times[\varphi(\rho_s-\rho)+\rho]\boldsymbol{f}+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{4-5-20}$$

消去 ρ_s，则为

$$\boldsymbol{r}\times\frac{\mathrm{D}}{\mathrm{D}t}(\rho\boldsymbol{v})+\varphi\left[\boldsymbol{r}\times\frac{\mathrm{D}}{\mathrm{D}t}\rho\left(\frac{T}{T_s}\boldsymbol{v}_s-\boldsymbol{v}\right)\right]-(\boldsymbol{r}\times\boldsymbol{v}_s)\left[\varphi\frac{\mathrm{D}}{\mathrm{D}t}\rho\left(\frac{T}{T_s}-1\right)\right]-(\boldsymbol{r}\times\boldsymbol{v})\frac{\mathrm{D}\rho}{\mathrm{D}t}$$
$$-[\boldsymbol{r}\times\rho(\boldsymbol{v}-\boldsymbol{v}_s)]\nabla\cdot\boldsymbol{v}=\boldsymbol{r}\times[\varphi(\rho_s-\rho)+\rho]\boldsymbol{f}+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{4-5-21}$$

利用式(4-5-11)消去 T，令 $\alpha=\frac{1}{\eta}$，则上式为

$$\boldsymbol{r}\times\left(\rho\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}+\boldsymbol{v}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right)+\varphi\boldsymbol{r}\times\left[\rho\frac{\mathrm{D}}{\mathrm{D}t}(\alpha\boldsymbol{v}_s-\boldsymbol{v})-(\alpha\boldsymbol{v}_s-\boldsymbol{v})\frac{\mathrm{D}\rho}{\mathrm{D}t}\right]-(\boldsymbol{r}\times\boldsymbol{v}_s)\left[\varphi(\alpha-1)\frac{\mathrm{D}\rho}{\mathrm{D}t}-\frac{\mathrm{D}\rho}{\mathrm{D}t}\right]$$
$$-[\boldsymbol{r}\times\rho(\boldsymbol{v}-\boldsymbol{v}_s)]\nabla\cdot\boldsymbol{v}=\boldsymbol{r}\times\rho[1+(\varphi\alpha-1)]\boldsymbol{f}+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{4-5-22}$$

除以 ρ，写成

$$\boldsymbol{r}\times\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}+\frac{\boldsymbol{v}}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right)+\varphi\boldsymbol{r}\times\left[\frac{\mathrm{D}}{\mathrm{D}t}(\alpha\boldsymbol{v}_s-\boldsymbol{v})-(\alpha\boldsymbol{v}_s-\boldsymbol{v})\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}\right]-(\boldsymbol{r}\times\boldsymbol{v}_s)[\varphi(\alpha-1)-1]\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}$$
$$-\boldsymbol{r}\times(\boldsymbol{v}\cdot\nabla)(\boldsymbol{v}-\boldsymbol{v}_s)=\boldsymbol{r}\times[1+\varphi(\alpha-1)]\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{4-5-23}$$

按三个转轴分开列出：

$$
\begin{aligned}
& r_y\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{w}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right)-r_z\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{v}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right)\\
& +\varphi\bigg\{(\alpha-1)\left(r_y\frac{\partial w}{\partial t}-r_z\frac{\partial v}{\partial t}\right)-\frac{\nu}{2}\left[r_y\frac{\mathrm{D}}{\mathrm{D}t}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)-r_z\frac{\mathrm{D}}{\mathrm{D}t}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\right]\bigg\}\\
& -\{(r_yw-r_zv)[\varphi(\alpha-1)-1]\}\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}\\
& -\frac{\nu}{2}\bigg\{r_y\bigg[u\frac{\partial}{\partial x}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)+v\frac{\partial}{\partial y}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)+w\frac{\partial}{\partial z}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\bigg]\\
& -r_z\bigg[u\frac{\partial}{\partial x}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+v\frac{\partial}{\partial y}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+w\frac{\partial}{\partial z}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\right)\bigg]\bigg\}\\
& =[1+(\varphi-1)](r_yf_z-r_zf_y)+\frac{1}{\rho}\left(r_y\frac{\partial p}{\partial z}-r_z\frac{\partial p}{\partial y}\right)+r_y\bigg\{\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{2}{3}\frac{\partial^2 w}{\partial z^2}+\frac{1}{3}\frac{\partial^2 w}{\partial z\partial x}\right)\\
& +\varphi^{\frac{2}{3}}\bigg[2\nu t'\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y\partial x}\right)-\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y\partial x}\right)\bigg]\bigg\}-r_z\bigg\{\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{2}{3}\frac{\partial^2 v}{\partial y^2}+\frac{1}{3}\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 v}{\partial z\partial x}\right)\\
& +\varphi^{\frac{2}{3}}\bigg[2\nu t'\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial x}\right)-\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial x}\right)\bigg]\bigg\}
\end{aligned}
\tag{4-5-24}
$$

$$
\begin{aligned}
& r_z\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{v}{\rho}\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}\right)-r_x\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{w}{\rho}\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}\right)\\
& +\varphi\bigg\{(\alpha-1)\left(r_z\frac{\partial u}{\partial t}-r_x\frac{\partial w}{\partial t}\right)-\frac{\nu}{2}\left[r_z\frac{\mathrm{D}}{\mathrm{D}t}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)-r_x\frac{\mathrm{D}}{\mathrm{D}t}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]\bigg\}\\
& -\{(r_zu-r_xw)[\varphi(\alpha-1)-1]\}\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}\\
& -\frac{\nu}{2}\bigg\{r_z\bigg[u\frac{\partial}{\partial x}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+v\frac{\partial}{\partial y}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+w\frac{\partial}{\partial z}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\bigg]\\
& -r_x\bigg[u\frac{\partial}{\partial x}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)+v\frac{\partial}{\partial y}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)+w\frac{\partial}{\partial z}\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\bigg]\bigg\}\\
& =[1+(\varphi-1)](r_zf_x-r_xf_z)+\frac{1}{\rho}\left(r_z\frac{\partial p}{\partial x}-r_x\frac{\partial p}{\partial z}\right)+r_z\bigg\{\nu\left(\frac{2}{3}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z\partial y}++\frac{1}{3}\frac{\partial^2 u}{\partial x\partial y}\right)\\
& +\varphi^{\frac{2}{3}}\bigg[2\nu t'\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial y}\right)-\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z\partial y}-\frac{\partial^2 u}{\partial y\partial x}\right)\bigg]\bigg\}-r_x\bigg\{\nu\bigg(\frac{\partial^2 w}{\partial y^2}+\frac{2}{3}\frac{\partial^2 w}{\partial z^2}\\
& +\frac{1}{3}\frac{\partial^2 w}{\partial z\partial y}+\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial^2 w}{\partial x\partial z}\bigg)+\varphi^{\frac{2}{3}}\bigg[2\nu t'\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial z}\right)\\
& -\nu\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w}{\partial x\partial z}-\frac{\partial^2 w}{\partial y\partial z}\right)\bigg]\bigg\}
\end{aligned}
\tag{4-5-25}
$$

$$
\begin{aligned}
& r_x\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{v}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right)-r_y\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{u}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right)\\
& +\varphi\bigg\{(\alpha-1)\left(r_x\frac{\partial v}{\partial t}-r_y\frac{\partial u}{\partial t}\right)-\frac{\nu}{2}\left[r_x\frac{\mathrm{D}}{\mathrm{D}t}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)-r_y\frac{\mathrm{D}}{\mathrm{D}t}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\right]\bigg\}\\
& -\{(r_xv-r_yu)[\varphi(\alpha-1)-1]\}\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}\\
& -\frac{\nu}{2}\bigg\{r_x\bigg[u\frac{\partial}{\partial x}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+v\frac{\partial}{\partial y}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)+w\frac{\partial}{\partial z}\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\bigg]\\
& -r_y\bigg[u\frac{\partial}{\partial x}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+v\frac{\partial}{\partial y}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+w\frac{\partial}{\partial z}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\bigg]\bigg\}
\end{aligned}
$$

$$=[1+(\varphi-1)](r_x f_y - r_y f_x)+\frac{1}{\rho}\left(r_x\frac{\partial p}{\partial y}-r_y\frac{\partial p}{\partial x}\right)+r_x\left\{\nu\left(\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 v}{\partial z^2}+\frac{2}{3}\frac{\partial^2 v}{\partial y^2}+\frac{1}{3}\frac{\partial^2 v}{\partial y\partial z}-\frac{\partial^2 v}{\partial y\partial z}\right)\right.$$
$$+\varphi^{\frac{2}{3}}\left[2\nu t'\left(\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial z}\right)-\nu\left(\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial z\partial y}\right)\right]\Bigg\}-r_y\left\{\nu\left(\frac{2}{3}\frac{\partial^2 u}{\partial x^2}+\frac{1}{3}\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial y\partial z}\right)\right.$$
$$\left.+\varphi^{\frac{2}{3}}\left[2\nu t'\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right)-\nu\left(\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial z^2}\right)\right]\right\} \tag{4-5-26}$$

式(4-5-24)至(4-5-26)就是可压缩湍流动量矩在直角坐标系下的运动微分方程。定常流时，去$\frac{D\rho}{Dt}$项中$\frac{\partial\rho}{\partial t}$，$\frac{\partial u}{\partial t}$，$\frac{\partial v}{\partial t}$，$\frac{\partial w}{\partial t}$有关项。

4.5.4　能量微分方程

依式(3-7-11)，有

$$\varphi\frac{D}{Dt}\left[\rho_s\left(e_s+\frac{V_s^2}{2}\right)\right]+\rho_s\left(e_s+\frac{V_s^2}{2}\right)\nabla\cdot\boldsymbol{V}_s+\varphi_1\frac{D}{Dt}\left[\rho\left(e+\frac{V^2}{2}\right)\right]+\rho\left(e+\frac{V^2}{2}\right)\nabla\cdot\boldsymbol{V}$$
$$=\nabla\cdot(\lambda\nabla T)+\rho q_R+\boldsymbol{f}\cdot(\varphi\rho_s\boldsymbol{V}_s+\varphi_1\rho\boldsymbol{V})+\nabla\cdot\boldsymbol{P}(\varphi^{\frac{2}{3}}\boldsymbol{V}_s+\varphi_2\boldsymbol{V}) \tag{4-5-27}$$

将式(3-7-2)代入，有

$$\nabla\cdot\boldsymbol{V}_s=-\frac{1}{\rho_s}\left[\varphi\frac{D\rho_s}{Dt}+(1-\varphi)\frac{D\rho}{Dt}+\rho\nabla\cdot\boldsymbol{V}\right] \tag{4-5-28}$$

$$\varphi\frac{D}{Dt}\left[\rho_s\left(e_s+\frac{V_s^2}{2}\right)\right]-\left(e_s+\frac{V_s^2}{2}\right)\left[\varphi\frac{D\rho_s}{Dt}+(1-\varphi)\frac{D\rho}{Dt}+\rho\nabla\cdot\boldsymbol{V}\right]+(1-\varphi)\frac{D}{Dt}\left[\rho\left(e+\frac{V^2}{2}\right)\right]$$
$$+\rho\left(e+\frac{V^2}{2}\right)\nabla\cdot\boldsymbol{V}=\nabla\cdot(\lambda\nabla T)+\rho q_R+\boldsymbol{f}\cdot[\varphi\rho_s\boldsymbol{V}_s+(1-\varphi)\rho\boldsymbol{V}]+\nabla\cdot\boldsymbol{P}(\varphi^{\frac{2}{3}}\boldsymbol{V}_s+\varphi_2\boldsymbol{V}) \tag{4-5-29}$$

化简：

$$\frac{D}{Dt}\left[\rho\left(e+\frac{V^2}{2}\right)\right]+\left[(e-e_s)+\left(\frac{V^2}{2}-\frac{V_s^2}{2}\right)\right]\rho\nabla\cdot\boldsymbol{V}+\varphi\frac{D}{Dt}\left[\rho_s\left(e_s+\frac{V_s^2}{2}\right)-\rho\left(e+\frac{V^2}{2}\right)\right]$$
$$+\varphi\left(e_s+\frac{V_s^2}{2}\right)\frac{D}{Dt}(\rho-\rho_s)-\left(e_s+\frac{V_s^2}{2}\right)\frac{D\rho}{Dt}$$
$$=\nabla\cdot(\lambda\nabla T)+\rho q_R+\boldsymbol{f}\cdot[\varphi\rho_s\boldsymbol{V}_s+(1-\varphi)\rho\boldsymbol{V}]+\nabla\cdot\boldsymbol{P}\cdot[\varphi^{\frac{2}{3}}\boldsymbol{V}_s+(1-\varphi^{\frac{2}{3}})\boldsymbol{V}] \tag{4-5-30}$$

将 $\rho_s=\frac{\rho T}{T_s}$，$e=c_pT$，$e_s=c_pT_s$ 代入：

$$\frac{D}{Dt}\left[\rho\left(c_pT+\frac{V^2}{2}\right)\right]+\left[c_p(T-T_s)+\frac{1}{2}(V^2-V_s^2)\right]\rho\nabla\cdot\boldsymbol{V}+\varphi\frac{D}{Dt}\left[\frac{\rho T}{T_s}\left(c_pT_s+\frac{V_s^2}{2}\right)\right.$$
$$\left.-\rho\left(c_pT+\frac{V^2}{2}\right)\right]+\varphi\left(c_pT_s+\frac{V_s^2}{2}\right)\frac{D}{Dt}c_p(T-T_s)-\left(c_pT_s+\frac{V_s^2}{2}\right)\frac{D\rho}{Dt}$$
$$=\nabla\cdot(\lambda\nabla T)+\rho q_R+\boldsymbol{f}\cdot\rho\left[\boldsymbol{V}+\varphi\left(\frac{T}{T_s}\boldsymbol{V}_s-\boldsymbol{V}\right)\right]+\nabla\cdot\boldsymbol{P}\cdot[\varphi^{\frac{2}{3}}(\boldsymbol{V}_s-\boldsymbol{V})+\boldsymbol{V}] \tag{4-5-31}$$

将$\frac{T}{T_s}=\alpha$，$T_s=\eta T$ 代入，则有

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho\left(c_{\mathrm{p}}T+\frac{V^2}{2}\right)\right]+\left[c_{\mathrm{p}}T(1-\eta)+\frac{1}{2}(V^2-V_{\mathrm{s}}^2)\right]\rho\nabla\cdot\boldsymbol{V}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}\rho\left[\alpha\left(c_{\mathrm{p}}T\eta+\frac{V_{\mathrm{s}}^2}{2}\right)\right.$$
$$\left.-\left(c_{\mathrm{p}}T+\frac{V^2}{2}\right)\right]+\varphi\left(c_{\mathrm{p}}T\eta+\frac{V_{\mathrm{s}}^2}{2}\right)\frac{\mathrm{D}}{\mathrm{D}t}c_{\mathrm{p}}T(1-\eta)-\left(c_{\mathrm{p}}T\eta+\frac{V_{\mathrm{s}}^2}{2}\right)\frac{\mathrm{D}\rho}{\mathrm{D}t}$$
$$=\nabla\cdot(\lambda\nabla T)+\rho q_{\mathrm{R}}+\boldsymbol{f}\cdot\rho\left[\boldsymbol{V}+\varphi(\alpha\boldsymbol{V}_{\mathrm{s}}-\boldsymbol{V})\right]+\nabla\cdot\boldsymbol{P}\cdot\left[\varphi^{\frac{2}{3}}(\boldsymbol{V}_{\mathrm{s}}-\boldsymbol{V})+\boldsymbol{V}\right]\qquad(4\text{-}5\text{-}32)$$

进一步整理为

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(c_{\mathrm{p}}T+\frac{V^2}{2}\right)+\varphi\rho\frac{\mathrm{D}}{\mathrm{D}t}\left[\alpha(\eta-1)c_{\mathrm{p}}T+\frac{1}{2}(\alpha V_{\mathrm{s}}^2-V^2)\right]+\left[c_{\mathrm{p}}(1-\eta)T+\frac{1}{2}(V^2-V_{\mathrm{s}}^2)\right]\rho\nabla\cdot\boldsymbol{V}$$
$$+\varphi c_{\mathrm{p}}(1-\eta)\left(c_{\mathrm{p}}\eta T+\frac{V_{\mathrm{s}}^2}{2}\right)\frac{\mathrm{D}T}{\mathrm{D}t}+\left\{c_{\mathrm{p}}T+\frac{V^2}{2}+\varphi\left[\alpha\left(c_{\mathrm{p}}\eta T+\frac{V_{\mathrm{s}}^2}{2}\right)-\left(c_{\mathrm{p}}T+\frac{V^2}{2}\right)\right]-\left(c_{\mathrm{p}}\eta T+\frac{V_{\mathrm{s}}^2}{2}\right)\right\}\frac{\mathrm{D}\rho}{\mathrm{D}t}$$
$$=\nabla\cdot(\lambda\nabla T)+\rho q_{\mathrm{R}}+\boldsymbol{f}\cdot\rho\left[\boldsymbol{V}+\varphi(\alpha\boldsymbol{V}_{\mathrm{s}}-\boldsymbol{V})\right]+\nabla\cdot\boldsymbol{P}\cdot\left[\varphi^{\frac{2}{3}}(\boldsymbol{V}_{\mathrm{s}}-\boldsymbol{V})+\boldsymbol{V}\right]\qquad(4\text{-}5\text{-}33)$$

除以 ρ，化简整理：

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(c_{\mathrm{p}}T+\frac{V^2}{2}\right)+\varphi\frac{\mathrm{D}}{\mathrm{D}t}\left[\alpha(\eta-1)c_{\mathrm{p}}T+\frac{1}{2}(\alpha V_{\mathrm{s}}^2-V^2)\right]+\left[c_{\mathrm{p}}(1-\eta)T+\frac{1}{2}(V^2-V_{\mathrm{s}}^2)\right]\nabla\cdot\boldsymbol{V}$$
$$+\frac{1}{\rho}\varphi c_{\mathrm{p}}(1-\eta)\left(c_{\mathrm{p}}\eta T+\frac{V_{\mathrm{s}}^2}{2}\right)\frac{\mathrm{D}T}{\mathrm{D}t}+\left[c_{\mathrm{p}}\eta(\varphi\alpha-1)T+(\varphi-1)\frac{V_{\mathrm{s}}^2}{2}\right]\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}$$
$$=\frac{1}{\rho}\nabla\cdot(\lambda\nabla T)+q_{\mathrm{R}}+\boldsymbol{f}\cdot\left[\boldsymbol{V}+\varphi(\alpha\boldsymbol{V}_{\mathrm{s}}-\boldsymbol{V})\right]+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot\left[\varphi^{\frac{2}{3}}(\boldsymbol{V}_{\mathrm{s}}-\boldsymbol{V})+\boldsymbol{V}\right]\qquad(4\text{-}5\text{-}34)$$

化掉式(4-5-34)中的 V_{s}^2。式中，V^2 是连续相标量速度，其值为

$$V^2=u^2+v^2+w^2\qquad(4\text{-}5\text{-}35)$$

V_{s}^2 是涡旋微团标量速度，其值为

$$V_{\mathrm{s}}^2=u_{\mathrm{s}}^2+v_{\mathrm{s}}^2+w_{\mathrm{s}}^2\qquad(4\text{-}5\text{-}36)$$

由式(4-3-49)至式(4-4-51)可知

$$u_{\mathrm{s}}^2=\left[u-\frac{\nu}{2u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\right]^2=u^2-\nu\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{\nu^2}{4u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)^2\qquad(4\text{-}5\text{-}37)$$

取近似值：

$$u_{\mathrm{s}}^2=u^2-\nu\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\qquad(4\text{-}5\text{-}38)$$

同理：

$$v_{\mathrm{s}}^2=v^2-\nu\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\qquad(4\text{-}5\text{-}39)$$

$$w_{\mathrm{s}}^2=w^2-\nu\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\qquad(4\text{-}5\text{-}40)$$

将式(4-5-38)至式(4-5-40)代入式(4-5-34)，则有

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[c_{\mathrm{p}}T+\frac{1}{2}(u^2+v^2+w^2)\right]+\frac{1}{2}\varphi\left[(\alpha-1)(u^2+v^2+w^2)-\alpha\nu\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}+\frac{\partial v}{\partial z}+\frac{\partial v}{\partial x}\right.\right.$$
$$\left.\left.+\frac{\partial w}{\partial y}+\frac{\partial w}{\partial x}\right)\right]\frac{\mathrm{D}p}{\rho\mathrm{D}t}+\frac{1}{2}\varphi\frac{\mathrm{D}}{\mathrm{D}t}\left[(\alpha-1)(u^2+v^2+w^2)-2\nu\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]$$

$$+(1-n)c_{\mathrm{p}}T+\frac{1}{2}\left[(1-\alpha)(u^2+v^2+w^2)-2\nu\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]\nabla\cdot\mathbf{V}$$

$$+[\varphi(1-\alpha)-1]\eta c_{\mathrm{p}}T\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}-\frac{1}{2}\varphi\alpha\left[(u^2+v^2+w^2)-\nu\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}$$

$$=\frac{\lambda}{\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+q+f_x u+f_y v+f_z w+f_x\left[(\alpha-1)u-\frac{\alpha\nu}{2u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\right]$$

$$+f_y\left[(\alpha-1)-\frac{\alpha\nu}{2v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\right]+f_z\left[(\alpha-1)u-\frac{\alpha\nu}{2w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\right]-\frac{u}{\rho}\frac{\partial p}{\partial x}$$

$$+\nu\left[u\left(\frac{4}{3}\frac{\partial^2 u}{\partial x^2}-\frac{2}{3}\frac{\partial^2 v}{\partial y\partial x}-\frac{2}{3}\frac{\partial^2 w}{\partial z\partial x}\right)+v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial y\partial x}\right)+w\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)\right]-\frac{p}{\rho}\frac{\partial u}{\partial x}$$

$$+\nu\left[\left(\frac{4}{3}\frac{\partial u}{\partial x}-\frac{2}{3}\frac{\partial v}{\partial y}-\frac{2}{3}\frac{\partial w}{\partial z}\right)\frac{\partial u}{\partial x}+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial x}+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\frac{\partial w}{\partial x}\right]-\frac{v}{\rho}\frac{\partial p}{\partial y}$$

$$+\nu\left[u\left(\frac{\partial^2 u}{\partial y^2}-\frac{\partial v}{\partial x\partial y}\right)+v\left(-\frac{2}{3}\frac{\partial^2 u}{\partial x\partial y}+\frac{4}{3}\frac{\partial^2 v}{\partial y^2}-\frac{2}{3}\frac{\partial^2 w}{\partial z\partial y}\right)+w\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 v}{\partial z\partial y}\right)\right]-\frac{p}{\rho}\frac{\partial v}{\partial y}$$

$$+\nu\left[\left(-\frac{2}{3}\frac{\partial u}{\partial x}+\frac{4}{3}\frac{\partial v}{\partial y}-\frac{2}{3}\frac{\partial w}{\partial z}\right)\frac{\partial v}{\partial y}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial y}+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\frac{\partial w}{\partial y}\right]-\frac{w}{\rho}\frac{\partial p}{\partial z}$$

$$+\nu\left[u\left(\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}\right)+v\left(\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}\right)+w\left(-\frac{2}{3}\frac{\partial^2 u}{\partial x\partial z}-\frac{2}{3}\frac{\partial^2 v}{\partial y\partial z}+\frac{4}{3}\frac{\partial^2 w}{\partial z^2}\right)\right]-\frac{p}{\rho}\frac{\partial w}{\partial z}$$

$$+\nu\left[\left(-\frac{2}{3}\frac{\partial u}{\partial x}-\frac{2}{3}\frac{\partial v}{\partial y}+\frac{4}{3}\frac{\partial w}{\partial z}\right)\frac{\partial w}{\partial z}+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\frac{\partial u}{\partial z}+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\frac{\partial v}{\partial z}\right]$$

$$-\frac{\nu}{2}\varphi^{\frac{2}{3}}\left\{\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\left[-\frac{\partial p}{\rho\partial x}+\nu\left(\frac{4}{3}\frac{\partial^2 u}{\partial x^2}-\frac{2}{3}\frac{\partial^2 v}{\partial y\partial x}-\frac{2}{3}\frac{\partial^2 w}{\partial z\partial x}\right)\right]\right.$$

$$+\left[-\frac{p}{\rho}+\nu\left(\frac{4}{3}\frac{\partial u}{\partial x}-\frac{2}{3}\frac{\partial v}{\partial y}-\frac{2}{3}\frac{\partial w}{\partial z}\right)\right]\left[\frac{1}{u}\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial z\partial x}\right)-\frac{1}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial x}\right]$$

$$+\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\left\{\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial u}{\partial y}\right)+\varphi^{\frac{2}{3}}\left[2\nu t'\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\right)-\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial y\partial x}\right)\right]\right\}$$

$$+\left\{\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\varphi^{\frac{2}{3}}\left[\nu t'\left(\frac{\partial v}{\partial x}\right)^2-\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]\right\}\left[\frac{1}{v}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z\partial x}\right)-\frac{1}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial x}\right]$$

$$+\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\left\{\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial z\partial x}\right)+\varphi^{\frac{2}{3}}\left[2\nu t'\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)-\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial z\partial x}\right)\right]\right\}$$

$$+\left\{\nu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+\varphi^{\frac{2}{3}}\left[\nu t'\left(\frac{\partial w}{\partial x}\right)^2-\nu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]\right\}\left[\frac{1}{w}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y\partial x}\right)-\frac{1}{w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial x}\right]$$

$$+\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\left\{\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+\varphi^{\frac{2}{3}}\left[2\nu t'\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right)-\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)\right]\right\}$$

$$+\left\{\nu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\varphi^{\frac{2}{3}}\left[\nu t'\left(\frac{\partial u}{\partial y}\right)^2-\nu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\right\}\left[\frac{1}{u}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z\partial y}\right)-\frac{1}{u^2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial y}\right]$$

$$+\frac{1}{v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\left[-\frac{\partial p}{\rho\partial y}+\nu\left(\frac{4}{3}\frac{\partial^2 v}{\partial y^2}-\frac{2}{3}\frac{\partial^2 u}{\partial x\partial y}-\frac{2}{3}\frac{\partial^2 w}{\partial z\partial y}\right)\right]+\left[-\frac{p}{\rho}+\nu\left(\frac{4}{3}\frac{\partial v}{\partial y}-\frac{2}{3}\frac{\partial u}{\partial x}\right.\right.$$

$$\left.\left.-\frac{2}{3}\frac{\partial w}{\partial z}\right)\right]\left[\frac{1}{v}\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 v}{\partial z\partial y}\right)-\frac{1}{v^2}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial y}\right]+\frac{1}{w}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\left\{\nu\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 v}{\partial z\partial y}\right)\right.$$

$$\left.+\varphi^{\frac{2}{3}}\left[2\nu t\left(\frac{\partial w}{\partial y}\right)\frac{\partial^2 w}{\partial y^2}-\nu\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 v}{\partial z\partial y}\right)\right]\right\}+\left\{\nu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\varphi^{\frac{2}{3}}\left[\nu t\left(\frac{\partial w}{\partial y}\right)^2\right.\right.$$

$$
\begin{aligned}
&-\nu\bigg(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\bigg)\bigg]\bigg\}\bigg[\frac{1}{w}\bigg(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\bigg)-\frac{1}{w^2}\bigg(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\bigg)\frac{\partial w}{\partial y}\bigg]+\frac{1}{u}\bigg(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\bigg)\bigg\{\nu\bigg(\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}\bigg)\\
&+\varphi^{\frac{2}{3}}\bigg[2\nu t'\bigg(\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\bigg)-\nu\bigg(\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}\bigg)\bigg]\bigg\}+\bigg\{\nu\bigg(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\bigg)+\varphi^{\frac{2}{3}}\bigg[\nu t'\bigg(\frac{\partial u}{\partial z}\bigg)^2\\
&-\nu\bigg(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\bigg)\bigg]\bigg\}\bigg[\frac{1}{u}\bigg(\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial z^2}\bigg)-\frac{1}{u^2}\bigg(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\bigg)\frac{\partial u}{\partial z}\bigg]+\frac{1}{v}\bigg(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\bigg)\bigg\{\nu\bigg(\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}\bigg)\\
&+\varphi^{\frac{2}{3}}\bigg[2\nu t\bigg(\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}\bigg)-\nu\bigg(\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}\bigg)\bigg]\bigg\}+\bigg\{\nu\bigg(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\bigg)+\varphi^{\frac{2}{3}}\bigg[\nu t'\bigg(\frac{\partial v}{\partial z}\bigg)^2\\
&-\nu\bigg(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\bigg)\bigg]\bigg\}\bigg[\frac{1}{v}\bigg(\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 v}{\partial z^2}\bigg)-\frac{1}{v^2}\bigg(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\bigg)\frac{\partial v}{\partial z}\bigg]+\frac{1}{w}\bigg(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\bigg)\\
&\quad\bigg[-\frac{\partial p}{\rho\partial z}+\nu\bigg(\frac{4}{3}\frac{\partial^2 w}{\partial z^2}-\frac{2}{3}\frac{\partial^2 u}{\partial x\partial z}-\frac{2}{3}\frac{\partial^2 v}{\partial y\partial z}\bigg)\bigg]+\bigg[-\frac{p}{\rho}+\nu\bigg(\frac{4}{3}\frac{\partial w}{\partial z}-\frac{2}{3}\frac{\partial u}{\partial x}-\frac{2}{3}\frac{\partial v}{\partial y}\bigg)\bigg]\\
&\quad\bigg[\frac{1}{w}\bigg(\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 w}{\partial y\partial z}\bigg)-\frac{1}{w^2}\bigg(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\bigg)\frac{\partial w}{\partial z}\bigg]\bigg\} \qquad (4\text{-}5\text{-}41)
\end{aligned}
$$

式(4-5-41)就是可压缩湍流在直角坐标系下的能量微分方程，当定常流时，可去掉含$\frac{\partial}{\partial t}$项。

第 5 章　有压管道湍流运动

本章讨论充分发展湍流管道(限定为定常流)以及不可压缩流体运动。研究内容为圆形与矩形管道断面速度分布、涡旋体积分数 φ 的计算方法、沿程阻力损失计算公式。

5.1　湍流圆形管道断面速度分布

圆形管道湍流运动为一维流动、一元变化。依据第 4 章导出的湍流动量微分方程(4-4-10)，结合具体问题，忽略重力影响下，简化为圆形管道湍流运动控制微分方程

$$\frac{1}{\rho}\cdot\frac{\mathrm{d}p}{\mathrm{d}x}=2\nu\left[(1-\varphi^{\frac{2}{3}})+6t\varphi^{\frac{2}{3}}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)\right]\frac{\mathrm{d}^2u}{\mathrm{d}r^2} \tag{5-1-1}$$

边界条件为

$$u\big|_{r=R}=0 \tag{5-1-2}$$

$$\left.\frac{\mathrm{d}u}{\mathrm{d}r}\right|_{r=0}=0 \tag{5-1-3}$$

式(5-1-1)至式(5-1-3)为二阶非线性常微分方程组。用自创的“逆向法”对其求解。为此，先将其化成无因次方程组。如图 5-1 所示，取

$$\frac{r}{r_0}=R\quad \frac{u}{v_0}=U \tag{5-1-4}$$

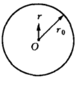


图 5-1　原点示意图

将式(5-1-4)代入式(5-1-1)至式(5-1-3)组成的方程组，则有

$$K_1=\left(1+K_2\frac{\mathrm{d}U}{\mathrm{d}R}\right)\frac{\mathrm{d}^2U}{\mathrm{d}R^2} \tag{5-1-5}$$

$$U\big|_{R=1}=0 \tag{5-1-6}$$

$$\left.\frac{\mathrm{d}U}{\mathrm{d}R}\right|_{R=0}=0 \tag{5-1-7}$$

式中：

$$K_1=\frac{r_0^2}{2(1-\varphi^{\frac{2}{3}})\mu v_0}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{5-1-8}$$

$$K_2=\frac{6\varphi^{\frac{2}{3}}tv_0}{(1-\varphi^{\frac{2}{3}})r_0} \tag{5-1-9}$$

根据边界条件，选择边界函数为

$$U'=1-R^2 \tag{5-1-10}$$

则待定函数为

$$U=\eta(1-R^2) \tag{5-1-11}$$

将式(5-1-11)代入式(5-1-5)得

$$K_1=(1+K_2\eta 2R)2\eta=4K_2R\eta^2+2\eta$$

解

$$4K_2R\eta^2+2\eta-K_1=0 \tag{5-1-12}$$

得

$$\eta=\frac{K_1}{2}=\frac{r_0^2}{4\mu v_0(1-\varphi^{\frac{2}{3}})}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{5-1-13}$$

将式(5-1-13)代入式(5-1-5)，并有因次式，得

$$u=\frac{1}{4\mu(1-\varphi^{\frac{2}{3}})}\frac{\mathrm{d}p}{\mathrm{d}x}(r_0^2-r^2) \tag{5-1-14}$$

式(5-1-1)至式(5-1-3)组成的方程组还可以用直接积分方法求解，将式(5-1-1)进行第一次积分：

$$\frac{\mathrm{d}p}{2\mu\mathrm{d}x}r+c_1=\left[(1-\varphi^{\frac{2}{3}})+6t\varphi^{\frac{2}{3}}\frac{\mathrm{d}u}{\mathrm{d}r}\right]\frac{\mathrm{d}u}{\mathrm{d}r}$$

利用边界条件式(5-1-3)，定出 $c_1=0$，则

$$\frac{\mathrm{d}p}{2\mu\mathrm{d}x}r=(1-\varphi^{\frac{2}{3}})\frac{\mathrm{d}u}{\mathrm{d}r}+6t\varphi^{\frac{2}{3}}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 \tag{5-1-15}$$

由式(5-1-15)解得

$$\frac{\mathrm{d}u}{\mathrm{d}r}=\frac{r}{2(1-\varphi^{\frac{2}{3}})\mu}\cdot\frac{\mathrm{d}p}{\mathrm{d}x} \tag{5-1-16}$$

由图 5-1 可知，$\frac{\mathrm{d}u}{\mathrm{d}r}$是距离增加，速度减少，则式(5-1-16)应为

$$\frac{\mathrm{d}u}{\mathrm{d}r}=\frac{-r}{2(1-\varphi^{\frac{2}{3}})\mu}\cdot\frac{\mathrm{d}p}{\mathrm{d}x} \tag{5-1-17}$$

对式(5-1-17)进行积分得

$$u=\frac{-\mathrm{d}p}{4(1-\varphi^{\frac{2}{3}})\mathrm{d}x\mu}r^2+c_2 \tag{5-1-18}$$

利用边界条件式(5-1-2)，定出 c_2：

$$c_2=\frac{r_0^2}{4(1-\varphi^{\frac{2}{3}})\mu}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{5-1-19}$$

将式(5-1-19)代入，得

$$u=\frac{1}{4\mu(1-\varphi^{\frac{2}{3}})}\frac{\mathrm{d}p}{\mathrm{d}x}(r_0^2-r^2) \tag{5-1-20}$$

式(5-1-20)与式(5-1-14)完全一样，说明“逆向法”是可行的。式(5-1-14)就是有压管道圆形断面湍流速度分布公式，当 $\varphi=0$ 时，与层流圆形断面速度分布一致，说明理论正确。

φ 值的大小反映湍流强度，φ 值愈大，说明速度 u 愈大。这就与物理现象吻合。

5.2　湍流圆形管道涡旋体积分数 φ 的分析

在第 4 章结束时，讨论湍流运动微分方程组封闭，尚留下一个未知函数涡旋体积分数 φ 没有解决。由于它在处于某一流态下是常数，具有参变常数的性质，所以在推导速度分布时把它作为常数处理。圆形湍流管道断面速度分布已经导出，要应用它，必须解决 φ 的计算问题。

5.2.1　涡旋产生横向运动的条件

在第 1 章边层流理论分析过程中，曾谈过涡旋运动分为 3 种形式：一是紧贴壁面就地旋转的涡旋运动；二是边层流区内一边平移一边旋转的成层运动涡旋运动；三是活跃在边外区(湍流区)除旋转外，还有沿主流方向平移与垂直于主流方向运动的涡旋，称之为具有横向运动分速的涡旋。

具有横向运动分速的涡旋，其产生条件有 3 个：①产生这种涡旋的地带，速度分布必须连续可导，而且速度梯度有一定的强度；②必须是有旋转的流体微团；③微团周围具有相适应的绕流运动。此外，壁面粗糙高度形成流层弯曲，当涡旋微团经过它时，受到离心力作用，对涡旋产生横向运动，起着瞬时推动作用。

5.2.2　涡旋产生横向运动地带

如图 5-2 所示，壁面粗糙 Δ_1 处，因断面缩小，速度分布形成的速度梯度比周边点也大，而且各点不同，最大发生在边层流界面上。

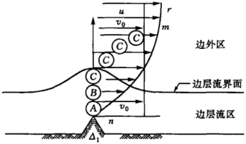


图 5-2　涡旋产生横向运动与其频率确定示意图

紧贴壁面上流体微团Ⓐ以平均速度全部转化的旋转而就地转动；微团Ⓑ一边平移一边转动地前进，但它不具备绕流条件，无法引起升力对其作用，也就不会产生横向运动；微团Ⓒ有平移与相应的转动，周围有绕流存在，具备引起升力的条件，而且该点形成的速度最大，再加上离心力的促动作用，在某一个流态下，使其在垂直于主流方向受的力足以克服自身的惯性力而产生横向运动，从而脱离所在流层，进入外边的边外区。通过上述分析，得出涡旋

产生横向运动的地带是在边层流界面上的结论。

5.2.3 涡旋产生横向运动的频率 f 的确定

涡旋微团©，脱离所在流层，使该层出现裂断，造成不连续现象。©向斜上方运动，所到之处，都会形成瞬时不连续现象。这样，原有的$\overset{\frown}{nm}$速度分布曲线就出现在一段时间内，总有不连续点的存在，使得在粗糙 Δ_1 的顶部失去产生涡旋横向运动的条件。

要使 Δ_1 顶部再次出现涡旋产生横向运动，必须使$\overset{\frown}{nm}$速度分布曲线恢复到稳定连续状态。这需要一段时间。从理论上规定速度分布$\overset{\frown}{nm}$曲线的范围，这个范围如何规定呢？认定涡旋微团©沿法线方向走出自身直径的 2 倍，用它定为$\overset{\frown}{nm}$曲线的范围。也就是说，当涡旋微团©沿法线走出 $2d_s$ 后，则$\overset{\frown}{nm}$曲线重新恢复到连续状态。因此，它又能使 Δ_1 顶部涡旋微团产生横向运动的条件。这段时间可以依据涡旋横向运动速度 u^* 除以 $2d_s$ 来得到。那么，在 1s 内，在 Δ_1 顶部产生横向运动涡旋的次数，即频率 f，就可以由下式表达：

$$f=\frac{u^*}{2d_s} \tag{5-2-1}$$

5.2.4 涡旋体积分数 φ 的计算公式的推导

从图 5-2 可以看出壁面粗糙对涡旋产生横向运动的影响。当层流转化为湍流时，首先，最高的粗糙起作用，随着湍流度增加，边层流厚度 δ 变薄，一批高的粗糙进入边外区，失去作用；较低粗糙出现在边层流界面下，开始对涡旋产生横向运动起作用。依此类推。

φ 的定义如下：

$$\varphi=\frac{\text{涡旋体积}}{\text{流量}}=\frac{nf\dfrac{\pi}{6}d_s^3}{\dfrac{\pi}{4}v_0D_0^2} \tag{5-2-2}$$

式中，n 是速度 v_0 的壁面上能产生涡旋的粗糙 Δ 的数目。在粗糙密布情况下，同样高度的粗糙 Δ，只要其中一个在某流态下产生涡旋，周围的其他粗糙就不会再产生涡旋。因为涡旋产生横向运动是受升力作用的，而引起升力的一个重要条件是涡旋周围必须有绕流。所以规定其绕流范围为 $2d_s\times2d_s=4d_s^2$。由此，n 可以写成：

$$n=\frac{\pi Dv_0}{4d_s^2} \tag{5-2-3}$$

将式(5-2-1)与式(5-2-3)代入式(5-2-2)，则有

$$\varphi=\frac{[\text{秒}]\pi u^*}{12D} \tag{5-2-4}$$

式(5-2-4)表明涡旋体积分数 φ 与涡旋横向速度 u^* 成正比，与管道直径 D 成反比。

将式(4-3-27)简化，可得

$$u^* = 3\nu t\left(\frac{\partial u}{\partial y}\right)^2 u^{-1} \tag{5-2-5}$$

结合管道特点，涡旋产生横向运动速度 u^* 的位置在边层流界面上，而它正是断面平均速度 v_0 与断面速度分布曲线的交点。这样，式(4-3-27)可写成

$$u^* = 3\nu t\left(\left.\frac{\mathrm{d}u}{\mathrm{d}r}\right|_{r=r_B}\right)^2 v_0^{-1} \tag{5-2-6}$$

由式(5-1-20)，得

$$\left.\frac{\mathrm{d}u}{\mathrm{d}r}\right|_{r=r_B} = \frac{r_B}{2\mu(1-\varphi^{\frac{2}{3}})}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{5-2-7}$$

将式(5-2-7)代入式(5-2-6)，得

$$u^* = 3\nu t\left[\frac{r_B\dfrac{\mathrm{d}p}{\mathrm{d}x}}{2\mu(1-\varphi^{\frac{2}{3}})}\right]^2 v_0^{-1} \tag{5-2-8}$$

式中，r_B 可由式(5-1-20)计算，从而

$$r_B^2 = r_0^2 - \frac{4v_0\mu(1-\varphi^{\frac{2}{3}})}{\dfrac{\mathrm{d}p}{\mathrm{d}x}} \tag{5-2-9}$$

将式(5-2-9)代入式(5-2-8)，得

$$u^* = \frac{3\nu t\dfrac{\mathrm{d}p}{\mathrm{d}x}\left[r_0^2\dfrac{\mathrm{d}p}{\mathrm{d}x} - 4\mu v_0(1-\varphi^{\frac{2}{3}})\right]}{4\mu^2(1-\varphi^{\frac{2}{3}})^2 v_0} \tag{5-2-10}$$

将式(5-2-10)代入式(5-2-4)，得

$$\varphi = \frac{[秒]\pi}{12D}\left\{\frac{3\nu t\dfrac{\mathrm{d}p}{\mathrm{d}x}\left[r_0^2\dfrac{\mathrm{d}p}{\mathrm{d}x} - 4\mu v_0(1-\varphi^{\frac{2}{3}})\right]}{4\mu^2(1-\varphi^{\frac{2}{3}})^2 v_0}\right\} \tag{5-2-11}$$

在具体问题中，式(5-2-11)只有 φ 是未知的，其他都是已知数，因而可以计算出 φ 值。为计算方便，将它分为 y_1 与 y_2。令

$$y_1 = 12D[4\mu^2(1-\varphi^{\frac{2}{3}})^2 v_0]\varphi \tag{5-2-12}$$

$$y_2 = [秒]\pi\left\{3\nu t\frac{\mathrm{d}p}{\mathrm{d}x}\left[r_0^2\frac{\mathrm{d}p}{\mathrm{d}x} - 4\mu v_0(1-\varphi^{\frac{2}{3}})\right]\right\} \tag{5-2-13}$$

分别计算 y_1 与 y_2，绘出 y_1 与 y_2 曲线的交点，其就是 φ 值，见图 5-3。

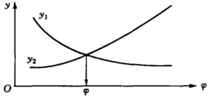


图 5-3　计算 φ 值用图

如图 5-2 所示，稳定连续速度分布曲线 $\overset{\frown}{nm}$ 的范围是由涡旋沿法线方向走出自身直径 3 倍距离为限，则该点重复产生涡旋频率 f 为

$$f=\frac{[秒]u^*}{3d_s} \tag{5-2-14}$$

产生涡旋横向运动的另一个重要条件是涡旋区周围绕流的范围，若定它为

$$3d_s\times 3d_s=9d_s^2 \tag{5-2-15}$$

则管内壁 πDv_0 面上产生涡旋数目 n 为

$$n=\frac{\pi Dv_0}{9d_s^2} \tag{5-2-16}$$

涡旋体积分数 φ 的定义式为

$$\varphi=\frac{涡旋体积}{流量}=\frac{nf\frac{\pi}{6}d_s^3}{\frac{\pi}{4}Dv_0} \tag{5-2-17}$$

将式(5-2-14)与式(5-2-16)代入式(5-2-17)，则有

$$\varphi=\frac{2[秒]\pi u^*}{81D} \tag{5-2-18}$$

式(5-2-4)与式(5-2-18)不同之处只是$\frac{1}{12}$与$\frac{2}{81}$，究竟为多少应由实验决定。

5.3　圆形管道湍流度的确定方法

首先确定涡旋以横向运动分速度 u^* 在准定常流时间 T 所走过的路程 s^*，现推导如下。

结合管道特点，将式(4-3-27)化简为

$$u_{yz}^*{}^- =12\nu\omega_{yz}^*{}^- u^{-1}t \tag{5-3-1}$$

由于 $u_{yz}^*{}^-$ 是在边层流界面产生的，所以 $u^{-1}=v_0^{-1}$，T 为湍流准定常时间，流态一定时为常数。$\omega_{yz}{}^-$ 在圆形管道中为 ω_r，因产生在边层流界面上，故 $\omega_{r=r_B}$。涡旋产生后，有消失过程，一般很短，但不会超过 4s。它所走过的距离为

$$s^*=u_0^*\left(\sum_{t=0}^{n}e^{-t}\right) \tag{5-3-2}$$

湍流强度定义为

$$\alpha^*=\frac{s^*}{r_0} \tag{5-3-3}$$

例 5-1　有压湍流圆形管道长 50m，直径 D 为 0.1m，水的运动黏性系数 ν 为 1.141×10^{-6} m^2/s，运动黏性系数 μ 为 $1.14\times10^{-3}N\cdot s/m^2$，水头损失观测为 12.742m，通过的流量为 $0.04m^3/s$，求其涡旋体积分数 φ 与准定常时间 T 各为多少？然后依此确定边层流厚度 δ，边层流界面上涡旋强度 ω，涡旋直径 d_s，横向运动分速度 u^*，传播距离 s^*，以及湍流度 α^*。

解　确定断面平均流速 v_0：

$$v_0=\frac{Q}{A}=\frac{0.04}{\frac{\pi}{4}D^2}=5m/s$$

确定压力梯度

$$\frac{\mathrm{d}p}{\mathrm{d}x}=12.74\mathrm{N/m^3}$$

判别流态，计算雷诺数 Re：

$$Re=\frac{v_0D}{\nu}=\frac{5\times0.1\times10^6}{1.141}=4.36\times10^{-6}$$

计算涡旋体积分数 φ，由式(5-2-12)与式(5-2-13)，有

$$\begin{aligned}y_1&=12D[4\mu^2(1-\varphi^{\frac{2}{3}})^2]v_0\varphi=48\times1.14^2\times10^{-6}\times5\varphi(1-\varphi^{\frac{2}{3}})^2\\&=31.19\times10^{-6}(1-\varphi^{\frac{2}{3}})\varphi=0.32\times10^{-6}(1-\varphi^{\frac{2}{3}})\varphi\end{aligned}\tag{5-3-4}$$

$$\begin{aligned}y_2&=3\pi\nu\frac{\mathrm{d}p}{\mathrm{d}x}\left[r_0^2\frac{\mathrm{d}p}{\mathrm{d}x}-4\mu v_0(1-\varphi^{\frac{2}{3}})\right]T_K\\&=3\times3.1416\times1.141\times10^{-6}\times12.74[0.05^2\times12.74-4\times1.14\times10^{-3}\times5(1-\varphi^{\frac{2}{3}})]T_K\\&=0.137\times10^{-3}[0.0318-0.023(1-\varphi^{\frac{2}{3}})]T_K\end{aligned}\tag{5-3-5}$$

设不同的 φ 值，将计算结果列入表 5-1 中，然后绘成相交曲线，其交点则为 T_K 与 φ_K。

表 5-1　y_1 与 y_2 的值

φ	$\varphi^{\frac{2}{3}}$	$1-\varphi^{\frac{2}{3}}$	$(1-\varphi^{\frac{2}{3}})^2$	$\varphi(1-\varphi^{\frac{2}{3}})^2$	$y_1/\times10^{-5}$	$y_2(T_K=1\mathrm{s})/\times10^{-5}$
0.01	0.046	0.954	0.910	0.0091	0.0291	0.1356
0.05	0.136	0.864	0.746	0.0373	0.1190	0.1515
0.10	0.215	0.758	0.575	0.0575	0.1840	0.1959
0.15	0.280	0.720	0.518	0.0777	0.2486	0.2165
0.20	0.342	0.658	0.433	0.0866	0.2771	0.2284
0.25	0.397	0.603	0.364	0.0910	0.2912	0.2480
0.30	0.448	0.552	0.304	0.0912	0.3260	0.2617

将 y_1，y_2 绘成曲线，如图 5-4 所示，交点 $\varphi_K=0.088$，$T_K=1\mathrm{s}$。由此可以计算其他各值。

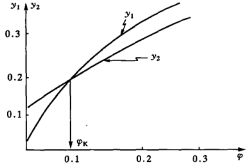


图 5-4　例 5-1 计算结果示意图($\varphi_K=0.088$，$T_K=1\mathrm{s}$)

计算边层流界面位置，由式(5-2-9)，得

$$r_B=\sqrt{r_0^2-\frac{4\mu v_0(1-\varphi^{\frac{2}{3}})}{\frac{\mathrm{d}p}{\mathrm{d}x}}}=\sqrt{0.05^2-\frac{4\times1.14\times10^{-3}\times5(1-0.088^{\frac{2}{3}})}{12.74}}$$

$$=\sqrt{0.0025-\frac{0.0228\times0.802}{12.74}}=\sqrt{0.0025-0.00143}=0.0327\mathrm{m}$$

边层流区厚度 δ：

$$\delta=r_0-r_B=0.05-0.0327=0.0173\mathrm{m}$$

计算边层流界面上产生的涡旋 ω 的强度及大小，由式(5-2-7)，有

$$\omega=\frac{r_B\frac{\mathrm{d}p}{\mathrm{d}x}}{4\mu(1-\varphi^{\frac{2}{3}})}=\frac{0.0327\times12.74\times10^{-3}}{4\times1.14\times(1-0.088^{\frac{2}{3}})}=114\mathrm{rad/s}$$

涡旋直径：

$$d_s=2r_s=\sqrt{\frac{10\nu}{\omega}}\times2=2\sqrt{\frac{10\times1.141\times10^{-6}}{114}}=\frac{2}{10^4}\sqrt{\frac{1.141}{1.14}}$$

$$=\frac{2}{10^4}\sqrt{1.009}=\frac{2\times1.0045}{10^4}=0.0002\mathrm{m}$$

计算涡旋横向运动分速度 u^*，由式(5-2-10)，有

$$u^*=\frac{3\nu T_K\frac{\mathrm{d}p}{\mathrm{d}x}\left[r_0^2\frac{\mathrm{d}p}{\mathrm{d}x}-4\mu v_0(1-\varphi^{\frac{2}{3}})\right]}{4\mu^2(1-\varphi^{\frac{2}{3}})^2v_0}$$

$$=\frac{3\times1.141\times10^{-6}\times1\times12.74[0.05^2\times12.74-4\times1.14\times10^{-3}\times5(1-0.088^{\frac{2}{3}})]}{4\times1.14^2\times10^{-6}(1-0.088^{\frac{2}{3}})^2\times5}$$

$$=\frac{43.609\times(0.03185-0.0228\times0.802)}{16.718}=\frac{43.609\times0.0135}{15.85}=\frac{0.5909}{16.718}=0.035\mathrm{m/s}$$

计算涡旋垂直于边层流界面方向的运动时间与距离。由式(5-3-2)，已知 $u_0^*=0.035$ m/s，若 $n=2$s，则有

$$s^*=u_0^*(\mathrm{e}^{-0}+\mathrm{e}^{-1}+\mathrm{e}^{-2})\approx0.035(1+0.368+0.135)=0.035\times1.503=0.0526\mathrm{m}>0.05\mathrm{m}$$

说明 2s 时涡旋已超过轴线，而此时 u^* 减少为 0.0047m/s，直径也变得大得多，转速已很小，近似消失。

计算湍流度 α^*，由式(5-3-3)，有

$$\alpha^*=\frac{s^*}{r_0}=\frac{0.0526}{0.05}=1.012$$

说明湍流度不高。

5.4 矩形管道湍流断面速度分析

矩形管道湍流是一维流动二维变化，由动量微分方程式(4-4-10)，在定常流、不可压缩

流体运动条件下，可简化为其控制微分方程：

$$\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}=(1-\varphi^{\frac{2}{3}})\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+6\nu t\varphi^{\frac{2}{3}}\left[\frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial z}\left(\frac{\partial^2 u}{\partial z^2}\right)\right] \tag{5-4-1}$$

分析：选断面上坐标位置如图 5-5 所示。为便于数学分析，将式(5-4-1)无因次化，为此令

$$U=\frac{u}{v_0}\quad \frac{z}{H}=Z\quad \frac{y}{B}=Y$$

代入式(5-4-1)，则

$$\frac{1}{\mu}\frac{\mathrm{d}pH^2}{\mathrm{d}x\,v_0}=(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right]+\varphi^{\frac{2}{3}}\left[\frac{6tv_0}{B}\left(\frac{H}{B}\right)^2\frac{\partial U}{\partial Y}\left(\frac{\partial^2 U}{\partial Y^2}\right)+\frac{6tv_0}{H}\left(\frac{\partial U}{\partial Z}\right)\left(\frac{\partial^2 U}{\partial Z^2}\right)\right] \tag{5-4-2}$$

令

$$K_1=\frac{1}{\mu}\frac{\mathrm{d}pH^2}{\mathrm{d}x\,v_0} \tag{5-4-3}$$

$$K_2=\frac{6v_0t}{B} \tag{5-4-4}$$

$$K_3=\frac{6v_0t}{H} \tag{5-4-5}$$

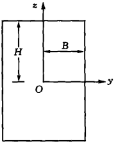


图 5-5　原点位置

则

$$K_1=(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right]+\varphi^{\frac{2}{3}}\left[K_2\left(\frac{H}{B}\right)^2\frac{\partial U}{\partial Y}\left(\frac{\partial^2 U}{\partial Y^2}\right)+K_3\frac{\partial U}{\partial Z}\left(\frac{\partial^2 U}{\partial Z^2}\right)\right] \tag{5-4-6}$$

边界条件：

$$U(Y,\ Z)\big|_{Z=1}=0 \tag{5-4-7}$$

$$U(Y,\ Z)\big|_{Y=0}=U(Z) \tag{5-4-8}$$

$$\left.\frac{\partial U}{\partial Z}\right|_{\substack{Z=0\\Y=0}}=0 \tag{5-4-9}$$

$$\left.\frac{\partial U}{\partial Y}\right|_{\substack{Z=0\\Y=0}}=0 \tag{5-4-10}$$

根据边界条件，选含待定系函数 $\eta(Y,\ Z)$的函数为

$$U(Y,\ Z)=\eta(Y,\ Z)(1-Z^2)(1-Y^2) \tag{5-4-11}$$

$$\frac{\partial U}{\partial Z}=2\eta Z(1-Y^2) \tag{5-4-12}$$

$$\frac{\partial^2 U}{\partial Z^2}=2\eta(1-Y^2) \tag{5-4-13}$$

$$\frac{\partial U}{\partial Y}=2\eta Y(1-Z^2) \tag{5-4-14}$$

$$\frac{\partial^2 U}{\partial Y^2}=2\eta(1-Z^2) \tag{5-4-15}$$

将式(5-4-12)~式(5-4-15)代入式(5-4-6)，得

$$K_1=(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2 2\eta(1-Z^2)+2\eta(1-Y^2)\right]+\varphi^{\frac{2}{3}}\left[K_2\left(\frac{H}{B}\right)^2 2\eta Y(1-Z^2)^2 2\eta+K_3 2\eta Z(1-Y^2)^2 2\eta\right]$$

$$=2\eta(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(1-Z^2)+(1-Y^2)\right]+4\varphi^{\frac{2}{3}}\left[K_2\left(\frac{H}{B}\right)^2 Y(1-Z^2)^2+K_3 Z(1-Y^2)\right]\eta^2 \tag{5-4-16}$$

解式(5-4-16)，得

$$\eta=\frac{K_1}{2(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(1-Z^2)+(1-Y^2)\right]} \tag{5-4-17}$$

将式(5-4-17)代入式(5-4-11)，得

$$U=\frac{K_1(1-Z^2)(1-Y^2)}{2(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(1-Z^2)+(1-Y^2)\right]} \tag{5-4-18}$$

将式(5-4-18)变回有因次速度式为

$$u=\frac{\frac{\mathrm{d}p}{\mathrm{d}x}(H^2-z^2)(B^2-y^2)}{2\mu(1-\varphi^{\frac{2}{3}})(H^2-z^2+B^2-y^2)} \tag{5-4-19}$$

当 $H=B$ 时，则成为圆形断面，它的结果正是圆形管道断面上速度分布公式，所以方法正确、可靠。

由边界条件决定函数形状，而微分方程确定的函数表示强度，微分运算改变函数的形状，所以改变后的函数其强度仍由微分方程确定的函数来描述。积分决定总量问题，所以无论由边界条件或方程确定的函数均参加计算。根据这个原则，$\eta(Y,Z)$不参加微分运算，而参加积分运算。

5.5 矩形管道湍流涡旋体积分数 φ 的分析

矩形断面上速度分布是二元函数，形成各边的边层流厚度不同，产生的涡旋横向运动分速度也不同。因此分为 y，z 两个方向分别讨论其 φ 值。

5.5.1 涡旋产生频率 f

在 y 方向，壁面边层流界面上产生的涡旋频率为 f_y，则

$$f_y=\frac{u^*_{yz^-}}{2d_{sz^-}} \tag{5-5-1}$$

式中，d_{sz^-}-是涡旋 ω_{z^-}的直径；$u^*_{yz^-}$-是由 ω_{z^-}引起的涡旋在 y 方向的运动分速度。

$$f_z=\frac{u^*_{zy^+}}{2d_{sy^+}} \tag{5-5-2}$$

式中，d_{sy^+}是涡旋 ω_{y^+}的直径；$u^*_{zy^+}$是由 ω_{y^+}引起的涡旋在 z 方向的运动分速度。

5.5.2　粗糙分布密集型涡旋体积分数 φ 的公式

涡旋产生数量表达式分为 y 与 z 方向：

$$\eta_y = \frac{4v_0 H}{4d^2_{sz^-}} \qquad \eta_z = \frac{4v_0 B}{4d^2_{sy^+}}$$

涡旋体积分数 φ 的定义式为

$$\varphi = \varphi_y + \varphi_z = \frac{\pi(Hu^*_{yz^-} + Bu^*_{zy^+})}{4BHB} \tag{5-5-3}$$

涡旋横向运动分速表达式：由式(4-3-28)与式(4-3-29)简化为

$$u^*_{zy^+} = 3\nu t\left(\frac{\partial u}{\partial z}\right)^2 u^{-1} \tag{5-5-4}$$

$$u^*_{yz^-} = 3\nu t\left(\frac{\partial u}{\partial y}\right)^2 u^{-1} \tag{5-5-5}$$

研究问题是在边层流界面上，如图 5-6 所示。

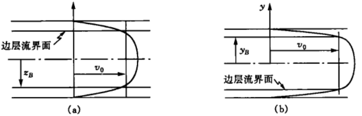


图 5-6　矩形断面上边层流界面示意图

结合矩形断面特点，式(5-5-4)与式(5-5-5)可写成

$$u^*_{zy^+} = 3t\nu\left(\omega_{zy^+}\Big|_{z=0}^{y=y_B}\right)^2 v_0^{-1} \tag{5-5-6}$$

$$u^*_{yz^-} = 3t\nu\left(\omega_{yz^-}\Big|_{z=z_B}^{y=0}\right)^2 v_0^{-1} \tag{5-5-7}$$

$$\omega_{yz^-} = \frac{1}{2}\frac{\partial u}{\partial y} = \frac{K_1}{2}\frac{v_0(1-Z^2)Y}{B(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(1-Z^2)+1-Y^2\right]} \tag{5-5-8}$$

$$\omega_{yz^-}\Big|_{z=0}^{y=y_B} = \frac{K_1 v_0 Y_B}{2B(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1-Y_B^2\right]} \tag{5-5-9}$$

同理

$$\omega_{zy^+}\Big|_{y=0}^{z=z_B} = \frac{K_1 v_0 Z_B}{2H(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(1-Z_B^2)+1\right]} \tag{5-5-10}$$

由式(5-4-18)计算 Y_B，Z_B。令 $U=1$，$Z=0$，得 Y_B；令 $Y=0$，得 Z_B。

$$Z_B=\sqrt{1-\frac{2(1-\varphi^{\frac{2}{3}})}{K_1-2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2}} \tag{5-5-11}$$

$$Y_B=\sqrt{1-\frac{2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2}{K_1-2(1-\varphi^{\frac{2}{3}})}} \tag{5-5-12}$$

例 5-2 矩形管道湍流通过的流量 $Q=0.04\text{m}^3/\text{s}$，管径高 $2H$，宽 $2B$，$H=0.0425\text{m}$，$B=0.0475\text{m}$，管长为 50m，水头损失 $h_f=14.158\text{m}$，水的运动黏性系数 $\nu=1.141\times10^{-6}$ m^2/s，动力黏性系数 $\mu=1.14\times10^{-3}\text{N}\cdot\text{s/m}^2$。求涡旋体积分数 φ，准定常时间 T，边层流厚度 δ，涡旋强度 ω，涡旋直径 d_s，涡旋横向运动速度 u^*，涡旋横向运动所走过的距离 s^*，湍流度 α^*。

解 计算断面平均速度：

$$v_0=\frac{Q}{A}=\frac{0.04}{2H\cdot2B}=\frac{0.04}{0.085\times0.095}=\frac{0.04}{0.008}=5\text{m/s}$$

计算雷诺数 Re：

$$Re=\frac{v_0D'}{\nu}=\frac{5\times0.008^{\frac{1}{2}}}{1.14\times10^{-6}}=3.944\times10^6$$

$$\frac{\mathrm{d}p}{\mathrm{d}x}=\frac{rh_f}{l}=14.16\text{N/m}^3$$

$$K_1=\frac{\mathrm{d}p}{\mu\mathrm{d}x}\frac{H^2}{v_0}=\frac{14.16\times10^3\times0.0425^2}{1.14\times5}=\frac{25.57}{5.05}=5.06$$

$$\left(\frac{H}{B}\right)^2=\left(\frac{0.0425}{0.0475}\right)^2=0.8947^2=0.801$$

由式(5-5-3)，令

$$y_1=48HB\varphi=48\times0.0425\times0.0475\varphi=0.0969$$

$$y_2=[\text{秒}]\pi(Hu^*_{yz^-}+Bu^*_{zy^+}) \tag{5-5-13}$$

由式(5-5-7)与式(5-5-6)，计算 $u^*_{zy^+}$ 与 $u^*_{yz^-}$：

$$u^*_{zy^+}=3T_K\nu\left(\omega_{zy^+}\Big|_{\substack{y=y_B\\z=0}}\right)^2v_0^{-1}=3\times1.141\times10^{-6}\left(\omega_{zy^+}\Big|_{\substack{y=y_B\\z=0}}\right)^2\times5^{-1}$$

$$=0.684\left(\omega_{zy^+}\Big|_{\substack{y=y_B\\z=0}}\right)^2\times10^{-6}T_K \tag{5-5-14}$$

$$u^*_{yz^-}=0.684\left(\omega_{yz^-}\Big|_{\substack{z=z_B\\y=0}}\right)^2\times10^{-6}T_K \tag{5-5-15}$$

由式(5-5-9)

$$\omega_{yz^-}\Big|_{\substack{z=z_B\\y=0}}=\frac{K_1v_0Y_B}{2B(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1-Y_B^2\right]}=\frac{25.3Y_B}{0.095(1-\varphi^{\frac{2}{3}})(1.801-Y_B^2)} \tag{5-5-16}$$

$$\omega_{xy^+}\Big|_{\substack{y=y_B\\z=0}} = \frac{25.3Z_B}{0.085(1-\varphi^{\frac{2}{3}})[0.801(1-Z_B^2)+1]} \tag{5-5-17}$$

由式(5-5-12)

$$Y_B = \sqrt{1-\frac{2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2}{K_1-2(1-\varphi^{\frac{2}{3}})}} = \sqrt{1-\frac{2\times0.801(1-\varphi^{\frac{2}{3}})}{5.06-2(1-\varphi^{\frac{2}{3}})}} = \sqrt{1-\frac{1.602(1-\varphi^{\frac{2}{3}})}{5.06-2(1-\varphi^{\frac{2}{3}})}} \tag{5-5-18}$$

由式(5-5-11)

$$Z_B = \sqrt{1-\frac{2(1-\varphi^{\frac{2}{3}})}{K_1-2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2}} = \sqrt{1-\frac{2(1-\varphi^{\frac{2}{3}})}{5.06-2(1-\varphi^{\frac{2}{3}})\times0.801}}$$

$$= \sqrt{1-\frac{2(1-\varphi^{\frac{2}{3}})}{5.06-1.602(1-\varphi^{\frac{2}{3}})}} \tag{5-5-19}$$

设不同的 φ 值，列表计算，结果见表 5-2。

表 5-2　　例 5-2 中 y_1，y_2 的计算结果

φ	$\varphi^{\frac{2}{3}}$	$1-\varphi^{\frac{2}{3}}$	Y_B	Z_B	ω_{yx^-}	ω_{xy^+}	$\omega^*_{yx^-}$ T_K	$\omega^*_{xy^+}$ T_K	$Hu^*_{yx^-}$ $/10^{-3}T_K$	$Bu^*_{xy^+}$ $/10^{-3}T_K$	y_2 $/10^{-2}T_K$	y_1 $/10^{-2}T_K$
0.01	0.046	0.954	0.717	0.678	156	148	0.017	0.015	0.723	0.713	0.451	0.097
0.05	0.136	0.864	0.756	0.728	192	182	0.025	0.023	1.063	1.093	0.679	0.485
0.10	0.215	0.758	0.817	0.778	226	232	0.035	0.037	1.488	1.758	1.020	0.979
0.15	0.280	0.720	0.825	0.795	272	258	0.051	0.046	2.168	2.185	1.368	1.450
0.20	0.342	0.658	0.848	0.819	317	293	0.069	0.059	2.933	2.803	1.802	1.940
0.25	0.397	0.603	0.865	0.840	363	335	0.090	0.077	3.825	3.658	2.351	2.420
0.30	0.448	0.552	0.881	0.858	415	382	0.118	0.100	5.015	4.750	3.068	2.910

将表 5-2 中 y_1 与 $y_2(T=1\text{s})$绘成图 5-7，发现有两个交点，不合理；另选 $T_K=0.9\text{s}$，结果列于表 5-3，并绘在图 5-7 中，有一个交点，$\varphi_K=0.086$，$T_K=0.9\text{s}$。

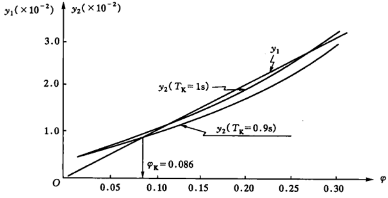


图 5-7　例 5-2 计算结果($T_K=0.9\text{s}$，$\varphi_K=0.086$)

表 5-3

φ	0.01	0.05	0.10	0.15	0.20	0.25	0.30
$y_2(T_K=0.9s)$	0.40	0.611	0.910	1.224	1.626	2.116	2.76

断面速度分布公式

$$u=12.974\times\frac{(1-Z^2)(1-Y^2)}{0.801(1-Z^2)+(1-Y^2)}$$

确定管底边层流位置，由式(5-5-19)

$$Z_B=\sqrt{1-\frac{2(1-0.086^{\frac{2}{3}})}{5.06-2(1-0.086^{\frac{2}{3}})0.801}}=\sqrt{1-\frac{2\times0.805}{5.06-1.602\times0.801}}=\sqrt{1-\frac{1.61}{5.06-1.2896}}$$

$$=\sqrt{1-\frac{1.61}{3.77}}=\sqrt{0.427}=0.653$$

$$z_B=Z_BH=0.653\times0.0425=0.0278\text{m}$$

管底边层流厚度

$$\delta=H-z_B=0.0425-0.0278=0.0147\text{m}=14.7\text{mm}$$

确定管侧壁边层流位置，由式(5-5-18)

$$Y_B=\sqrt{1-\frac{1.602(1-0.086^{\frac{2}{3}})}{5.06-2(1-0.086^{\frac{2}{3}})}}=\sqrt{1-\frac{1.29}{5.06-1.61}}=\sqrt{1-\frac{1.29}{3.45}}$$

$$=\sqrt{1-0.374}=\sqrt{0.626}=0.791$$

$$y_B=Y_BB=0.791\times0.0475=0.0376\text{m}$$

管侧壁边层流厚度

$$\delta=B-y_B=0.0475-0.0376=0.0099\text{m}\approx10\text{mm}$$

确定管侧壁边层流界面产生的代表性的涡旋强度与大小。由式(5-5-16)

$$\omega_{yz^-}=\frac{K_1v_0Y_B}{2B(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1-Y_B^2\right]}=\frac{5.06\times5\times0.791}{2\times0.0475(1-0.086^{\frac{2}{3}})(0.801+1-0.791^2)}$$

$$=\frac{2.0012}{0.095\times0.805\times1.175}=\frac{20.012}{0.0899}=222.603\text{rad/s}$$

$$d_{syz^-}=2r_s=2\sqrt{\frac{10\nu}{\omega_{yz}^-}}=2\sqrt{\frac{10\times1.141\times10^{-6}}{222.603}}=\frac{2}{10^4}\sqrt{\frac{1.141}{0.223}}$$

$$=\frac{2}{10^4}\sqrt{5.117}=\frac{2\times2.262}{10^4}=\frac{4.524}{10^4}=0.00045\text{m}=0.45\text{mm}$$

确定管底边层流界面上产生的代表性的涡旋强度与大小。由式(5-5-17)

$$\omega_{zy^+}=\frac{25.3Z_B}{0.085(1-\varphi^{\frac{2}{3}})[0.801(1-Z_B^2)+1]}=\frac{25.3\times0.653}{0.085(1-0.086^{\frac{2}{3}})[0.801(1-0.653)^2+1]}$$

$$=\frac{16.521}{0.085\times0.805(0.801\times0.574+1)}=\frac{16.521}{0.068\times1.46}=\frac{16.521}{0.0999}=165.375\text{rad/s}$$

涡旋直径

$$d_s=\frac{2}{10^4}\sqrt{\frac{1.141}{0.165}}=\frac{2}{10^4}\sqrt{7.869}=\frac{2\times 2.805}{10^4}=5.61\times 10^4=0.00056\text{m}=0.56\text{mm}$$

确定管侧壁边层流界面上涡旋横向分速度 $u^*_{yz^-}$ 的大小。由式(5-5-15)

$$u^*_{yz^-}=0.684\left(\omega_{yz^-}\Big|_{\substack{y=0\\ z=z_B}}\right)^2\times 10^{-6}T_K=0.684\times 222.603^2\times 10^{-6}\times 0.9$$

$$=0.684\times 49552.096\times 0.9\times 10^{-6}=30504.27\times 10^{-6}\approx 0.031\text{m/s}$$

确定 ω_{yz^-} 在 y 方向的运动范围 s^*_y。由式(5-3-2)

$$s^*_y=u^*_{yz^-}\left(\sum_{t=0}^{n}e^{-\frac{t}{T_K}}\right)\approx 0.031\times\left(1+e^{-\frac{1}{0.9}}+e^{-\frac{2}{0.9}}\right)=0.031(1+0.329+0.108)$$

$$=0.03\times 1.437=0.0445\text{m}<0.0475\text{m}$$

确定管底边层流界面上涡旋横向运动分速度 $u^*_{zy^+}$。由式(5-5-14)

$$u^*_{zy^+}=0.684\times\omega^2_{zy^+}\times 10^{-6}T_K=0.684\times 165.375^2\times 10^{-6}\times 0.9$$

$$=0.684\times 27348.89\times 10^{-6}\times 0.9=0.0295\text{m/s}$$

确定 ω_{zy^+} 在 z 方向的运动范围 s^*_z。由式(5-3-2)

$$s^*_z=u^*_{zy^+}\left(\sum_{t=0}^{n}e^{-\frac{t}{T_K}}\right)\approx 0.0295\left(1+e^{-1.11}+e^{-2.22}\right)=0.0295\times 1.437$$

$$=0.0435\text{m}>0.0425\text{m}$$

计算结果表明，在 y 方向中心未达到湍流，但若 $t=3\text{s}$ 时，可以达到。在 z 方向 $t=2\text{s}$ 时，湍流达到表面。综合起来，整个管内已达到湍流状态。湍流度由式(5-3-3)

$$\alpha^*_y=\frac{s^*_y}{B}=\frac{0.0445}{0.0475}=0.894$$

$$\alpha^*_z=\frac{s^*_z}{H}=\frac{0.0435}{0.0425}=1.023$$

综合起来，湍流度近似为 1。

第6章 明渠湍流

6.1 明渠湍流控制方程

这里谈的明渠湍流，与第5章所谈的有压管道湍流一样，断面充满的是湍流，没有理想流体运动存在。这里只讨论矩形明渠湍流，若要讨论层流湍流，只需将公式中的涡旋体积分数 φ 取为0即可。为保证整个断面均充满湍流，对其断面尺寸要作一些要求：一种矩形是水面很宽，但水层很浅；另一种是水很深，水面很窄。如图6-1所示。

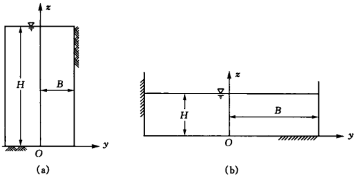


图6-1 矩形明渠断面要求示意图

明渠湍流运动控制微分方程由式(4-4-11)并结合明渠特点，应写成

$$g\sin\alpha=(1-\varphi^{\frac{2}{3}})\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+6\nu T\varphi^{\frac{2}{3}}\left(\frac{\partial u}{\partial y}\cdot\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial z}\cdot\frac{\partial^2 u}{\partial z^2}\right) \tag{6-1-1}$$

结合矩形明渠将其无因次化，为此令

$$\frac{y}{B}=Y,\quad \frac{z}{H}=Z,\quad \frac{u}{v_0}=U$$

并将它们代入式(6-1-1)，简化为

$$K_1=(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right]+\varphi^{\frac{2}{3}}\left[K_2\left(\frac{H}{B}\right)^2\frac{\partial U}{\partial Y}\left(\frac{\partial^2 U}{\partial Y^2}\right)+K_3\left(\frac{\partial U}{\partial Z}\right)\frac{\partial^2 U}{\partial Z^2}\right] \tag{6-1-2}$$

式中：

$$K_1=\frac{g\sin\alpha H^2}{\nu v_0} \tag{6-1-3}$$

$$K_2=\frac{6v_0 T}{B} \tag{6-1-4}$$

$$K_3=\frac{6v_0 T}{H} \tag{6-1-5}$$

6.2　边界条件分析

流体运动轨迹或形状，是由边界形状而定的。渠道弯曲，水流也弯曲，矩形断面边界对流体速度分布影响如图 6-2 所示，由此得出边界条件为

$$\left.\frac{\partial U}{\partial Z}\right|_{\substack{Z=1\\Y=0}}=0 \tag{6-2-1}$$

$$\left.\frac{\partial U}{\partial Y}\right|_{\substack{Z=Z_i\\Y=0}}=0 \tag{6-2-2}$$

$$U|_{Z=0}=0 \tag{6-2-3}$$

$$U|_{Y=0}=0 \tag{6-2-4}$$

$$\frac{\partial^2 U}{\partial Y^2}<0 \tag{6-2-5}$$

$$\frac{\partial^2 U}{\partial Z^2}<0 \tag{6-2-6}$$

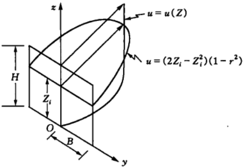


图 6-2　湍流矩形明渠边界影响示意图

6.3　确定无因次速度分布

根据边界条件，选择形函数为

$$U'=U_1(Z)U_2(Y)=(2Z-Z^2)(1-Y^2) \tag{6-3-1}$$

待定速度 U 为

$$U=\eta U'=\eta(2Z-Z^2)(1-Y^2) \tag{6-3-2}$$

确定参变常数 η，为此将式(6-3-2)代入式(6-1-2)，整理结果为

$$K_1=2(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(2Z-Z^2)+(1-Y^2)\right]\eta$$
$$+4\varphi^{\frac{2}{3}}\left[K_2\left(\frac{H}{B}\right)^2(2Z-Z^2)Y+K_2(1-Z)(1-Y^2)^2\right]\eta^2 \tag{6-3-3}$$

取 η 的近似表达式

$$\eta=\frac{K_1}{2(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(2Z-Z^2)+(1-Y^2)\right]} \tag{6-3-4}$$

无因次速度分布

$$U=\frac{K_1(2Z-Z^2)(1-Y^2)}{2(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(2Z-Z^2)+(1-Y^2)\right]} \tag{6-3-5}$$

6.4　边层流界面位置确定公式

确定边层流界面位置，分侧壁与渠底两个方面。如图 6-3 所示。

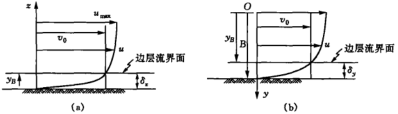


图 6-3　边层流位置与厚度示意图

侧壁边层流界面位置 Y_B，依式(6-3-5)，令 $U=1$，$Y=Y_B$，则有

$$2(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(2Z-Z^2)+(1-Y_B^2)\right]-K_1(2Z-Z^2)(1-Y_B^2)=0$$

化简，解出

$$Y_B=\sqrt{\frac{\left[2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2-K_1\right](2Z-Z^2)+2(1-\varphi^{\frac{2}{3}})}{2(1-\varphi^{\frac{2}{3}})-K_1(2Z-Z^2)}} \tag{6-4-1}$$

式(6-4-1)表明，侧壁边层流位置是 Z 的函数。为计算当量涡旋体积分数 φ，特选择 Z 的一个代表点 Z_K，下面利用积分找到 Z_K。

$$\int_0^1(2Z-Z^2)\mathrm{d}Z=\frac{2}{3} \tag{6-4-2}$$

利用积分中值定理，有

$$1\cdot(2Z_Z-Z_K^2)=\frac{2}{3} \tag{6-4-3}$$

由式(6-4-3)得 $Z_K=0.423$，将 Z_K 代入式(6-4-1)，得

$$Y_B=\sqrt{\frac{\left[2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2-K_1\right]0.667+2(1-\varphi^{\frac{2}{3}})}{2(1-\varphi^{\frac{2}{3}})-0.667K_1}} \tag{6-4-4}$$

渠底边层流界面位置 Z_B，仍利用式(6-3-5)，$U=1$，$Z=Z_B$，则有

$$1=\frac{K_1(2Z_B-Z_B^2)(1-Y^2)}{2(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(2Z_B-2Z_B^2)+(1-Y^2)\right]}$$

展开，得

$$2(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(2Z_B-Z_B^2)+(1-Y^2)\right]$$

$$=K_1(2Z_B-2Z_B^2)(1-Y^2)\left[2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2-K_1(1-Y^2)\right](2Z_B-Z_B^2)+2(1-\varphi^{\frac{2}{3}})(1-Y^2)$$

$$=0$$

$$2Z_B-Z_B^2+\frac{2(1-\varphi^{\frac{2}{3}})(1-Y^2)}{2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2-K_1(1-Y^2)}=0$$

$$Z_B^2-2Z_B-\frac{2(1-\varphi^{\frac{2}{3}})(1-Y^2)}{2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2-K_1(1-Y^2)}=0$$

则 Z_B 为

$$Z_B=\frac{2\pm\sqrt{Z^2-4\dfrac{2(1-\varphi^{\frac{2}{3}})(1-Y^2)}{K_1(1-Y^2)-2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2}}}{2}$$

化简，得

$$Z_B=1-\sqrt{1-\frac{2(1-\varphi^{\frac{2}{3}})(1-Y^2)}{K_1(1-Y^2)-2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2}} \tag{6-4-5}$$

近似解

$$Z_B=\frac{(1-\varphi^{\frac{2}{3}})(1-Y^2)}{K_1(1-Y^2)-2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2} \tag{6-4-6}$$

式(6-4-5)与式(6-4-6)中，Z_B 均是 Y 的函数。同理，找到一个 Y 变化的代表点 Y_K，为此利用积分中值定理

$$\int_0^1(1-Y^2)\mathrm{d}Y=\frac{2}{3} \tag{6-4-7}$$

$$1(1-Y_K^2)=\frac{2}{3} \tag{6-4-8}$$

由式(6-4-8)解得 $Y_K=0.577$，将 Y_K 代入式(6-4-5)与式(6-4-6)，则有

$$Z_B=1-\sqrt{1-\frac{1.334(1-\varphi^{\frac{2}{3}})}{K_1\times 1.334-2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2}} \tag{6-4-9}$$

$$Z_B=\frac{1.334(1-\varphi^{\frac{2}{3}})}{1.334K_1-2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2} \tag{6-4-10}$$

6.5 确定侧壁边层流界面上产生的涡旋强度

由式(6-3-5)计算无因次涡旋强度 $\overline{\omega}$：

$$\overline{\omega}\,|_{YZ^-}=\frac{K_1(2Z-Z^2)Y_B}{2(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(2Z-Z^2)+(1-Y^2)\right]}=\frac{0.423K_1Y_B}{2(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2 0.423+1-Y_B^2\right]}$$

$$=\frac{0.423K_1Y_B}{2(1-\varphi^{\frac{2}{3}})\left[0.423\left(\frac{H}{B}\right)^2+1-Y_B^2\right]} \tag{6-5-1}$$

确定渠底边界流界面上产生的涡旋强度。由式(6-3-5)

$$\overline{\omega}_{ZY^+}=\frac{1}{2}\left(\frac{\partial U}{\partial Z}\right)\Big|_{Z=Z_B}=\eta(1-Z_B)(1-Y^2)=\frac{K_1(1-Z_B)(1-Y^2)}{2(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(2Z_B-Z_B^2)+(1-Y^2)\right]}$$

$$=\frac{0.667K_1(1-Z_B)}{2(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(2Z_B-Z_B^2)+0.667\right]} \tag{6-5-2}$$

将无因次涡旋转为有因次涡旋

$$\overline{\omega}_{ZY^+}=\frac{1}{2}\left(\frac{\partial U}{\partial Z}\right)=\frac{1}{2}\frac{H\partial u}{v_0\partial z}=\frac{H}{v_0}\omega_{zy^+} \tag{6-5-3}$$

$$\overline{\omega}\,|_{YZ^-}=\frac{1}{2}\frac{\partial U}{\partial Y}=\frac{1}{2}\frac{B}{v_0}\frac{\partial u}{\partial y}=\frac{B}{v_0}\omega_{yz^-} \tag{6-5-4}$$

将式(6-5-4)代入式(6-5-1)，则有

$$\omega\,|_{yz^-}=\frac{0.423g\sin\alpha\left(\frac{H}{B}\right)HY_B}{2\mu(1-\varphi^{\frac{2}{3}})\left[0.423\left(\frac{H}{B}\right)^2+1-Y_B^2\right]} \tag{6-5-5}$$

将式(6-5-3)代入式(6-5-2)

$$\omega\,|_{zy^+}=\frac{0.667g\sin\alpha H(1-Z_B)}{2\mu(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(2Z_B-Z_B^2)+0.667\right]} \tag{6-5-6}$$

式(6-5-5)是侧壁边层流界面产生的涡旋强度表达式，其直径为

$$d_{syz^-}=2\sqrt{\frac{10\nu}{\omega_{yz^-}\big|_{y=y_B}}} \tag{6-5-7}$$

式(6-5-6)是渠底边层流界面上产生的涡旋强度表达式，其直径为

$$d_{szy^+}=2\sqrt{\frac{10\nu}{\omega_{zy^+}\big|_{z=z_B}}} \tag{6-5-8}$$

侧壁边层流界面上涡旋横向运动速度 $u^*_{yz^-}$，由式(4-3-28)

$$u^*_{yz^-}\Big|_{\substack{y=y_B\\z=z_K}}=12T_K\nu\left(\omega_{yz^-}\Big|_{\substack{y=y_B\\z=z_K}}\right)^2 v_0^{-1} \tag{6-5-9}$$

渠底边层流界面上涡旋横向运动速度 $u^*_{zy^+}$，由式(4-3-29)并结合边层流特点，简化为

$$u^*_{zy^+}\Big|_{\substack{z=z_B\\y=y_K}}=12T_K\nu\left(\omega_{zy^+}\Big|_{\substack{z=z_B\\y=y_K}}\right)^2 v_0^{-1} \tag{6-5-10}$$

6.6 确定涡旋体积分数 φ 计算公式

结合矩形明渠特点，其涡旋体积分数定义为

$$\varphi=\frac{\text{涡旋体积}}{\text{流量}}=\frac{\frac{\pi}{6}\left(n_y f_y d^3_{syz^-}+n_z f_z d^3_{szy^+}\right)}{2HBv_0} \tag{6-6-1}$$

根据式(6-5-1)与式(6-5-2)

$$f_y=\frac{u^*_{yz^-}\Big|_{\substack{y=y_B\\z=z_K}}}{2d_{yz^-}} \tag{6-6-2}$$

$$f_z=\frac{u^*_{zy^+}\Big|_{\substack{z=z_B\\y=y_K}}}{2d_{szy^+}} \tag{6-6-3}$$

$$n_y=\frac{2Hv_0}{\pi d^2_{syz^-}} \tag{6-6-4}$$

$$n_z=\frac{2Bv_0}{\pi d^2_{szy^+}} \tag{6-6-5}$$

将以上四式代入式(6-6-1)

$$\varphi=\frac{\left(Hu^*_{yz^-}\Big|_{\substack{y=y_B\\z=z_K}}+Bu^*_{zy^+}\Big|_{\substack{z=z_B\\y=y_K}}\right)}{6HB} \tag{6-6-6}$$

将式(6-5-9)与式(6-5-10)代入式(6-6-6)

$$\varphi=\frac{12T_K\nu\left[\left(\omega_{yz^-}\Big|_{\substack{y=y_B\\z=z_K}}\right)^2H+\left(\omega_{zy^+}\Big|_{\substack{z=z_B\\y=y_K}}\right)^2B\right]}{6HBv_0} \tag{6-6-7}$$

式(6-6-7)就是矩形湍流明渠涡旋体积分数 φ 的定义公式。

例 6-1　一矩形湍流明渠如图 6-1(b)所示，渠底坡 $i=0.0001$，$B=3\text{m}$，$H=0.4\text{m}$，水动力黏性系数 $\mu=1.14\times10^{-3}\text{N·s/m}^2$，水运动黏性系数 $\nu=1.141\times10^{-6}\text{m}^2/\text{s}$，要求确定涡旋体积分数 φ，准定常流时间 T_K，渠底与侧壁边层流界面位置，产生涡旋的强度与大小，涡旋在边层流界面上横向运动速度。(已知 $\lambda=0.022$)

解　计算谢才系数

$$c=\sqrt{\frac{8g}{\lambda}}=\sqrt{\frac{8\times9.81}{0.022}}=59.28\text{m}^{\frac{1}{2}}/\text{s}$$

水力半径

$$R=\frac{HB}{H+B}=\frac{0.4\times3}{0.4+3}=0.353\text{m}$$

断面平均流速

$$v_0=c\sqrt{Ri}=59.28\sqrt{0.353\times0.0001}=0.352\text{m/s}$$

宽高比

$$\left(\frac{H}{B}\right)^2=\left(\frac{0.4}{3}\right)^2=0.018$$

无因次常数 K_1

$$K_1=\frac{g\sin\alpha H^2}{\mu v_0}=\frac{1000\times0.0001\times0.4^2}{1.14\times10^{-3}\times0.353}=40$$

由式(6-6-7)

$$y_1=6HBv_0\varphi=6\times0.4\times3\times0.352\varphi=2.534\varphi \tag{a}$$

$$y_2=[\text{秒}]12T_K\nu\left(H\omega_{yz}^2{}^-+B\omega_{zy}^2{}^+\right) \tag{b}$$

首先由式(a)和式(b)确定涡旋体积分数 φ_K 与准定常时间 T_K。

设不同的 φ 值，计算 y_1，例如 $\varphi=0.01$，代入式(a)，$y\approx0.101$，其他结果列入表 6-1 中。

表 6-1　　例 6-1 中 y_1，y_2 的计算结果

y_1	φ	$1-\varphi^{\frac{2}{3}}$	Y_B	Z_B	$\omega_{yz}{}^-$	$\omega_{zy}{}^+$	y_2 ($T_K=1\text{s}$)
0.025	0.01	0.954	0.9735	0.012	30	18	0.018
0.127	0.05	0.864	0.9760	0.110	20	20	0.018
0.25	0.10	0.758	0.9781	0.010	25	22	0.032
0.38	0.15	0.720	0.9804	0.009	29	24	0.035
0.50	0.20	0.658	0.9822	0.008	34	26	0.049

计算 y_2 值比较麻烦，需分几步计算，计算 Y_B，由式(6-4-4)

$$Y_B=\sqrt{\frac{\left[2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2-K_1\right]0.667+2(1-\varphi^{\frac{2}{3}})}{2(1-\varphi^{\frac{2}{3}})-0.667K_1}}=\sqrt{\frac{\left[2(1-\varphi^{\frac{2}{3}})0.018-40\right]0.667+2(1-\varphi^{\frac{2}{3}})}{2(1-\varphi^{\frac{2}{3}})-0.667\times40}}$$

$$=\sqrt{\frac{[2\times0.954\times0.018-40]0.667+2\times0.954}{2(1-\varphi^{\frac{2}{3}})-26.68}}=\sqrt{\frac{-23.477}{-24.772}}=\sqrt{0.9477}=0.9735$$

其他不同 φ 的值计算的 Y_B，均列入表 6-1 中。

同样计算 Z_B，由式(6-4-9)

$$Z_B=1-\sqrt{1-\frac{1.334(1-\varphi^{\frac{2}{3}})}{1.334K_1-2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2}}=1-\sqrt{1-\frac{1.334\times0.954}{40\times1.334-2\times0.954\times0.018}}$$

$$=1-\sqrt{1-\frac{1.2726}{53.36-0.0343}}=1-\sqrt{1-\frac{1.2726}{53.325}}=1-\sqrt{0.976}$$

$$=1-0.988=0.012$$

设其他 φ 值同样计算，其结果列入表 6-1 中。

计算侧壁边层流界面上产生的涡旋强度 ω_{yz^-}，由式(6-5-5)，结合具体问题，写成

$$\omega_{yz^-}\Big|_{Y=Y_B}=\frac{0.423g\sin\alpha\left(\frac{H}{B}\right)HY_B}{2\mu(1-\varphi^{\frac{2}{3}})\left[0.423\left(\frac{H}{B}\right)^2+1-Y_B^2\right]}=\frac{0.423\times1000\times0.0001\left(\frac{0.4}{3}\right)0.4Y_B}{2\times1.14\times10^{-3}(1-\varphi^{\frac{2}{3}})(0.423\times0.018+1-Y_B^2)}$$

$$=\frac{0.002256\times10^3Y_B}{2.28(1-\varphi^{\frac{2}{3}})(1.0076-Y_B^2)}=\frac{2.256Y_B}{2.28(1-\varphi^{\frac{2}{3}})(1.0076-Y_B^2)}\tag{6-6-8}$$

将 $1-\varphi^{\frac{2}{3}}=0.954$，$Y_B=0.9735$ 代入式(6-6-8)，有

$$\omega_{yz^-}\Big|_{Y=Y_B=0.9735}=\frac{2.256\times0.9735}{2.28\times0.954(1.0076-0.9735^2)}=\frac{2.256\times0.9735}{2.175\times0.0332}$$

$$=\frac{2.196}{0.072}=30\text{rad/s}$$

设不同的 φ 值与不同的 Y_B，计算 ω_{yz^-} 的值列入表 6-1 中。

计算渠底边层流界面上产生的涡旋强度 ω_{zy^+}，由式(6-5-6)

$$\omega_{zy^+}=\frac{0.667g\sin\alpha H(1-Z_B)}{2\mu(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(2Z_B-Z_B^2)+0.667\right]}$$

$$=\frac{0.667\times1000\times0.0001\times0.4(1-Z_B)}{2\times1.14\times10^{-3}(1-\varphi^{\frac{2}{3}})[0.018(2Z_B-Z_B^2)+0.667]}$$

$$=\frac{0.0266\times10^3(1-Z_B)}{2.28(1-\varphi^{\frac{2}{3}})[0.018(2Z_B-Z_B^2)+0.667]}$$

$$=\frac{26.68(1-0.012)}{2.28\times0.954(0.018\times2\times0.012-0.012^2+0.667)}=\frac{26.68\times0.988}{2.182\times0.6674}$$

$$=\frac{26.3598}{1.4563}=18.10\text{rad/s}$$

同理，计算结果列入表 6-1 中。

计算 y_2，由式(b)

$$y_2=[秒]12\times1.141\times10^{-6}T_K(0.4\omega_{yz^-}^2+B\omega_{zy^+}^2)$$
$$=13.692\times10^{-6}(0.4\times30^2+3\times18^2)T_K=0.0182T_K$$

其他计算列入表 6-1 中。

计算 y_1，由式(a)

$$y_1=2.534\varphi=2.534\times0.01=0.025$$

其他计算列入表 6-1 中。

将 y_1 与 y_2 绘在图 6-4 上，交点 $\varphi_K=0.01$，$T_K=1\mathrm{s}$。

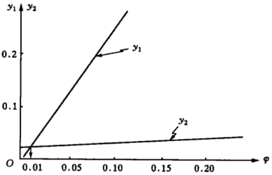


图 6-4　例 6-1 图

有了 φ_K，T_K，就可以确定有关的计算，首先确定 Y_B 的位置，由式(6-4-4)

$$Y_B=\sqrt{\frac{[2(1-0.01^{\frac{2}{3}})0.018-40]0.667}{2(1-0.01^{\frac{2}{3}})}}=0.9735$$
$$y_B=0.9735\times3=2.9205\mathrm{m}$$

侧壁边层流代表厚度

$$B-y_B=3-2.9205=0.0795\mathrm{m}\approx8\mathrm{cm}$$

渠底边层流位置已在表 6-1 找到，$\varphi_K=0.01$ 时，$Z_B=0.012$，$z_B=0.012\mathrm{m}$(厚度)。

渠底与侧壁边层流界面上产生的涡旋强度对应于 $\varphi_K=0.01$ 时，$\omega_{yz^-}=30\mathrm{rad/s}$，$\omega_{zy^+}=18\mathrm{rad/s}$，很小，其对应的直径

$$d_{syz^-}=2\sqrt{\frac{10\nu}{30}}=2\sqrt{\frac{1.141}{30\times10^5}}=\frac{2}{10^2}\sqrt{\frac{1.141}{300}}=\frac{2}{10^2}\sqrt{0.0038}$$
$$=\frac{2\times0.06}{10^2}=\frac{0.12}{10^2}=0.0012\mathrm{m}=1.2\mathrm{mm}$$
$$d_{szy^+}=\frac{2}{10^2}\sqrt{\frac{1.141}{180}}=\frac{2}{10^2}\sqrt{0.0863}=\frac{2\times0.6796}{10^2}=\frac{0.159}{10^2}=0.0016\mathrm{m}=1.6\mathrm{mm}$$

计算侧壁边层流界面上涡旋横向运动速度

$$u_{yz^-}^*\Big|_{\substack{y=y_B\\z=z_K}}=12T_K\nu\omega_{yz^-}^2v_0^{-1}=\frac{12\times1\times1.141\times10^{-6}\times30^2}{0.352}=0.035\mathrm{m/s}$$

计算渠底边层流界面涡旋横向运动速度

$$u_{zy}^{*+}\Big|_{\substack{z=z_B\\ y=y_K}}=12T_K\nu\omega_{zy^+}^2v_0^{-1}=\frac{12\times1\times1.141\times10^{-6}\times18^2}{0.352}=0.0126\text{m/s}$$

例 6-2 湍流矩形明渠如图 6-1(a)所示，底坡 $i=0.001$，$B=0.2\text{m}$，$H=3\text{m}$，水动力黏性系数 $\mu=1.14\times10^{-3}\text{N}\cdot\text{s/m}^2$，水运动黏性系数 $\nu=1.141\text{m}^2/\text{s}$，要求确定涡旋体积分数 φ_K、准定常时间 T_K、渠底与侧壁边层流界面位置、产生涡旋强度及大小、涡旋在边层流界面上横向运动速度。(已知 $\lambda=0.022$)

解 计算前准备工作：谢才系数 $c=59.28\text{m}^{\frac{1}{2}}/\text{s}$，水力半径

$$R=\frac{HB}{2\times H+B}=\frac{0.4\times3}{2\times3+0.4}=0.188\text{m}$$

断面平均速度

$$v_0=59.28\sqrt{Ri}=59.28\sqrt{0.188\times0.001}=0.812\text{m/s}$$

无因次常数 K_1

$$K_1=\frac{\gamma\sin\alpha H^2}{\mu v_0}=\frac{1000\times0.001\times3^2\times10^3}{1.14\times0.812}=7919$$

$$\frac{H}{B}=\frac{3}{0.2}=15$$

首先确定侧壁边层流位置，由式(6-4-4)

$$Y_B=\sqrt{\frac{\left[2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2-K_1\right]0.667+2(1-\varphi^{\frac{2}{3}})}{2(1-\varphi^{\frac{2}{3}})-0.667K_1}}$$

$$=\sqrt{\frac{[2(1-\varphi^{\frac{2}{3}})\times15^2-9719]0.667+2(1-\varphi^{\frac{2}{3}})}{2(1-\varphi^{\frac{2}{3}})-0.667\times9719}}$$

$$=\sqrt{\frac{(2\times0.954\times225-9719)0.667+2\times0.954}{2\times0.954-0.667\times9719}}=\sqrt{\frac{6221.708}{6480.84}}$$

$$=\sqrt{0.96}=0.978$$

$(1-\varphi^{\frac{2}{3}})$的计算结果列入表 6-2 中。

表 6-2 例 6-2 计算结果

y_1	φ	$1-\varphi^{\frac{2}{3}}$	Y_B	Z_B	ω_{yz}^-	ω_{xy}^+	y_2
0.029	0.01	0.954	0.978	0.000075	89.910	1306	$4.70T_K$
0.146	0.05	0.864	0.980	0.000070	99.550	1450	$6.305T_K$
0.292	0.10	0.758	0.983	0.000061	114.000	1668	$8.152T_K$
0.438	0.15	0.72	0.984	0.00006	116.000	1755	$8.905T_K$
0.584	0.20	0.658	0.997	0.00005	133.150	1933	$10.474T_K$

计算渠底边层流界面位置，由式(6-4-9)

$$Z_K=1-\sqrt{1-\frac{1.334(1-\varphi^{\frac{2}{3}})}{1.334K_1-2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2}}=1-\sqrt{1-\frac{1.334\times0.954}{1.334\times9719-2\times0.954\times225}}$$

$$=1-\sqrt{1-\frac{1.273}{12965-429}}=1-\sqrt{1-\frac{1.273}{12535}}=1-\sqrt{1-0.0001}$$

$$=1-\sqrt{0.9999}=1-0.99995=0.00005$$

其他$(1-\varphi^{\frac{2}{3}})$值的计算结果列入表 6-2 中。

确定侧壁边层流界面上产生的涡旋强度 ω_{yz^-}，由式(6-5-5)，结合例题，写成

$$\omega_{yz^-}\big|_{Y=Y_B}=\frac{0.423\gamma\sin\alpha\left(\frac{H}{B}\right)HY_B}{2\mu(1-\varphi^{\frac{2}{3}})\left[0.423\left(\frac{H}{B}\right)^2+1-Y_B^2\right]}$$

$$=\frac{0.423\times1000\times0.001\left(\frac{3}{0.2}\right)\times3\times0.978}{2\times1.14\times10^{-3}(1-\varphi^{\frac{2}{3}})(0.423\times225+1-0.978^2)}$$

$$=\frac{18616}{2.28\times95.22\times0.954}=\frac{18616}{207.105}=89.88\text{rad/s}$$

$(1-\varphi^{\frac{2}{3}})$为其他值时，计算结果列入表 6-2 中。

确定渠底边层流界面上产生涡旋强度 ω_{zy^+}，由式(6-5-6)，结合例题，写成

$$\omega_{zy^+}\big|_{Z=Z_B}=\frac{0.667\gamma\sin\alpha H(1-Z_B)}{2\mu(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(2Z_B-Z_B^2)+0.667\right]}$$

$$=\frac{0.667\times1000\times0.001\times3(1-0.000075)}{2\times1.14\times10^{-3}(1-\varphi^{\frac{2}{3}})[225(2\times0.000075)-(0.000075)^2+0.667]}$$

$$=\frac{1991}{2.28\times0.954\times0.704}=\frac{1991}{1.531}=1306\text{rad/s}$$

其他$(1-\varphi^{\frac{2}{3}})$值的计算结果列入表 6-2 中。

计算 y_1，由式(a)

$$y_1=6HBv_0\varphi=6\times0.2\times3\times0.812\varphi=2.923\varphi$$

当 $\varphi=0.01$ 时

$$y_1=2.923\times0.01=0.029$$

列入表 6-2 中，其他 φ 值计算结果均列入表 6-2 中。

$$y_2=[秒]12\times1.141\times10^{-6}T_K(H\omega_{yz^-}^2+B\omega_{zy^+}^2)=13.692\times10^{-6}T_K(3\omega_{yz^-}^2+0.2\omega_{zy^0}^2)$$

$$=13.692\times10^{-6}(3\times89.91^2+0.2\times1306^2)=4703913\times T_K\times10^{-6}=4.704T_K$$

将其他$(1-\varphi^{\frac{2}{3}})$值的计算结果列入表 6-2 中。

因为 y_1 与 y_2 必须相等，而 T_K 正好是它们的调整系数。实际问题中，y_1 与 y_2 只有一点相交，这就为调整 T_K 提供了方便。设 $T_K=0.03$，则 y_2 值见表 6-3。

表 6-3　y_1，y_2 计算结果

y_1	0.029	0.146	0.292	0.438	0.584
y_2	0.141	0.189	0.245	0.267	0.314
φ	0.01	0.05	0.10	0.15	0.20

将表 6-3 中 y_1 与 y_2 绘在图 6-5 上，其交点对应为 $\varphi_K = 0.076$，其他问题就可以迎刃而解。

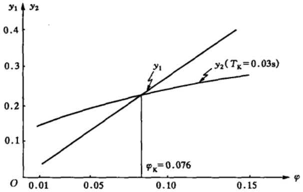


图 6-5　例 6-2 图

第一个问题，确定侧壁边层流界面位置与代表性厚度，由式(6-4-4)

$$Y_B = \sqrt{\frac{\left[2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2 - K_1\right]0.667 + 2(1-\varphi^{\frac{2}{3}})}{2(1-\varphi^{\frac{2}{3}}) - 0.667K_1}}$$

$$= \sqrt{\frac{\left[2(1-0.076^{\frac{2}{3}})\left(\frac{3}{0.2}\right)^2 - 9719\right]0.667 + 2(1-0.076^{\frac{2}{3}})}{2(1-0.076^{\frac{2}{3}}) - 0.667\times 9719}}$$

$$= \sqrt{\frac{(2\times 0.821\times 225 - 9719)0.667 + 2\times 0.821}{2\times 0.821 - 6482.573}} = \sqrt{\frac{6234.51}{6480.53}}$$

$$= \sqrt{0.962} = 0.981$$

$$y_B = 0.981\times 0.2 = 0.196\text{m}$$

$$\delta_y = 0.2 - 0.196 = 0.004\text{m} = 4\text{mm}$$

第二个问题，确定渠底边层流界面位置及其厚度，由式(6-4-9)

$$Z_B = 1 - \sqrt{1 - \frac{1.334(1-\varphi^{\frac{2}{3}})}{1.334K_1 - 2(1-\varphi^{\frac{2}{3}})\left(\frac{H}{B}\right)^2}} = 1 - \sqrt{1 - \frac{1.334\times(1-0.076^{\frac{2}{3}})}{1.334\times 9719 - 2(1-0.076^{\frac{2}{3}})\left(\frac{3}{0.2}\right)^2}}$$

$$= 1 - \sqrt{1 - \frac{1.334\times 0.821}{12965.14 - 2\times 0.821\times 225}} = 1 - \sqrt{1 - \frac{1.095}{12595.696}}$$

$$=1-\sqrt{1-0.000845}=1-\sqrt{0.99915}=1-0.99958=0.00042$$

$$z_B=0.00042\times 3=0.00136\text{m}=1.36\text{mm}$$

第三个问题，确定侧壁边层流界面产生的涡旋强度及大小，由式(6-5-6)

$$\omega_{zy^+}\Big|_{z=z_B}=\frac{0.667\gamma\sin\alpha H(1-Z_B)}{2\mu(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(2Z_B-Z_B^2)+0.667\right]}$$

$$=\frac{0.667\times 1000\times 0.001\times 3(1-0.00042)}{2\times 1.14\times 10^{-3}(1-0.076^{\frac{2}{3}})\left[\left(\frac{3}{0.2}\right)^2(2\times 0.00042)-0.00042^2+0.667\right]}$$

$$=\frac{0.667\times 3\times 0.99958\times 10^3}{2.28\times 0.821[225\times 0.00084+0.667]}=\frac{2000}{1.602}=1248\text{rad/s}$$

第四个问题，涡旋直径 d_s：

$$d_s=2r_s=2\sqrt{\frac{10\nu}{\omega_{zy^+}}}=\frac{2}{10^2}\sqrt{\frac{1.141}{12480}}=\frac{2}{10^2}\sqrt{0.000914}$$

$$=\frac{2\times 0.00956}{10^2}=\frac{0.019}{10^2}=0.00019\approx 0.0002\text{m}=0.2\text{mm}$$

第五个问题，侧壁边层流界面上涡旋横向运动速度，由式(6-5-9)，T_K 取 1s，则

$$u^*_{yz^-}\Big|_{\substack{y=y_B\\ z=z_K}}=12T_K\nu(\omega^2_{yz^-})v_0^{-1}=\frac{12\times 1\times 1.141\times 10^{-6}\times 105^2}{0.812}$$

$$=\frac{13.692\times 10^{-6}\times 11025}{0.812}=16.862\times 10^{-6}\times 11025$$

$$=185903\times 10^{-6}=0.186\text{m/s}$$

说明涡旋能达到渠中心线，全部为湍流。

第六个问题，渠底边层流界面上产生涡旋横向速度，由式(6-5-5)，T_K 取 0.03s，则

$$u^*_{zy^+}\Big|_{z=z_B}=12T_K\nu(\omega_{zy^+})^2v_0^{-1}=\frac{12\times 0.03\times 1.141\times 10^{-6}\times 1248^2}{0.812}$$

$$=\frac{0.411\times 10^{-6}\times 1557504}{0.812}=0.506\times 10^{-6}\times 1557504$$

$$=0.506\times 1.558=0.788\text{m/s}$$

讨论问题的指导思想，所求的湍流速度是假设涡旋分布是连续的，实际是不连续的，由此建立起的湍流速度分布是不合理的，因此得到的速度分布就不会符合实际情况。而我们关心的断面湍流速度分布是在边层流界面附近，在此处速度梯度分布是稳定的，是连续的，由此得到该层附近速度分布是符合实际的，从而得到边层流界面位置、厚度、涡旋强度及大小、横向运动速度均是正确的，而这些正是湍流所关心的问题，也是与其他学科相关的问题。

关于准定常流时间 T_K，它是一个调整参数，一旦 T_K 值一定，则它变动下限就限定了，上限不能超过 1s，这样在每个具体问题中进行调整，能使问题得到满足实际情况的解决。

第 7 章　不可压缩流体管道进口段

在定常流，忽略质量力条件下，研究层流与湍流在圆形与矩形管道进口段中的运动。

7.1　层流圆形管道进口段

坐标系及原点的选定如图 7-1 所示。

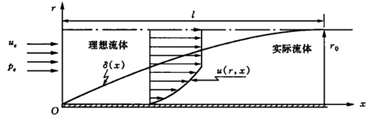


图 7-1　层流不可压缩圆形管进口段示意图

7.1.1　运动控制方程与边界条件

由式(4-1-10)，结合讨论的问题，可得

$$u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial r^2}\right)\tag{7-1-1}$$

$$u(x,\ r)\big|_{\substack{x=0\\r=0}}=u_e\tag{7-1-2}$$

$$u(x,\ r)\big|_{r=\delta(x)}=u(x)\tag{7-1-3}$$

$$u(x,\ r)\big|_{\substack{r=r_0\\x=l}}=u_{\max}\tag{7-1-4}$$

$$\left.\frac{\partial u(x,\ r)}{\partial r}\right|_{r=\delta(x)}=0\tag{7-1-5}$$

$$\frac{\partial^2 u(x,\ r)}{\partial r^2}<0\tag{7-1-6}$$

$$P(x,\ r)\big|_{x=0}=p_e\tag{7-1-7}$$

将方程与边界条件无因次化，为此取

$$\frac{u}{u_e}=U,\quad \frac{r}{\delta(x)}=R,\quad \frac{x}{r_0}=X,\quad \frac{\nu}{\nu_e}=\nu_0,\quad \frac{\rho}{\rho_e}=\rho_0,\quad \frac{p}{p_e}=P_0$$

式中：u_e——管道进口处速度；

$\delta(x)$——实际流体与理想流体分界线；

r_0——管道半径；

p_e——管道进口处流体压力；

ν_e——管道进口处流体运动黏性系数。

将以上比值代入式(7-1-1)，进行整理后，得

$$ReU\frac{\partial U}{\partial X}=-K_1\frac{\partial P_0}{\rho_0\partial X}+\nu_0\left[\frac{\partial^2 U}{\partial X^2}+\left(\frac{r_0}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial R^2}\right] \tag{7-1-8}$$

式中

$$Re=\frac{u_e r_0}{\nu_e} \tag{7-1-9}$$

$$K_1=\frac{p_e r_0}{\rho_e\nu_e u_e} \tag{7-1-10}$$

$$U(X,\ R)\big|_{\substack{X=0\\R=0}}=1 \tag{7-1-11}$$

$$U(X,\ R)\big|_{R=1}=U(X) \tag{7-1-12}$$

$$U(X,\ R)\big|_{\substack{R=1\\X=L}}=U_{\max} \tag{7-1-13}$$

$$\left.\frac{\partial U(X,\ R)}{\partial R}\right|_{R=1}=0 \tag{7-1-14}$$

$$\frac{\partial^2 U(X,\ R)}{\partial R^2}<0 \tag{7-1-15}$$

$$P_0(X,\ R)\big|_{X=0}=1 \tag{7-1-16}$$

7.1.2 初步确定进口段速度分布

理想流体运动微分方程为

$$u_0\frac{\mathrm{d}u_0}{\mathrm{d}x}=-\frac{\mathrm{d}p}{\rho\mathrm{d}x} \tag{7-1-17}$$

式中：u_0——理想流体运动速度。

将式(7-1-17)无因次化，则有

$$U_0\frac{\mathrm{d}U_0}{\mathrm{d}X}=-K_3\frac{\mathrm{d}P_0}{\rho_0\mathrm{d}X} \tag{7-1-18}$$

$$K_3=\frac{p_e}{u_e^2\rho_e} \tag{7-1-19}$$

将式(7-1-18)代入式(7-1-8)，则有

$$U\frac{\partial U}{\partial X}=\frac{K_1}{K_3}U_0\frac{\mathrm{d}U_0}{\mathrm{d}X}+\nu_0\left[\frac{\partial^2 U}{\partial X^2}+\left(\frac{r_0}{\delta}\right)^2\frac{\partial^2 U}{\partial R^2}\right] \tag{7-1-20}$$

根据边界条件，选择黏性流区速度分布为

$$U=X^{f(X)}(2R-R^2) \tag{7-1-21}$$

理想流体运动区速度分布选为

$$U_0 = \left(1 + \frac{X}{L}\right)^{f(X)} \tag{7-1-22}$$

$$U_0 = \frac{u(x)}{v_0} \tag{7-1-23}$$

$$U = \frac{u(x,\ r)}{u(x)} \tag{7-1-24}$$

将式(7-1-21)，式(7-1-22)代入式(7-1-20)，得

$$Re(2R - R^2)^2 X^{f(X)} f(X) X^{f(X)-1}$$
$$= \frac{K_1}{K_3}\left(1+\frac{X}{L}\right)^{f(X)} f(X)\left(1+\frac{X}{L}\right)^{f(X)-1} + \nu_0\left[(2R-R^2)f(X)(f(X)-1)X^{f(X)-2} - 2\left(\frac{r_0}{\delta}\right)^2 X^{f(X)}\right] \tag{7-1-21$'$}$$

式(7-1-21)′中，$\frac{K_1}{K_3} = Re$，$\nu_0 = 1$。当 $R = 1$ 时，则有

$$ReX^{2f(X)-1} f(X) = Ref(X)\left(1+\frac{X}{L}\right)^{2f(X)-1} + \left[(f(X)-1)f(X)X^{f(X)-2} - 2\left(\frac{r_0}{\delta}\right)^2 X^{f(X)}\right] \tag{7-1-25}$$

取对数，化简，得

$$Re(2f(X)-1) = \frac{Re(2f(X)-1)\ln\left(1+\frac{X}{L}\right)}{\ln X + f^2(X) - 3f(X) + 2 - 2\left(\frac{r_0}{\delta}\right)^2} \tag{7-1-26}$$

当 $X \to 0$ 时，$\ln X \to \infty$，$\delta(x) \to 0$，$\frac{r_0}{\delta(x)} \to \infty$ 没有物理意义，去掉两项。

$$f^2(X) - (2Re+3)f(X) + 2 + Re = 0 \tag{7-1-27}$$

近似确定 $f(X)$，可为

$$f(X) = \frac{Re+2}{2Re+3} = \frac{1+\frac{2}{Re}}{2+\frac{3}{Re}} \approx \frac{1}{2} \tag{7-1-28}$$

则黏性流区速度分布为

$$U = Z^{\left(\frac{Re+2}{2Re+3}\right)}(2R - R^2) \tag{7-1-29}$$

结合边界条件，当 $X = L$ 时，$U = 2$，则式(7-1-29)变为

$$U = 2\left(\frac{X}{L}\right)^{\left(\frac{Re+2}{2Re+3}\right)}(2R - R^2) \tag{7-1-30}$$

若取近似式，则有

$$U = 2\left(\frac{X}{L}\right)^{\frac{1}{2}}(2R - R^2) \tag{7-1-31}$$

理想流区速度分布

$$U_0=\left(1+\frac{X}{L}\right)^{\left(\frac{Re+2}{2Re+3}\right)} \tag{7-1-32}$$

结合边界条件，$X=L$，$U_0=2$，近似式为

$$U_0=\left(1+3\frac{X}{L}\right)^{\frac{1}{2}} \tag{7-1-33}$$

7.1.3 确定进口段长度

根据实际问题的要求，将式(7-1-20)写成

$$ReU\frac{\partial U}{\partial X}=ReU_0\frac{\mathrm{d}U_0}{\mathrm{d}X}+\nu_0\left[\frac{\partial^2 U}{\partial X^2}+\left|\left(\frac{r_0}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial R^2}\right|\right] \tag{7-1-21″}$$

将式(7-1-31)与式(7-1-33)代入式(7-1-21)″

$$2\frac{Re}{L}(2R-R^2)=1.5\frac{Re}{L}+\nu_0\left[-\frac{1}{2}\frac{2R-R^2}{L^2}\left(\frac{X}{L}\right)^{-\frac{3}{2}}+4\left(\frac{r_0}{\delta(x)}\right)^2\left(\frac{X}{L}\right)^{\frac{1}{2}}\right] \tag{A}$$

因为是求 L，所以 $R=1$，$\nu_0=1$，$X=L$。而 $X\to 0$ 时，$\left(\frac{X}{L}\right)^{-\frac{3}{2}}\to\infty$，故式（A）简化为

$$\frac{0.5Re}{L}=4 \tag{B}$$

$$L=0.125Re \tag{7-1-34}$$

7.1.4 确定理想与实际流体运动分界线 $\delta(x)$

由于是确定 $\delta(x)$，则一定取 $R=1$，则式(A)变为

$$\frac{0.5}{L}Re=4\left(\frac{r_0}{\delta(x)}\right)^2\left(\frac{X}{L}\right)^{\frac{1}{2}} \tag{C}$$

由于 $L=0.125Re$，代入式(C)，则

$$\delta(x)=r_0\sqrt{\left(\frac{X}{L}\right)^{\frac{1}{2}}}=r_0\sqrt[4]{\frac{X}{L}} \tag{7-1-35}$$

7.1.5 最终确定速度分布

实际流体运动区速度分布在分界线上应与理想流体运动区速度相等，也就是说边界条件式(7-1-12)还没有利用。

实际流体运动区速度(7-1-31)要与理想流体运动区速度在边界上相等，必须有一个调整系数，选其为 Re^{α}，依此

$$\left(1+3\frac{X}{L}\right)^{\frac{1}{2}}=Re^{\alpha}2\left(\frac{X}{L}\right)^{\frac{1}{2}} \tag{D}$$

将式(7-1-33)代入式(D)

$$\left[1+3\left(\frac{X}{0.125Re}\right)\right]^{\frac{1}{2}}=Re^{\alpha}2\left(\frac{X}{0.125Re}\right)^{\frac{1}{2}} \tag{7-1-36}$$

从式(7-1-36)可以看出，当 Re 一定时，不同 X 有不同 α，例如，当 $Re=1160$ 时，求出不同 X 所对应的不同 α，计算结果列入表 7-1。

表 7-1　　$Re=1160$，$L=145$ 时的 α 值

X	10	20	50	100	$L=145$
U_0	1.0986	1.1840	1.4264	1.7518	2
α	0.1044	0.0665	0.0275	0.0075	0

同理，当 Re 为已知则可以求出 α 值，则说明实际流体速度为

$$U=2Re^{\alpha}\left(\frac{X}{L}\right)^{\frac{1}{2}}(2R-R^2) \tag{7-1-37}$$

理想流体运动区速度分布仍为式(7-1-33)。

7.1.6　压力变化公式

将理想流体速度式(7-1-33)代入式(7-1-18)，积分后，得

$$p=p_c\left(1-\frac{3}{2}X\right) \tag{7-1-38}$$

7.2　层流矩形管道进口段流体运动情况分析

管道进口段，是指流体运动进入管道后，受管道影响的范围逐渐增加，一直沿流程扩展到管道中心为止。这段长度称为进口段。段内中间是理想流体运动区，其外是实际流体运动区。图 7-2(a)示意矩形断面尺寸；图 7-2(b)示意管道进口段长 l，中心下半剖面表示理想与实际流体分界是空间曲面，其顶点就是矩形断面中心最大速度 u_{max}，到此管道流速得到充分发展；图 7-2(c)示出 $z=H$ 半平面上理想与实际流体运动分界 $\delta_y(x,\ y,\ H)$；图 7-2(d)示意 $y=0$，沿长度 l 理想与实际流体运动分界线 $\delta_z(x,\ 0,\ z)$；图 7-2(e)示意沿长度 l，当 $z=H$ 时平面上速度分布。

7.2.1　运动控制方程及边界条件

首先是运动物理方程及边界条件，它们均是根据物理定律推导出来的，是有因次的，为了进行分析，运用数学工具，必须将其转化为无因次方程与边界条件。

(1) 有因次控制方程与边界条件

理想流体运动区动量微分方程：

$$u_0\frac{du_0}{dx}=-\frac{1}{\rho}\frac{dp}{dx} \tag{7-2-1}$$

边界条件：

$$u_0(x)\big|_{x=0}=u_c \tag{7-2-2}$$

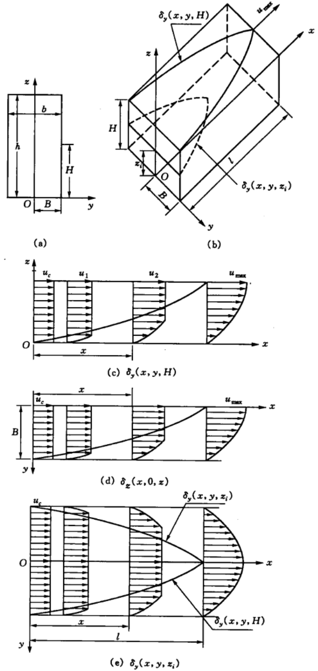


图 7-2　进口段示意图

$$u_0(x)\big|_{x=l,y=0,z=H}=u_{\max} \tag{7-2-3}$$

实际流体运动区动量微分方程：根据式(4-1-10)，结合本题

$$u\frac{\mathrm{d}u}{\mathrm{d}x}=-\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{7-2-4}$$

边界条件：

$$u(x,\ y,\ z)\big|_{x=0,y=0,z=0}=0 \tag{7-2-5}$$

$$u(x,\ y,\ z)\big|_{x=l,y=0,z=H}=u_{\max} \tag{7-2-6}$$

$$u(x,\ y,\ z)\big|_{y=\delta_y,z=\delta_z}=u(x) \tag{7-2-7}$$

$$u(x,\ y,\ z)\big|_{y=B}=0 \tag{7-2-8}$$

$$\left.\frac{\partial u(x,\ y,\ z)}{\partial y}\right|_{y=\delta_y}=0 \tag{7-2-9}$$

$$\frac{\partial^2 u(x,\ y,\ z)}{\partial y^2}<0 \tag{7-2-10}$$

$$\left.\frac{\partial u(x,\ y,\ z)}{\partial z}\right|_{z=\delta_z}=0 \tag{7-2-11}$$

$$\frac{\partial^2 u(x,\ y,\ z)}{\partial z^2}<0 \tag{7-2-12}$$

$$p(x,\ y,\ z)\big|_{z=0}=p_c \tag{7-2-13}$$

(2) 无因次控制方程与边界条件

取

$$\frac{u}{u_c}=U,\ \frac{u_0}{u_c}=U_0,\ \frac{x}{H}=X,\ \frac{B-y}{\delta_y}=Y,\ \frac{z}{\delta_z}=Z,\ \frac{p}{p_c}=P$$

将以上比值代入以上各微分方程与边界条件，理想流体运动区：

$$U_0\frac{\mathrm{d}U_0}{\mathrm{d}X}=-K_1\frac{\mathrm{d}P}{\mathrm{d}X} \tag{7-2-14}$$

$$K_1=\frac{p_c}{\rho_c u_c^2} \tag{7-2-15}$$

$$U_0(X)\big|_{X=0}=1 \tag{7-2-16}$$

$$U_0(X)\big|_{X=L,Y=1,Z=1}=U_{\max} \tag{7-2-17}$$

实际流体运动区：将有关比值代入式(7-2-4)

$$\frac{u_c^2}{H}U\frac{\partial U}{\partial X}=-\frac{p_c}{\rho_c H\rho_0}\frac{\mathrm{d}P}{\mathrm{d}X}+\frac{u_c\nu_c}{H^2}\nu_0\left(\frac{\partial^2 U}{\partial X^2}-\frac{H^2}{\delta y^2}\frac{\partial^2 U}{\partial Y^2}+\frac{H^2}{\delta z^2}\frac{\partial^2 U}{\partial Z^2}\right) \tag{7-2-4$'$}$$

全式除以$\dfrac{u_c\nu_c}{H^2}$，则有

$$ReU\frac{\partial U}{\partial X}=-K_2\frac{\mathrm{d}P}{\rho_0\mathrm{d}X}+\nu_0\left(\frac{\partial^2 U}{\partial X^2}-\frac{H^2}{\delta y^2}\frac{\partial^2 U}{\partial Y^2}+\frac{H^2}{\delta z^2}\frac{\partial^2 U}{\partial Z^2}\right) \tag{7-2-18}$$

$$K_2=\frac{p_c H}{\rho_c \nu_c u_c} \tag{7-2-19}$$

无因次边界条件：

$$U(X,\ Y,\ Z)\big|_{X=0,\ Y=\frac{B}{\delta_y},\ Z=0}=0 \tag{7-2-20}$$

$$U(X,\ Y,\ Z)\big|_{X=L,\ Y=\frac{B}{\delta_y},\ Z=1}=U_{\max} \tag{7-2-21}$$

$$U(X,\ Y,\ Z)\big|_{Y=1,\ Z=1}=U_0(X) \tag{7-2-22}$$

$$\left.\frac{\partial U}{\partial Y}\right|_{Y=1}=0 \tag{7-2-23}$$

$$\frac{\partial^2 U}{\partial Y^2}<0 \tag{7-2-24}$$

$$\left.\frac{\partial U}{\partial Z}\right|_{Z=1}=0 \tag{7-2-25}$$

$$\frac{\partial^2 U}{\partial Z^2}<0 \tag{7-2-26}$$

$$P\big|_{X=0}=1 \tag{7-2-27}$$

7.2.2 实际流体运动区速度分布初选

根据实际流体运动区流体运动形成的边界条件，初选对其适合的两个速度分布式：

$$U=X^{\frac{1}{2}}(2Y-Y^{\frac{3}{2}})(2Z-Z^{\frac{3}{2}}) \tag{7-2-28}$$

$$U=X(2Y-Y^2)(2Z-Z^2) \tag{7-2-29}$$

以上两式均满足实际流体运动区的边界条件。然而，确定该区内的速度分布，还必须通过运动控制方程来决定。如果将速度分布式代入方程后使其出现奇点，或者说出现不合理现象，则该速度分布应被选掉。

将式(7-2-28)代入式(7-2-18)，则有

$$\begin{aligned}&\frac{1}{2}Re(2Y-Y^{\frac{3}{2}})^2(2Z-Z^{\frac{3}{2}})^2\\&=-K_2\frac{\mathrm{d}P}{\rho_0\mathrm{d}X}+\nu_0\left[\frac{1}{2}X^{-\frac{3}{2}}(2Y-Y^{\frac{3}{2}})(2Z-Z^{\frac{3}{2}})+\frac{H^2}{\delta y^2}(2Z-Z^{\frac{3}{2}})Y^{-\frac{1}{2}}X^{-\frac{1}{2}}-\frac{H^2}{\delta z^2}(2Y-Y^{\frac{3}{2}})X^{-\frac{1}{2}}\right]\end{aligned} \tag{7-2-18$'$}$$

对式(7-2-18)′作定性分析，当 $X=0$，$Y=0$，$Z=0$ 时，均出现奇点，认为现象不合理，故该速度分布舍掉。

将式(7-2-29)代入动量方程式(7-2-18)，得

$$ReX(2Y-Y^2)^2(2Z-Z^2)^2=-K_2\frac{\mathrm{d}P}{\rho_0\mathrm{d}X}+2\nu_0X\left[(2Z-Z^2)\frac{H^2}{\delta y^2}-(2Y-Y^2)\frac{H^2}{\delta z^2}\right] \tag{7-2-30}$$

没有出现不合理现象，故选定实际流体运动速度分布为式(7-2-29)。

7.2.3　理想与实际流体分界线方程

利用流量不变，建立理想与实际流体分界线方程，矩形断面是对称的，取其四分之一作为研究对象。

进口流量

$$Q_c = u_c HB \tag{a}$$

理想流区

$$Q_{理} = u_0(H-\delta_z)(B-\delta_y) = u_0(HB - B\delta_z - H\delta_y + \delta_z\delta_y) \tag{b}$$

实际流区

$$Q_{实} = u_c X\iint(2Y-Y^2)(2Z-Z^2)\mathrm{d}y\mathrm{d}z = u_c X\delta_y\delta_z\int_0^1(2Y-Y^2)\mathrm{d}Y\int_0^1(2Z-Z^2)\mathrm{d}Z$$

$$= \frac{4}{9}u_c X\delta_y\delta_z \tag{c}$$

根据各断面通过的流量不变，有

$$Q_{进} = Q_{理} + Q_{实} \tag{d}$$

$$u_c HB = u_0(HB - B\delta_z - H\delta_y + \delta_z\delta_y) + \frac{4}{9}u_c X\delta_y\delta_z \tag{e}$$

无因次化，得

$$1 = U_0\left(1 - \frac{\delta_z}{H} - \frac{\delta_y}{B} + \frac{\delta_z}{H}\frac{\delta_y}{B}\right) + \frac{4}{9}X\frac{\delta_z}{H}\frac{\delta_y}{B} \tag{f}$$

因为在进口段末端中心速度最大 u_{max} 处，存在条件：

$$\frac{\delta_y}{B} = \frac{\delta_z}{H} \tag{g}$$

将关系式(g)代入式(f)

$$1 = U_0\left(1 - \frac{\delta_z}{H}\right)^2 + \frac{4}{9}X\left(\frac{\delta_z}{H}\right)^2 \tag{h}$$

$$U_0 = \frac{1 - \frac{4}{9}X\left(\frac{\delta_z}{H}\right)^2}{\left(1 - \frac{\delta_z}{H}\right)^2} \tag{7-2-31}$$

式(7-2-31)含有 δ_z，只要得到它，则 δ_y 也就可知，故称之为理想与实际流体分界线方程。

$$U_0\frac{\mathrm{d}U_0}{\mathrm{d}X} = \frac{-\left[1 - \frac{4}{9}X\left(\frac{\delta_z}{H}\right)^2\right]\frac{4}{9}\left(\frac{\delta_z}{H}\right)^2}{\left(1 - \frac{\delta_z}{H}\right)^4} \tag{7-2-32}$$

7.2.4　确定理想与实际流体分界线 δ_y，δ_z

实际流体速度分布式(7-2-29)中，$Z = \frac{z}{\delta_z}$，$Y = \frac{B-y}{\delta_y}$，因此，只有先确定分界线 δ_z，

δ_y，该式才能应用。

因为在整个断面压力是不变的，所以可以将理想流体运动区的无因次微分方程(7-2-14)代入实际流区流体运动无因次方程式(7-2-30)。

$$ReX(2Y-Y^2)^2(2Z-Z^2)^2=ReU_0\frac{\mathrm{d}U_0}{\mathrm{d}X}+2\nu_0X\left[(2Z-Z^2)\frac{H^2}{\delta_y^2}-(2Y-Y^2)\frac{H^2}{\delta_z^2}\right] \tag{7-2-33}$$

将式(7-2-32)代入式(7-2-33)

$$ReX(2Y-Y^2)^2(2Z-Z^2)^2=-Re\frac{\left[1-0.444X\left(\frac{\delta_z}{H}\right)^2\right]0.444\left(\frac{\delta_z}{H}\right)^2}{\left(1-\frac{\delta_z}{H}\right)^4}+2\nu_0X\left[(2Z-Z^2)\frac{H^2}{\delta_y^2}-(2Y-Y^2)\frac{H^2}{\delta_z^2}\right] \tag{7-2-34}$$

因为是求 δ_z，δ_y，故上式取 $Z=1$，$Y=1$，则为

$$ReX=-Re\frac{\left[1-0.444X\left(\frac{\delta_z}{H}\right)^2\right]0.444\left(\frac{\delta_z}{H}\right)^2}{\left(1-\frac{\delta_z}{H}\right)^4}+2\nu_0X\frac{H^2}{\delta_z^2}\left[\left(\frac{H}{B}\right)^2-1\right] \tag{7-2-35}$$

近似式

$$\left(1-\frac{\delta_z}{H}\right)^4=1-4\frac{\delta_z}{H}+6\left(\frac{\delta_z}{H}\right)^2 \tag{i}$$

将式(i)代入式(7-2-35)，$\nu_0\approx1$，有

$$ReX\left[1-4\frac{\delta_z}{H}+6\left(\frac{\delta_z}{H}\right)^2\right]=-Re\left[1-0.444X\left(\frac{\delta_z}{H}\right)^2\right]0.444\left(\frac{\delta_z}{H}\right)^2+2Z\left(\frac{H}{\delta_z}\right)^2\left[\left(\frac{H}{B}\right)^2+1\right]\left[1-4\frac{\delta_z}{H}+6\left(\frac{\delta_z}{H}\right)^2\right] \tag{7-2-36}$$

将全式除以$\frac{H^2}{\delta_z^2}$，则为

$$ReX\left[\left(\frac{\delta_z}{H}\right)^2-4\left(\frac{\delta_z}{H}\right)^3+6\left(\frac{\delta_z}{H}\right)^4\right]+Re\left[1-0.444X\left(\frac{\delta_z}{H}\right)^2\right]0.444\left(\frac{\delta_z}{H}\right)^4-2X\left(\frac{H^2}{B^2}+1\right)\left[1-4\frac{\delta_z}{H}+6\left(\frac{\delta_z}{H}\right)^2\right]=0 \tag{7-2-36$'$}$$

展开

$$-ReX\left(\frac{\delta_z}{H}\right)^6\times0.444^2+(6X+0.444)Re\left(\frac{\delta_z}{H}\right)^4-4ReX\left(\frac{\delta_z}{H}\right)^3+\left[Re-12\left(\frac{H^2}{B^2}+1\right)\right]X\cdot\left(\frac{\delta_z}{H}\right)^2+8X\left(\frac{H^2}{B^2}+1\right)\frac{\delta_z}{H}-2X\left(\frac{H^2}{B^2}+1\right)=0 \tag{7-2-36$''$}$$

除以 $0.444^2 ReX$，则为

$$-0.444^2\left(\frac{\delta_z}{H}\right)^6+\left(6+\frac{0.444}{X}\right)\left(\frac{\delta_z}{H}\right)^4-4\left(\frac{\delta_z}{H}\right)^3+\left[1-\frac{12\left(\frac{H^2}{B^2}+1\right)}{Re}\right]\left(\frac{\delta_z}{H}\right)^2$$

$$+\frac{8}{Re}\left(\frac{H^2}{B^2}+1\right)\frac{\delta_z}{H}-\frac{2}{Re}\left(\frac{H^2}{B^2}+1\right)=0 \tag{7-2-36)'''$$

因为 $\frac{\delta_z}{H}$ 的最大值为 $\frac{\delta_z}{H}=1$，$0.444^2\left(\frac{\delta_z}{H}\right)^6$ 项最大值为 0.197，所以忽略它。则上式为

$$\left(\frac{6X+0.444}{X}\right)\left(\frac{\delta_z}{H}\right)^4-4\left(\frac{\delta_z}{H}\right)^2+\left[1-\frac{12}{Re}\left(\frac{H^2}{B^2}-1\right)\right]\left(\frac{\delta_z}{H}\right)^2+\frac{8}{Re}\left(\frac{H^2}{B^2}-1\right)\frac{\delta_z}{H}-\frac{2}{Re}\left(\frac{H^2}{B^2}-1\right)=0 \tag{7-2-37}$$

式(7-2-37)是 $\frac{\delta_z}{H}$ 的四次代数方程，可运用高等教育出版社于 2000 年 5 月第八次印刷的《数学手册》90 页介绍的方法进行求解。

将式(7-2-37)全式除以第一项系数，则为

$$\left(\frac{\delta_z}{H}\right)^4-\frac{4X}{6X+0.444}\left(\frac{\delta_z}{H}\right)^3+\frac{X}{6X+0.444}\left[1-\frac{12}{Re}\left(\frac{H^2}{B^2}-1\right)\right]\left(\frac{\delta_z}{H}\right)^2+\frac{6X\left(\frac{H^2}{B^2}-1\right)}{(6X+0.444)Re}\frac{\delta_z}{H}$$

$$-\frac{2X\left(\frac{H^2}{B^2}-1\right)}{(6X+0.444)Re}=0 \tag{7-2-38}$$

令

$$b=-\frac{X}{6X+0.444} \tag{7-2-39}$$

$$c=\frac{X}{6X+0.444}\left[1-\frac{12}{Re}\left(\frac{H^2}{B^2}-1\right)\right] \tag{7-2-40}$$

$$d=\frac{8X\left(\frac{H^2}{B^2}-1\right)}{(6X+0.444)Re} \tag{7-2-41}$$

$$e=-\frac{2X\left(\frac{H^2}{B^2}-1\right)}{(6X+0.444)Re} \tag{7-2-42}$$

将式(7-2-38)写成下式

$$\left(\frac{\delta_z}{H}\right)^4-b\left(\frac{\delta_z}{H}\right)^3+c\left(\frac{\delta_z}{H}\right)^2+d\,\frac{\delta_z}{H}-e=0 \tag{7-2-43}$$

应用上式找 δ_z，还必须利用进口段结束处 $\delta_z=H$，所以应用式(7-2-43)时，必须依边界条件要求，结合实际情况，将有关公式加以改造方可求出合理的 δ_z 分布并找到对应的进口段长度。

式(7-2-43)是 $\frac{\delta_z}{H}$ 的四次代数方程，理论上它有四个解，而实际情况只有一个解。经过

大量运算与多方面分析，选计算其解的公式为

$$\left(\frac{\delta_x}{H}\right)^2+\frac{1}{2}\left(b-\sqrt{8y_1+b^2-4c}\right)\frac{\delta_x}{H}+\left(y_2+\frac{|b|y_2-|d|}{\sqrt{8y_1+b^2-4c}}\right)=0 \tag{7-2-44}$$

其中 y_1 与 y_2 为下式的两个根：

$$y^2-0.5cy+(0.25b|d||e|)=0 \tag{7-2-45}$$

式中：

$$b=\frac{-X^{1.3}}{6X+0.444} \tag{7-2-46}$$

$$c=|b|c'=|b|\left[1-\frac{12}{Re}\left(\frac{H^2}{B^2}-1\right)\right] \tag{7-2-47}$$

$$|d|=|b|d'=|b|\frac{8}{Re}\left(\frac{H^2}{B^2}-1\right) \tag{7-2-48}$$

$$|e|=|be'|=|b|\frac{4}{Re}\left(\frac{H^2}{B^2}-1\right) \tag{7-2-49}$$

式中，e，d 取绝对值，是为保证公式能得到合理解。公式应用时，分$\frac{H}{B}>1$ 与$\frac{H}{B}<1$ 两种情况，每种情况又依不同 Re 值，计算$\frac{\delta_x}{H}$随 X 的变化曲线。

举例说明公式应用方法。设矩形断面 $B=0.3\text{m}$，$H=0.4\text{m}$，$Re=580$，绘出进口段理想流体运动与实际流体运动分界曲线$\left(\frac{\delta_x}{H}\right)$。首先计算 c'，d'，e'的值。

$$c'=1-\frac{12}{580}\left[\left(\frac{0.4}{0.3}\right)^2-1\right]=1-0.0207\times0.7769=1-0.0161=0.9839$$

$$|d'|=\frac{8}{580}\left[\left(\frac{0.4}{0.3}\right)^2-1\right]=0.0138\times0.7769=0.0107$$

$$|e'|=\frac{4}{580}\left[\left(\frac{0.4}{0.3}\right)^2-1\right]=0.0069\times0.7769=0.0054$$

设 $X=50$，计算得

$$|b|=\left|-\frac{X^{1.3}}{6X+0.444}\right|=\left|\frac{-50^{1.3}}{6\times50+0.444}\right|=\left|\frac{-161.6818}{300.444}\right|=|-0.5381|$$

$$c=|b|c'=0.5381\times0.9839=0.5294$$

$$|d|=|b||d'|=0.5381\times0.0107=0.0058$$

$$|e|=|b||e'|=0.5381\times0.0054=0.0029$$

将 b，c，$|d|$，$|e|$值代入式(7-2-45)

$$y^2-0.5cy+(0.25b|d|-|e|)=y^2-0.5\times0.5294y+[0.25\times(-0.05381)\times0.0058-0.0029]$$
$$=y^2-0.26979-0.0037=0 \tag{1}$$

$$y_1=\frac{0.2697+\sqrt{0.2697^2+4\times0.0037}}{2}=\frac{0.2697+\sqrt{0.0875}}{2}=0.2828$$

$$y_2=\frac{0.2697-\sqrt{0.0875}}{2}=\frac{0.2897-0.2958}{2}=\frac{-0.0261}{2}\approx-0.0131$$

将 y_1 与 y_2 代入式(7-2-44)

$$\left(\frac{\delta_z}{H}\right)^2+\frac{1}{2}\left(-0.5381-\sqrt{8\times0.2828+0.5381^2-4\times0.5294}\right)$$

$$+\left(-0.0131+\frac{0.538\times(-0.0131)-0.0058}{\sqrt{8\times0.2828+0.5381^2-4\times0.5294}}\right)$$

$$=\left(\frac{\delta_z}{H}\right)^2+\frac{1}{2}(-0.5381-0.6592)\left(\frac{\delta_z}{H}\right)+\left(-0.0131+\frac{-0.0072-0.0058}{0.6592}\right)$$

$$=\left(\frac{\delta_z}{H}\right)^2-0.5987\frac{\delta_z}{H}-0.0325=0 \tag{2}$$

解出

$$\frac{\delta_z}{H}=\frac{0.5987+\sqrt{0.5987^2+4\times0.00325}}{2}=\frac{0.5987+0.6989}{2}=0.6488$$

$$\delta_z=0.6488\times0.4=0.2595\text{m}$$

当 $X=100$ 时，求 δ_z 与 $\frac{\delta_z}{H}$。

$$b=-\frac{100^{1.3}}{6\times100+0.444}=-0.663$$

$$c=|b|c'=0.663\times0.9839=0.6523$$

$$|d|=|b||d'|=0.663\times0.0107=0.0071$$

$$|e|=|b||e'|=0.663\times0.0054=0.0036$$

$$y^2-0.5cy+(0.25b|d|-|e|)=y^2-0.5\times0.6523y+[0.25\times(-0.663)\times0.0071-0.0036]$$

$$=y^2-0.3262y-0.0048=0 \tag{3}$$

解出 y_1，y_2：

$$y_1=\frac{0.3262+\sqrt{0.3262^2+4\times0.0048}}{2}=\frac{0.3262+0.3544}{2}=0.3403$$

$$y_2=\frac{0.3262-0.3544}{2}=-0.0141$$

将 y_1 与 y_2 代入式(7-2-44)

$$\left(\frac{\delta_z}{H}\right)^2+\left(b-\sqrt{8y_1+b^2-4c}\right)\frac{1}{2}\frac{\delta_z}{H}+\left(y_2+\frac{|b|y-|d|}{\sqrt{8y_1+b^2-4c}}\right)$$

$$=\left(\frac{\delta_z}{H}\right)^2\left(-0.663-\sqrt{8\times0.3403+0.663^2-4\times0.6523}\right)\frac{1}{2}\frac{\delta_z}{H}$$

$$+\left(-0.0141+\frac{0.663\times(-0.0141)-0.0071}{\sqrt{8\times0.3403+0.663^2-4\times0.6523}}\right)$$

$$=\left(\frac{\delta_z}{H}\right)^2+(-0.663-0.7435)\frac{1}{2}\frac{\delta_z}{H}+\left(-0.0141+\frac{-0.0093-0.0071}{0.7435}\right)$$

$$=\left(\frac{\delta_z}{H}\right)^2-0.7033\frac{\delta_z}{H}-0.0362=0 \tag{4}$$

解出

$$\frac{\delta_z}{H}=\frac{0.7033+\sqrt{0.7033^2+4\times0.0362}}{2}=\frac{0.7033+\sqrt{0.6395}}{2}$$

$$=\frac{0.7033+0.7997}{2}=0.7515$$

$$\delta_z=0.7515H=0.7515\times0.4=0.3006\text{m}$$

当 $X=200$ 时

$$b=-\frac{X^{1.3}}{6X+0.444}=\frac{200^{1.3}}{6\times200+0.444}=\frac{980.255}{1200.444}=-0.8166$$

当 $X=300$ 时

$$b=-\frac{300^{1.3}}{1800.444}=-0.9223$$

计算结果列入表 7-2 内。δ_z 变化曲线见图 7-3。

表 7-2　δ_z 曲线计算$\left(Re=580,\ \frac{H}{B}=1.333\right)$

X	0	20	50	100	200	300	320
x/m	0	8	20	40	80	120	128
$\frac{\delta_z}{H}$	0	0.451	0.649	0.752	0.893	0.997	1
δ_z/m	0	0.180	0.260	0.301	0.357	0.399	0.4

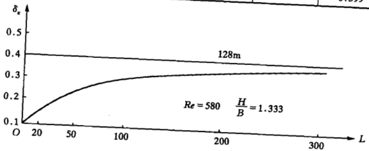


图 7-3　例题进口段 δ_z 变化曲线

7.2.5　确定进口段长度的方法

确定进口段长度 L 与确定理想和实际流体运动分界线$\frac{\delta_z}{H}$是紧密相连的。在设 X 计算不同$\frac{\delta_z}{H}$值、绘出$\frac{\delta_z}{H}$的曲线时，当$\frac{\delta_z}{H}=1$ 时，对应的 X 值就是 L。现分$\frac{H}{B}=1.333$，$\frac{H}{B}=0.5$ 两种情况，依 Re 不同计算$\frac{\delta_z}{H}$值，将结果列入表 7-3 和表 7-4 中，并绘成曲线，见图 7-4 和图 7-5。

表 7-3　$\frac{H}{B}=1.33$ 时 $\frac{\delta_z}{H}$ 值

Re \ X	0	20	50	100	200	300
580	0	0.45	0.65	0.75	0.89	0.99
400	0	0.53	0.67	0.82	0.94	1.02
200	0	0.60	0.72	0.88	0.96	1.10
100	0	0.81	0.92	0.97	1.09	

从曲线上可以看出，当 $Re=580$ 时，进口段长为 320；当 $Re=400$ 时，进口段长 $L=280$；当 $Re=200$，$L=220$；当 $Re=100$ 时，$L=110$。

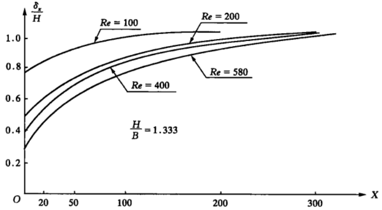


图 7-4　理想与实际流体运动分界线与进口段长度

因为 $\frac{H}{B}$ 比值与 Re 在矩形管道层流运动中均是已知的，所以当 $\frac{H}{B}=1.333$ 时，不同 Re 时进口段长度与 $\frac{\delta_z}{H}$ 变化情况均可以得到。

表 7-4　$\frac{H}{B}=0.5$ 时 $\frac{\delta_z}{H}$ 值

Re \ X	0	20	50	100	200	300
580	0	0.48	0.58	0.73	0.89	1.01
400	0	0.55	0.66	0.77	0.92	1.02
200	0	0.60	0.75	0.80	0.99	1.09
100	0	0.74	0.86	0.97	1.11	

7.2.6　流体速度分布最后确定

所谓速度分布，是指理想与实际流体运动区速度各自的分布。

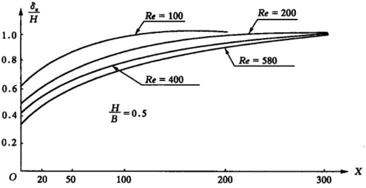


图 7-5 进口段长与分界线计算用图

(1) 理想流体速度分布

前边建立的理想与实际流体运动分界线方程式(7-2-31)，含有理想流体运动区速度 U_0，而且当 $X=0$ 时，$U_0=1$，已经具备其速度的基础。它实际上是一个平衡方程，将它写成

$$U_0=\frac{1+\frac{4}{9}X\left(\frac{\delta_z}{H}\right)^2}{\left(1+\frac{\delta_z}{H}\right)^2} \tag{7-2-50}$$

为使它满足边界条件，成为真正的速度分布，将它写成

$$U_0=a\left[\frac{1+\frac{4}{9}X\left(\frac{\delta_z}{H}\right)^2}{\left(1+\frac{\delta_z}{H}\right)^2}\right]+b \tag{7-2-51}$$

利用条件 $X=0$，$U_0=1$，$X=L$，$U_0(L, 0)=U_{max}=2$，确定 a，b 为

$$a=1-\left[\frac{2-\left(\frac{1}{4}+\frac{L}{9}\right)}{1-\left(\frac{1}{4}+\frac{L}{9}\right)}\right] \tag{7-2-52}$$

$$b=\frac{2-\left(\frac{1}{4}+\frac{L}{9}\right)}{1-\left(\frac{1}{4}+\frac{L}{9}\right)} \tag{7-2-53}$$

(2) 实际流体运动区速度分布

实际流体运动区速度分布式(7-2-29)是初选，它没有满足当 $X=L$ 时 $U=2$，也没有具体满足当 $Z=1$，$Y=1$ 时 $U_0=U$ 的条件。为此，利用边界条件式(7-2-22)

$$U(X, Y, Z)\big|_{Y=1,Z=1}=U_0(X)$$

取

$$a\left[\frac{1+\frac{4}{9}X\left(\frac{\delta_x}{H}\right)^2}{\left(1+\frac{\delta_x}{H}\right)^2}\right]+b=\frac{X}{R^a} \tag{7-2-54}$$

只要知道 Re 及$\frac{H}{B}$值，则$\frac{\delta_x}{H}$，L 均可以计算出来。由式(7-2-54)计算出 a，这个 a 就保证了实际流体运动区速度分布在分界线上与理想流体运动区速度相等。现分$\frac{H}{B}=1.333$ 与$\frac{H}{B}=0.5$ 两种情况下计算出 a 值。列成表(表 7-5～表 7-12)，绘成图（图 7-6 和图 7-7)。为以后计算理想流体运动区速度与实际流体运动区速度提供了方便。

表 7-5　$\frac{H}{B}=1.333$，$Re=580$，$L=320$，$a=0.0391$，$b=0.09709$

X	0	20	50	100	200	320
$\frac{\delta_x}{H}$	0	0.45	0.65	0.75	0.89	1
U_0	1	1.0114	1.0819	1.2177	1.5525	2.0127
a	0	0.469	0.6024	0.6928	0.7672	0.7976

$$U=\frac{X}{Re^a}(2Z-Z^2)(2Y-Y^2) \tag{7-2-55}$$

表 7-6　$\frac{H}{B}=1.333$，$Re=400$，$L=280$，$a=0.0332$，$b=0.9668$

X	0	20	50	100	200	280
$\frac{\delta_x}{H}$	0	0.530	0.666	0.824	0.888	1
U_0	1	1.016	1.097	1.278	1.629	2
a	0	0.497	0.638	0.728	0.802	0.825

表 7-7　$\frac{H}{B}=1.333$，$Re=200$，$L=220$，$a=0.0427$，$b=0.9573$

X	0	20	50	100	200	220
$\frac{\delta_x}{H}$	0	0.599	0.715	0.876	0.944	1
U_0	1	1.0261	1.1366	1.3829	1.8627	2.0107
a	0	0.5606	0.7142	0.808	0.8826	0.8872

表 7-8　$\frac{H}{B}=1.333$，$Re=100$，$L=130$，$a=0.0731$，$b=0.9269$

X	0	20	50	100	130
$\frac{\delta_x}{H}$	0	0.614	0.922	0.970	1
U_0	1	1.051	1.302	1.715	2
a	0	0.632	0.792	0.883	0.906

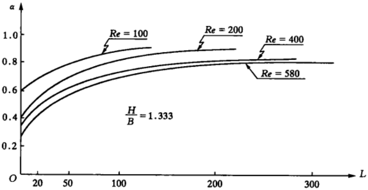


图 7-6 实际流体运动区速度公式 α 值$\left(\frac{H}{B}=1.333\right)$

表 7-9 $\frac{H}{B}=0.5$, $Re=580$, $L=280$, $a=0.0332$, $b=0.9668$

X	0	20	50	100	200	280
$\frac{\delta_x}{H}$	0	0.483	0.581	0.729	0.890	1
U_0	1	1.013	1.080	1.300	1.630	2
α	0	0.469	0.603	0.686	0.756	0.861

表 7-10 $\frac{H}{B}=0.5$, $Re=400$, $L=240$, $a=0.0386$, $b=0.9614$

X	0	20	50	100	200	240
$\frac{\delta_x}{H}$	0	0.55	0.663	0.772	0.918	1
U_0	1	1.021	1.111	1.299	1.721	2
α	0	0.497	0.635	0.725	0.794	0.799

表 7-11 $\frac{H}{B}=0.5$, $Re=200$, $L=205$, $a=0.046$, $b=0.954$

X	0	20	50	100	200	205
$\frac{\delta_x}{H}$	0	0.59	0.746	0.855	0.993	1
U_0	1	1.025	1.156	1.401	1.968	2
α	0	0.561	0.711	0.806	0.872	0.874

表 7-12 $\frac{H}{B}=0.5$, $Re=100$, $L=96$, $a=0.1008$, $b=0.8992$

X	0	20	50	96
$\frac{\delta_x}{H}$	0	0.740	0.856	1
U_0	1	1.095	1.405	2
α	0	0.725	0.892	0.967

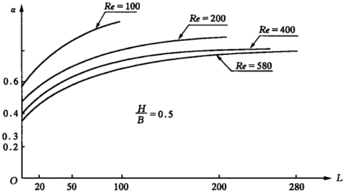


图 7-7　实际流体运动区速度公式 α 值$\left(\frac{H}{B}=0.5\right)$

通过表与图可以看出，α 值与$\frac{H}{B}$与 Re 有关，而且随 X 而变化。

(3) 压力变化

将式(7-2-51)代入式(7-2-14)，积分、定积分常数，得

$$P = 1 - \frac{1}{K_1}\left\{ \frac{a\cdot\frac{4}{9}\left(\frac{\delta_z}{H}\right)^2}{\left(1+\frac{\delta_z}{H}\right)^2}\left[\frac{a}{\left(1+\frac{\delta_z}{H}\right)^2}+b\right] + \frac{1}{2}\frac{a^2\left(\frac{4}{9}\right)^4\left(\frac{\delta_z}{H}\right)^4}{\left(1+\frac{\delta_z}{H}\right)^4}X^2\right\} \tag{7-2-56}$$

7.3　不可压缩湍流圆形管道进口段

坐标系与原点选定如图 7-8 所示。

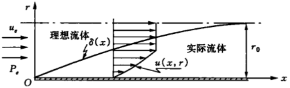


图 7-8　不可压缩湍流圆形管道进口段

7.3.1　运动控制方程与边界条件

由式(4-4-11)并结合本问题，可得

$$\frac{\nu}{2}\varphi_2\left(\frac{\partial^2 u}{\partial r\partial x} - \frac{1}{u}\frac{\partial u}{\partial r}\frac{\partial u}{\partial x}\right)$$
$$= -\frac{\partial p}{\rho\partial x} + \nu\left[2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial r\partial x} + \varphi_2\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial r\partial x} + 2t'\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial x}\right)\right] \tag{7-3-1}$$

边界条件

$$u(x,\ r)\big|_{\substack{x=0\\r=0}}=0 \tag{7-3-2}$$

$$u(x,\ r)\big|_{r=\delta(x)}=u(x) \tag{7-3-3}$$

$$u(x,\ r)\big|_{\substack{x=l\\r=r_0}}=u_{\max} \tag{7-3-4}$$

$$\left.\frac{\partial u(x,\ r)}{\partial r}\right|_{r=\delta(x)}=0 \tag{7-3-5}$$

$$\frac{\partial^2 u(x,\ r)}{\partial r^2}<0 \tag{7-3-6}$$

$$p(x,\ r)\big|_{x=0}=p_e \tag{7-3-7}$$

将方程与边界条件无因次化，为此取

$$\frac{u}{u_e}=U,\ \frac{r}{\delta(x)}=R,\ \frac{x}{r_0}=X,\ \frac{\nu}{\nu_e}=\nu_0,\ \frac{\rho}{\rho_e}=\rho_0,\ \frac{p}{p_e}=p_0$$

式中：u_e——管道进口流速；

$\delta(x)$——理想与黏性流体分界线；

r_0——管道半径；

p_e——管道进口处流体压力；

ν_e——管道进口处流体运动黏性系数；

ρ_e——管道进口处流体密度。

将以上比值代入式(7-3-1)，整理后得

$$\begin{aligned}&\frac{\nu_0}{2}\varphi_2\frac{u_e}{r_0\delta(x)}\left(\frac{\partial^2 U}{\partial X\partial R}-\frac{1}{U}\frac{\partial U}{\partial X}\frac{\partial U}{\partial R}\right)\\&=-\frac{p_e}{\rho_e r_0}\frac{\partial p_0}{\rho_0\partial X}+\nu_e\nu_0\left[\frac{u_e\partial^2 U}{r_0^2\partial X^2}+\frac{u_e}{\delta(x)r_0}\frac{\partial^2 U}{\partial R\partial X}+\varphi_2\left(\frac{u_e}{\delta(x)^2}\frac{\partial^2 U}{\partial R^2}+\frac{u_e}{\delta(x)r_0}\frac{\partial^2 U}{\partial R\partial X}\right.\right.\\&\left.\left.\quad+2t'\frac{u_e^2}{\delta(x)^2 r_0}\frac{\partial U}{\partial R}\frac{\partial^2 U}{\partial R\partial X}\right)\right]\end{aligned} \tag{7-3-1$'$}$$

除以$\frac{\nu_e u_e}{r_0^2}$，则有

$$\begin{aligned}&\frac{\nu_0\varphi_2}{2}\frac{r_0}{\delta(x)}\left(\frac{\partial^2 U}{\partial R\partial X}-\frac{1}{U}\frac{\partial U}{\partial R}\frac{\partial U}{\partial X}\right)\\&=-K_1\frac{\partial p_0}{\rho_0\partial X}+\nu_0\left\{\frac{\partial^2 U}{\partial X^2}\frac{r_0}{\delta(x)}\frac{\partial^2 U}{\partial R\partial X}+\varphi_2\left[\left(\frac{r_0}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial R^2}+\frac{r_0}{\delta(x)}\frac{\partial^2 U}{\partial R\partial X}+K_2\frac{\partial U}{\partial R}\frac{\partial^2 U}{\partial R\partial X}\right]\right\}\end{aligned} \tag{7-3-8}$$

式中：

$$K_1=\frac{r_0 p_e}{\nu_e u_e\rho_e} \tag{7-3-9}$$

$$K_2=\frac{2t'u_e r_0}{\delta(x)^2} \tag{7-3-10}$$

无因次边界条件：

$$U(X,\ R)|_{\substack{X=0\\R=0}}=0 \tag{7-3-11}$$

$$U(X,\ R)|_{R=1}=U_0(X) \tag{7-3-12}$$

$$\left.\frac{\partial U(X,\ R)}{\partial R}\right|_{R=1}=0 \tag{7-3-13}$$

$$U(X,\ R)|_{\substack{X=L\\R=1}}=U_{\max} \tag{7-3-14}$$

$$\frac{\partial^2 U(X,\ R)}{\partial R^2}>0 \tag{7-3-15}$$

$$p_0(X,\ R)|_{X=0}=1 \tag{7-3-16}$$

理想流体运动区，流体运动微分方程为

$$u_0\frac{\mathrm{d}u_0}{\mathrm{d}x}=-\frac{\mathrm{d}p}{\rho\mathrm{d}x} \tag{7-3-17}$$

理想流体运动边界条件：

$$u_0(x)|_{x=0}=u_e \tag{7-3-18}$$

式中：u_0——理想流体运动速度。

将式(7-3-17)与式(7-3-18)无因次化，取$\frac{u_0}{u_e}=U_0$，则

$$U_0\frac{\mathrm{d}U_0}{\mathrm{d}X}=-K_3\frac{\mathrm{d}p_0}{\rho_0\mathrm{d}X} \tag{7-3-19}$$

式中：

$$K_3=\frac{p_e}{u_e^2\rho_e} \tag{7-3-20}$$

将式(7-3-19)代入式(7-3-8)，则有

$$\begin{aligned}&\frac{\nu_0}{2}\varphi_2\frac{r_0}{\delta(x)}\left(\frac{\partial^2 U}{\partial R\partial X}-\frac{1}{U}\frac{\partial U}{\partial R}\frac{\partial U}{\partial X}\right)\\&=\frac{K_1}{K_3}U_0\frac{\mathrm{d}U_0}{\mathrm{d}X}+\nu_0\left\{\frac{\partial^2 U}{\partial X^2}+(1+\varphi_2)\frac{r_0}{\delta(x)}\frac{\partial^2 U}{\partial R\partial X}+\varphi_2\left[\left(\frac{r_0}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial R^2}+K_2\frac{\partial U}{\partial R}\frac{\partial^2 U}{\partial R\partial X}\right]\right\}\end{aligned} \tag{7-3-21}$$

7.3.2 进口段速度分布

根据边界条件，理想流体运动区无因次速度分布暂选

$$U_0=(1+X)^{F(X)} \tag{a}$$

黏性流体运动区，流体运动无因次速度暂选

$$U=X^{F(X)}(2R-R^2) \tag{b}$$

将式(a)与式(b)代入式(7-3-21)

$$\frac{\nu_0}{2}\varphi_2\frac{r_0}{\delta(x)}\left[F(X)X^{F(X)-1}2(1-R)-2(1-R)F(X)X^{F(X)-1}\right]$$

$$=Re(1+X)^{2F(X)-1}F(X)+\nu_0\Big\{(2R-R^2)F(X)(F(X)-1)X^{F(X)-1}(1+\varphi_2)\frac{r_0}{\delta(x)}$$

$$F(X)X^{F(X)-1}2(1-R)+\varphi_2\left[\left(\frac{r_0}{\delta(x)}\right)^2X^{F(X)}(-2)+4K_2(1-R)^2X^{2F(X)}\right]\Big\} \quad (7\text{-}3\text{-}21)'$$

当 $R=1$ 时，则式(7-3-21)′为

$$2\nu_0\varphi_2\left(\frac{r_0}{\delta(x)}\right)^2X^{F(X)}=Re(1+X)^{2F(X)-1}F(X)+\nu_0F(X)(F(X)-1)X^{F(X)-1} \quad (7\text{-}3\text{-}21)''$$

对 X 取对数，则有

$$2\nu_0\varphi_2\left(\frac{r_0}{\delta(x)}\right)^2F(X)\ln(X)=Re(2F(X)-1)\ln(1+X)F(X)+\nu_0F(X)(F(X)-1)^2\ln X \quad (1)$$

将式(1)全式除以 $F(X)\ln X$，则为

$$2\nu_0\varphi_2\left(\frac{r_0}{\delta(x)}\right)^2=Re(2F(X)-1)+\nu_0(F(X)-1)^2 \quad (2)$$

Re 是雷诺数，在湍流时，它是一个很大的标量；$\nu_0=\frac{\nu}{\nu_e}\approx1$；$\frac{r_0}{\delta(x)}$是与 $F(X)$无关的量，$\varphi_2<1$。所以式(2)可以简化为

$$F(X)^2+2(Re-1)F(X)=Re-1 \quad (3)$$

按代数解法，得

$$F(X)=\frac{-2(Re-1)\pm\sqrt{4(Re-1)^2+4(Re-1)}}{2} \quad (4)$$

将式(4)化简，取合理解

$$F(X)=\frac{-2(Re-1)+\sqrt{4(Re-1)^2+4\frac{(Re-1)^2}{Re-1}}}{2}=\frac{-2(Re-1)+2(Re-1)\sqrt{1+\frac{1}{Re-1}}}{2}$$

$$\approx0$$

说明按代数精确解法找到不合理的解 $F(X)=0$。只好放掉 $F(X)^2$ 项，用式(3)中的 $F(X)$ 一次式解，得

$$F(X)=\frac{Re-1}{2(Re-1)}=\frac{1}{2} \quad (7\text{-}3\text{-}22)$$

从理论上讲 $F(X)=\frac{1}{2}$是合理的，当然还可以通过实验证明它的合理性。

将式(7-3-22)代入式(a)，将其写成

$$U_0=a(1+X)^{\frac{1}{2}}+c \quad (\mathrm{a})'$$

利用理想流体运动区的边界条件，$X=0$，$U_0=1$；$X=L$，$U_0=U_{max}$，可以确定

$$a=\frac{1-U_{max}}{1-(1+L)^{\frac{1}{2}}} \quad (7\text{-}3\text{-}23)$$

$$c=\frac{U_{\max}-(1+L)^{\frac{1}{2}}}{1-(1+L)^{\frac{1}{2}}} \tag{7-3-24}$$

将 a，c 代入式(a)′，则得理想流体运动区无因次速度分布

$$U_0=\frac{1}{1-(1+L)^{\frac{1}{2}}}\left[(1-U_{\max})(1+X)^{\frac{1}{2}}+U_{\max}-(1+L)^{\frac{1}{2}}\right] \tag{7-3-25}$$

黏性流体运动区速度分布，除将 $F(X)=\frac{1}{2}$代入式(b)外，它也要满足 $X=L$ 时 $U=U_{\max}$的条件，因此式(b)应写成

$$U=Re^{\alpha}U_{\max}X^{\frac{1}{2}}(2R-R^2) \tag{7-3-26}$$

式(7-3-25)中有 L，式(7-3-26)中有 α。要应用这两个公式，必须先确定 L 与α。

7.3.3　确定进口段无因次长度 L 与α

壁面摩擦力做功计算如下：

$$\begin{aligned}F_2&=\int_0^l 2\pi r_0 x 2\mathrm{d}x=\int_0^l 2\pi r_0 x 2\mu Re^{\alpha}\frac{u_{\max}}{\delta(x)}X^{\frac{1}{2}}\mathrm{d}x\\&=4\pi r_0\mu Re^{\alpha}\frac{u_{\max}}{\delta(x)}\frac{1}{r_0^{\frac{1}{2}}}\frac{2}{5}x^{\frac{5}{2}}\bigg|_0^l=\frac{8}{5}\pi r_0^3\mu Re^{\alpha}\frac{u_{\max}}{\delta(x)}\left(\frac{l}{r_0}\right)^{\frac{1}{2}}\left(\frac{l}{r_0}\right)^2\\&=\frac{8\pi}{5}r_0^3\mu Re^{\alpha}\frac{u_{\max}}{\delta(x)}L^{\frac{5}{2}}=\frac{8\pi}{5}r_0^2\mu Re^{\alpha}u_{\max}L^{\frac{5}{2}}\end{aligned} \tag{A}$$

理想流体运动区压力下降与速度间关系

$$\Delta p=\rho\left(\frac{u_{\max}^2}{2}-\frac{u_e^2}{2}\right) \tag{B}$$

对 1kg 流体而言，$\rho=1\text{kg}$。由于式(A)与式(B)相等，即

$$1[\text{kg}]\cdot\left(\frac{u_{\max}^2}{2}-\frac{u_e^2}{2}\right)=\frac{8\pi}{5}r_0^2\mu Re^{\alpha}u_{\max}L^{\frac{5}{2}} \tag{C}$$

则式(C)可以写成

$$L=\left[\frac{5[\text{kg}](u_{\max}^2-u_e^2)}{16\mu u_{\max}r_0^2Re^{\alpha}}\right]^{\frac{1}{2.5}} \tag{7-3-27}$$

利用在理想与黏性流体分界上它们速度应相等的条件，将式(7-3-26)与式(7-3-25)等起来

$$\frac{1}{1-(1+L)^{\frac{1}{2}}}\left[(1-U_{\max})(1+X)^{\frac{1}{2}}+U_{\max}-(1+L)^{\frac{1}{2}}\right]=R^{\alpha}U_{\max}X^{\frac{1}{2}} \tag{7-3-28}$$

当 $X=L$ 时，式(7-3-28)应为

$$1=Re^{\alpha}L^{\frac{1}{2}} \tag{7-3-29}$$

利用(7-3-27)与式(7-3-29)两式联立可以求出 α 与L。这里的 α 只起到确定问题定义域之

一的 L 的作用。

7.3.4 最后确定实际流区速度分布

以前已经得到实际流体运动区速度分布公式(7-3-26)，它为确定问题定义域起到非常重要作用。但它要真的作为实际流区速度分布还必须满足理想与实际流体速度在分界线上相等的条件。所以式(7-3-26)应依此进一步加以改造为

$$U = Re^{\alpha} X^{\frac{1}{2}} (2R - R^2) \tag{7-3-30}$$

式中的 α 应依理想与实际流体在分界线上的相等条件加以确定。U_0 是理想流体速度，在确定无因次长度 L 后，它是可以根据不同 X 加以确定的。

7.3.5 确定理想与实际流体分界线 $\delta(x)$

将式(7-3-25)与式(7-3-26)代入式(7-3-21)，得

$$\begin{aligned} 0 = & \frac{1}{1-(1+L)^{\frac{1}{2}}}\left[(1-U_{\max})(1+X)^{\frac{1}{2}} + U_{\max} - (1+L)^{\frac{1}{2}}\right] \frac{Re}{1-(1+L)^{\frac{1}{2}}}\left[\frac{1}{2}(1-U_{\max})(1+X)^{-\frac{1}{2}}\right] \\ & - \frac{1}{4} Re^{\alpha}(2R-R^2) X^{-\frac{3}{2}} - 2(1-\varphi^{\frac{2}{3}}) Re^{\alpha} X^{\frac{1}{2}} \left(\frac{r_0}{\delta(x)}\right)^2 \end{aligned} \tag{7-3-21$'$}$$

因为要找 $\delta(x)$，则 $R=1$，$\nu_0=1$，去掉当 $X=0$ 时奇点，取 $X^{-\frac{3}{2}}$ 为 $(X+1)^{-\frac{3}{2}}$。则上式为

$$\left(\frac{r_0}{\delta(x)}\right)^2 = \frac{1}{2(1-\varphi^{\frac{2}{3}}) Re^{\alpha} X^{\frac{1}{2}}} \left\{ \frac{1-U_{\max}}{2[1-(1+L)^{\frac{1}{2}}]^2} \left\{ (1-U_{\max}) + \frac{[U_{\max}-(1+L)^{\frac{1}{2}}]}{(1+X)^{\frac{1}{2}}} \right\} - \frac{1}{4} Re^{\alpha}(1+X)^{-\frac{3}{2}} \right\} \tag{7-3-21$''$}$$

$$\frac{\delta(x)}{r_0} = \sqrt{\frac{2(1-\varphi^{\frac{2}{3}}) Re^{\alpha} X^{\frac{1}{2}}}{\left\{ \frac{(1-U_{\max}) Re}{2[1-(1+L)^{\frac{1}{2}}]^2} \left\{ (1-U_{\max}) + \frac{[U_{\max}-(1+L)^{\frac{1}{2}}]}{(1+X)^{\frac{1}{2}}} \right\} - \frac{1}{4} Re^{\alpha}(1+X)^{-\frac{3}{2}} \right\}}} \tag{7-3-31}$$

式(7-3-31)不满足边界条件 $X=L$，$\frac{\delta(x)}{r_0}=1$。所以应改造为

$$\frac{\delta(x)}{r_0} = a\sqrt{\frac{2(1-\varphi^{\frac{2}{3}}) Re^{\alpha} X^{\frac{1}{2}}}{\frac{(1-U_{\max}) Re}{2[1-(1+L)^{\frac{1}{2}}]^2} \left\{ (1-U_{\max}) + \frac{[U_{\max}-(1+L)^{\frac{1}{2}}]}{(1+X)^{\frac{1}{2}}} \right\} - \frac{1}{4} Re^{\alpha}(1+X)^{-\frac{3}{2}}}} \tag{7-3-32}$$

$$\alpha=\sqrt{\frac{\dfrac{(1-U_{\max})Re}{2[1-(1+L)^{\frac{1}{2}}]^2}\left\{(1-U_{\max})+\dfrac{[U_{\max}-(1+L)^{\frac{1}{2}}]}{(1+L)^{\frac{1}{2}}}\right\}-\dfrac{1}{4}Re^{\alpha}(1+L)^{-\frac{3}{2}}}{2(1-\varphi_0^{\frac{2}{3}})Re^{\alpha}L^{\frac{1}{2}}}} \tag{7-3-33}$$

式中，φ_0 是充分发展管道中湍流涡旋体积分数，在此，它应是已知值。进口段涡旋体积分数为

$$\varphi=X\frac{\varphi_0}{L} \tag{7-3-34}$$

例 7-1　有一水管，直径 0.6m，断面平均速度 $u_e=1\text{m/s}$，流体动力黏性系数 $\mu=1.005\times10^{-6}\text{N·s/m}^2$，运动黏性系数 $\nu=1.141\times10^{-6}\text{m}^2/\text{s}$，管中心最大速度 $u_{\max}=1.2\text{m/s}$。确定进口段长度 l 为多少?

解

$$Re=\frac{u_e r_0}{\nu}=\frac{1\times0.3\times10^6}{1.141}=262927$$

将式(7-2-27)与式(7-2-29)合并为

$$Re^{\frac{\alpha}{2.5}-2\alpha}=\sqrt[2.5]{\frac{5[\text{kg}](u_{\max}^2-u_e^2)}{16\mu u_{\max}r_0^2}} \tag{1}$$

将有关数据代入式(1)

$$Re^{\frac{\alpha}{2.5}-2\alpha}=\sqrt[2.5]{\frac{5(1.2^2-1^2)\times10^3}{16\times1.005\times1.2\times0.3^2}}=\sqrt[2.5]{\frac{5\times0.44\times1000}{16\times0.09\times1.206}}=\sqrt[2.5]{\frac{2200}{1.737}}=1266.55^{0.4}$$

$$\left(\frac{\alpha}{2.5}-2\alpha\right)\ln Re=\ln17.42$$

$$\left(\frac{\alpha}{2.5}-2\alpha\right)\ln262927=2.858$$

$$-1.6\alpha\times12.48=2.858$$

$$\alpha=-\frac{2.858}{19.967}=-0.143$$

将 α 代入式(7-3-29)

$$1=Re^{\alpha}L^{\frac{1}{2}}=(262927)^{-0.143}L^{\frac{1}{2}}=\frac{1}{5.957}L^{\frac{1}{2}} \tag{2}$$

解出

$$5.957=L^{\frac{1}{2}}$$

$$5.957^2=L$$

$$35.478=L$$

$$l=L\times0.3=35.487\times0.3=10.646\text{m}$$

例 7-2　题意同例 7-1，已知充分发展管道中涡旋体积分数，$\varphi=0.1$，绘出理想与实际流体运动分界线 $\delta(x)$ 速度分布图。

解　首先由式(7-3-25)计算断面下理想流体速度分布与实际流体运动公式。

当 $X=5$ 时

$$
\begin{aligned}
U_0 &= \frac{1}{1-(1+L)^{\frac{1}{2}}}\left[(1-U_{max})(1+X)^{\frac{1}{2}}+U_{max}-(1+L)^{\frac{1}{2}}\right] \\
&= \frac{1}{1-(1+35.487)^{\frac{1}{2}}}\left[(1-1.2)(1+5)^{\frac{1}{2}}+1.2-(1+35.487)^{\frac{1}{2}}\right] \\
&= \frac{1}{-5.04}(-0.2\times 2.4495+1.2-6.04)=\frac{-5.299}{-5.04}=1.0575
\end{aligned}
$$

$$u(x)=1.0575\times u_e=1.0575\times 1=1.0575\text{m/s}$$

同理，$X=10$，$X=15$，$X=20$，$X=25$，$X=30$ 计算结果均列入表 7-13。

表 7-13　　例 7-2 计算结果

X	5	10	15	20	25	30	35.487
U_0	1.0575	1.0919	1.1190	1.1422	1.1627	1.1813	1.2
α	−0.0600	−0.0852	−0.0995	−0.1094	−0.1169	−0.1229	−0.1284
$\frac{\delta(x)}{r_0}$	0.6573	0.7991	0.8165	0.9084	0.9450	0.9712	1.0000
$\delta(x)$	0.1972	0.2391	0.2585	0.2725	0.2835	0.2913	0.3000

现在由式(7-3-30)计算 X 为不同值时，实际流体速度分布式中 α 值，以 $X=5$ 为例说明其计算方法。

$$U_0=Re^{\alpha}X^{\frac{1}{2}} \tag{7-3-30)$'$ (}$$

$$
\begin{gathered}
1.0575=Re^{\alpha}5^{\frac{1}{2}}=Re^{\alpha}2.236 \\
\frac{1.0575}{2.236}=Re^{\alpha} \\
0.4729=Re^{\alpha} \\
\ln 0.4729=\alpha\ln Re=\alpha\ln 262927 \\
-0.7487=\alpha 12.4796 \\
\alpha=\frac{-0.7487}{12.4796}=-0.06
\end{gathered}
$$

其他不同 X 的 α 值计算方法一样，均得出，列入表 7-13。

实际流体运动区速度分布，依不同 X 值分别如下。

当 $X=5$ 时

$$U=Re^{-0.06}X^{\frac{1}{2}}(2R-R^2) \tag{1}$$

当 $X=10$ 时

$$U=Re^{-0.0852}X^{\frac{1}{2}}(2R-R^2) \tag{2}$$

当 $X=15$ 时

$$U=Re^{-0.0991}X^{\frac{1}{2}}(2R-R^2) \tag{3}$$

当 $X=20$ 时

$$U=Re^{-0.1094}X^{\frac{1}{2}}(2R-R^2) \tag{4}$$

当 $X=25$ 时

$$U=Re^{-0.1169}X^{\frac{1}{2}}(2R-R^2) \tag{5}$$

当 $X=30$ 时

$$U=Re^{-0.1229}X^{\frac{1}{2}}(2R-R^2) \tag{6}$$

当 $X=L=35.487$ 时

$$U=Re^{-0.1284}X^{\frac{1}{2}}(2R-R^2) \tag{7}$$

计算理想与实际流体运动分界线 $\delta(x)$，应用式(7-3-32)与式(7-3-33)计算，其中，$\frac{1}{4}R_e^a(1+L)^{\frac{1}{2}}$项，量级很小，可以略去。首先计算 a 值，由式(7-3-33)

$$a=\sqrt{\frac{(1-U_{\max})Re\left[1-U_{\max}+\dfrac{U_{\max}-(1+L)^{\frac{1}{2}}}{(1+L)^{\frac{1}{2}}}\right]}{4(1-\varphi_0^{\frac{2}{3}})Re^aL^{\frac{1}{2}}[1-(1+L)^{\frac{1}{2}}]^2}}$$

$$=\sqrt{\frac{(1-1.2)\times 262927\left[1-1.2+\dfrac{1.2-(1+35.487)^{\frac{1}{2}}}{(1+35.487)^{\frac{1}{2}}}\right]}{4(1-0.1^{\frac{2}{3}})\times 262927^{-0.1284}[1-(1+35.487)^{\frac{1}{2}}]^2}}$$

$$=\sqrt{\frac{-52585\left(-0.2+\dfrac{-4.84}{6.04}\right)}{\dfrac{4\times 0.785\times 5.9571}{4.9649}(1-6.09)^2}}=\sqrt{\frac{52654.649}{3.7675\times 25.408}}$$

$$=\sqrt{\frac{52654.649}{95.7248}}=\sqrt{550.0628}=23.4534$$

当 $X=30$ 时，由式(7-3-32)计算$\dfrac{\delta(x)}{r_0}$值：

$$\frac{\delta(x)}{r_0}=a\sqrt{\frac{4(1-\varphi^{\frac{2}{3}})Re^aX^{\frac{1}{2}}[1-(1+L)^{\frac{1}{2}}]^2}{Re(1-U_{\max})\left[1-U_{\max}+\dfrac{U_{\max}-(1+L)^{\frac{1}{2}}}{(1+X)^{\frac{1}{2}}}\right]}} \tag{7-3-32$'$}$$

应用式(7-3-32)′之前，首先应用式(7-3-34)计算对应的 φ 值：

$$\varphi=30\,\frac{\varphi_0}{L}=30\times\frac{0.1}{35.487}=30\times 0.0028=0.0845$$

$$\varphi^{\frac{2}{3}}=0.0845^{\frac{2}{3}}=0.197$$

$$(1-\varphi^{\frac{2}{3}})=(1-0.197)=0.803$$

$$Re^{a}=262927^{-0.1229}=\frac{1}{4.6355}=0.2157$$

将以上有关数据代入式(7-3-32)

$$\frac{\delta(x)}{r_0}=23.4532\sqrt{\frac{4\times0.803\times0.2157\times30^{\frac{1}{2}}[1-(1+35.487)^{\frac{1}{2}}]^2}{262927(1-1.2)\left[1-1.2+\frac{1.2-(1+35.487)^{\frac{1}{2}}}{(1+30)^{\frac{1}{2}}}\right]}}$$

$$=23.4532\sqrt{\frac{0.6928\times5.4772(1-6.04)^2}{-0.2\times262927\left(-0.2+\frac{-4.84}{5.5678}\right)}}$$

$$=23.4532\sqrt{\frac{3.7946\times25.408}{-52585\times[-0.2+(-0.8693)]}}=23.4532\sqrt{\frac{96.4132}{-52585\times(-1.0693)}}$$

$$=23.4532\sqrt{\frac{96.4132}{56229.1405}}=23.4532\sqrt{0.0017146}$$

$$=23.4532\times0.0414=0.9712$$

$$\delta(x)=0.9712\times r_0=0.9712\times0.3=0.2913\text{m}$$

同法，其他 $\delta(x)$ 计算结果均列入表 7-13 内。

以 $X=5$ 处为例，说明断面速度分布方法。

$$U=Re^{-0.06}X^{\frac{1}{2}}(2R-R^2) \tag{1}$$

式(1)是无因次式，将其变为有因次速度分布，则为

$$u(x,\ r)=u_{e}Re^{-0.06}X^{\frac{1}{2}}\left[2\frac{r}{\delta(x)}-\left(\frac{r}{\delta(x)}\right)^2\right] \tag{1$'$}$$

当 $r=0.05$m 时

$$u(1.5,\ 0.05)=1\times262927^{-0.06}5^{\frac{1}{2}}\left[2\frac{0.05}{0.197}-\left(\frac{0.05}{0.197}\right)^2\right]$$

$$=1\times1.0576(2\times0.2538-0.0644)=1\times1.0576(0.5076-0.0644)$$

$$=1.0576\times0.4432=0.4687\text{m/s}$$

当 $r=0.1$m 时

$$u(x,\ r)=u(1.5,\ 0.1)=1\times1.0576\left[2\times\frac{0.1}{0.197}-\left(\frac{0.1}{0.197}\right)^2\right]$$

$$=1.0576(2\times0.5076-0.2577)=1.0576\times0.7575=0.8011\text{m/s}$$

当 $r=0.15$m 时

$$u(1.5,\ 0.15)=1\times1.0576\left[2\times\frac{0.15}{0.197}-\left(\frac{0.15}{0.197}\right)^2\right]=1.0576\times(2\times0.7614-0.7614^2)$$

$$=1.0576\times(1.5228-0.5798)=1.0576\times0.943=0.9973\text{m/s}$$

7.3.6 压力变化(无因次)

压力与其所在断面上位置无关，它只是距离 x 的函数。因此可以利用理想流体运动微分方程求解之，将式(7-3-25)代入式(7-3-19)

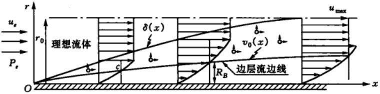


图 7-9　例 7-2 中 $\delta(x)$ 曲线

$$U_0\frac{dU_0}{dX}=-K_3\frac{dP_0}{\rho_0 dX}$$

$$\frac{1-U_{max}}{2[1-(1+L)^{\frac{1}{2}}]^2}\{(1-U_{max})X+2[U_{max}-(1+L)^{\frac{1}{2}}](1+X)^{\frac{1}{2}}\}+c=-\frac{K_3}{\rho_0}P_0 \tag{a}$$

当 $X=0$ 时，$P_0=1$，则定积分常数 $c=-\frac{K_3}{\rho_0}$。

$$P_0=1-\frac{K_3}{\rho_0}\frac{1-U_{max}}{2[1-(1+L)^{\frac{1}{2}}]^2}\{(1-U_{max})X+[U_{max}-(1+L)^{\frac{1}{2}}]2(1+X)^{\frac{1}{2}}\} \tag{7-3-35}$$

7.3.7　黏性(实际)流区断面平均速度与位置

计算断面平均速度 v_0 目的有二：一则其本身与计算涡旋径向分速度有关；二则是通过它可以得到平均速度在断面上的位置。有了这个位置，可以确定涡旋产生的地带。如图 7-10 所示。

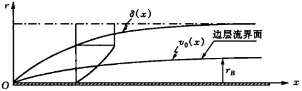


图 7-10　黏性流区断面平均速度 $v_0(x)$ 示意图

$dA=r_0 d\theta dr$，而$\frac{r}{\delta(x)}=R$，$dR=\frac{dr}{\delta(x)}$，$dr=\delta(x)dR$

$$dA=r_0\delta(x)d\theta dR \tag{b}$$

$$d\theta=u(x,\ r)dA$$

$$=u_e Re^a X^{\frac{1}{2}}(2R-R^2)dA \tag{c}$$

$$d\theta=u_e Re^a X^{\frac{1}{2}}(2R-R^2)r_0\delta(x)d\theta dR$$

$$\theta=u_e Re^a X^{\frac{1}{2}}r_0\delta(x)\int_0^{2\pi}d\theta\int_0^1(2R-R^2)dR$$

$$\theta=2\pi Re^{a}X^{\frac{1}{2}}r_0\delta(x)\frac{2}{3}u_e \tag{d}$$

$$v_0(x)=\frac{\theta}{A}=\frac{2\pi Re^{a}X^{\frac{1}{2}}u_e\frac{2}{3}r_0\delta(x)}{2\pi r_0\delta(x)}=\frac{2}{3}u_e Re^{a}X^{\frac{1}{2}} \tag{7-3-36}$$

式(7-3-36)就是计算黏性流区断面平均速度 $v_0(x)$ 的公式。其位置计算只需将其值代入速度分布式中，实际流区速度分布为

$$U=Re^{a}X^{\frac{1}{2}}(2R-R^2) \tag{7-3-30}$$

将其变为

$$u(x,\ r)=u_e Re^{a}X^{\frac{1}{2}}(2R-R^2) \tag{7-3-30$'$}$$

将平均速度式(7-3-36)代入式(7-3-30)′左边，对应的 R 就是平均速度的位置 R_B。如图 7-11 所示。

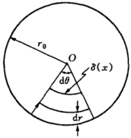


图 7-11 确定黏性流区断面平均速度 R_B 示意图

$$\frac{2}{3}Re^{a}X^{\frac{1}{2}}u_e=u_e Re^{a}X^{\frac{1}{2}}(2R_B-R_B^2)$$

$$\frac{2}{3}=2R_B-R_B^2 \tag{e}$$

解式(e)得

$$R_B=0.423 \tag{7-3-37}$$

$$r_B=0.423\delta(x) \tag{7-3-38}$$

平均速度 $v_0(x)$ 的位置，也就是边层流的界面，是涡旋产生的地方。

7.3.8 边层流界面上的涡旋强度

断面上黏性流体平均速度与速度分布曲线的交点组成的曲面，就是边层流界面。在这个面上会间歇地产生涡旋。有因次与无因次涡旋的关系如下：

$$[\omega_R]=\frac{1}{2}\frac{\partial U}{\partial R}=\frac{1}{2}\frac{\delta(x)}{u_e}\frac{\partial u}{\partial r}=\frac{\delta(x)}{u_e}\omega_r \tag{7-3-39}$$

$$\omega_r\Big|_{r=r_B}=\frac{u_e}{\delta(x)}\frac{1}{2}\frac{\partial U}{\partial R}=\frac{u_e}{\delta(x)}\frac{1}{2}Re^{a}X^{\frac{1}{2}}2(1-R)\Big|_{R=R_B}=\frac{u_e}{\delta(x)}Re^{a}X^{\frac{1}{2}}(1-R_B)$$

$$=\frac{u_e}{\delta(x)}Re^{a}X^{\frac{1}{2}}(1-0.423)$$

$$\omega\Big|_{r=r_B}=0.577\frac{u_e}{\delta(x)}Re^{a}X^{\frac{1}{2}} \tag{7-3-40}$$

7.3.9 边层流界面上涡旋径向速度 u_r^*

依第 4 章式(4-3-27)，结合本问题，则为

$$u_r^*=12\nu t\left(\omega\Big|_{r=r_B}\right)^2 v_0^{-1} \tag{7-3-41}$$

将式(7-3-40)与式(7-3-36)代入式(7-3-41)

$$u_r^*\big|_{r=r_B} = 12\nu t\,\frac{\left(0.577\,\dfrac{u_e}{\delta(x)}Re^a X^{\frac{1}{2}}\right)^2}{\dfrac{2}{3}u_e Re^a X^{\frac{1}{2}}} = 18\nu t\,\frac{0.577^2 u_e Re^a X^{\frac{1}{2}}}{\delta(x)^2}$$

$$=18\nu\left(\frac{\delta(x)-0.423\delta(x)}{u_r^*}\right)\frac{0.577^2 u_e Re^a X^{\frac{1}{2}}}{\delta(x)^2}$$

$$\left(u_r^*\big|_{r=r_B}\right)^2=\frac{18\nu\times 0.577^3 u_e Re^a X^{\frac{1}{2}}}{\delta(x)}$$

$$u^*_{r=r_B}=\sqrt{\frac{3.4578\nu u_e Re^a X^{\frac{1}{2}}}{\delta(x)}} \tag{7-3-42}$$

7.3.10　涡旋体积分数 φ

依第 4 章涡旋体积分数定义，结合本问题

$$\varphi=\frac{[秒]\pi u_r^*}{48\delta(x)} \tag{7-3-43}$$

7.4　不可压缩湍流矩形管道进口段

与前节不同之处，它是一维流动三维变化。坐标选定如图 7-12 所示。

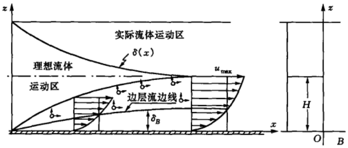


图 7-12　矩形管道进口段流动示意图

7.4.1　运动控制方程与边界条件

由式(4-4-11)，结合本问题，可得

$$\frac{\nu}{2}\varphi_2\left[\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial z\partial x}-\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial x}\right]$$

$$=-\frac{\partial p}{\rho\partial x}+\nu\left[2\frac{\partial^2 u}{\partial x^2}+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial y\partial x}\right)+(1-\varphi^{\frac{2}{3}})\frac{\partial^2 u}{\partial z\partial x}+2\varphi^{\frac{2}{3}}t'\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial y\partial x}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial z\partial x}\right)\right]$$

(7-4-1)

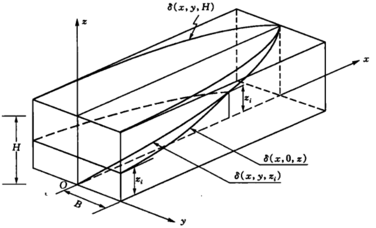


图 7-13 矩形进口段示意图

矩形管道进口段，理想与实际流体分界面比较复杂。如图 7-13 所示。

$$u(x, y, z)\big|_{x=0, y=0, z=0}=0 \tag{7-4-2}$$

$$u(x, y, z)\big|_{x=l, y=0, z=H}=u_{\max} \tag{7-4-3}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=B-\delta_y(x, z_\delta)}=0 \tag{7-4-4}$$

$$\left.\frac{\partial u}{\partial z}\right|_{z=\delta_z(x, y_\delta)}=0 \tag{7-4-5}$$

$$\frac{\partial^2 u}{\partial y^2}<0 \tag{7-4-6}$$

$$\frac{\partial^2 u}{\partial z^2}<0 \tag{7-4-7}$$

$$p(x, y, z)\big|_{x=0}=p_e \tag{7-4-8}$$

将方程与边界条件无因次化，为此取

$$\frac{u}{u_e}=U,\ \frac{x}{H}=X,\ \frac{B-y}{\delta_y(x, z_\delta)}=Y,\ \frac{z}{\delta_z(x, y_\delta)}=Z,\ \frac{p}{p_e}=p_0,\ \frac{\rho}{\rho_e}=\rho_0,\ \frac{\nu}{\nu_e}=\nu_0$$

将以上比值代入方程与边界条件，得

$$\begin{aligned}&\frac{\nu_0}{2}\varphi_2\frac{u_e\nu_e}{H^2}\left[\frac{H}{\partial y}\frac{\partial^2 U}{\partial Y\partial X}+\frac{H}{\partial z}\frac{\partial^2 U}{\partial Z\partial X}-\frac{1}{U}\left(\frac{H}{\partial y}\frac{\partial U}{\partial Y}+\frac{H}{\partial z}\frac{\partial U}{\partial Z}\right)\frac{\partial U}{\partial X}\right]\\&=-\frac{p_e}{H\rho_e}\frac{\partial p_0}{\rho_0\partial X}-\frac{\nu_e u_e}{H^2}\left\{2\frac{\partial^2 U}{\partial X^2}+(1+\varphi^{\frac{2}{3}})\left[\left(\frac{H}{\partial y}\right)^2\frac{\partial^2 U}{\partial Y^2}+\left(\frac{H}{\partial z}\right)^2\frac{\partial^2 U}{\partial Z^2}+\frac{H}{\partial y}\frac{\partial^2 U}{\partial Y\partial X}\right]\right.\\&\quad\left.+(\varphi^{\frac{2}{3}}-1)\frac{H}{\partial z}\frac{\partial^2 U}{\partial Z\partial X}+\frac{2t'u_e^2}{H^3}\left[\left(\frac{H}{\partial y}\right)^2\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y\partial X}+\left(\frac{H}{\partial z}\right)^2\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z\partial X}\right]\right\}\end{aligned} \tag{7-4-1$'$}$$

除以$\frac{\nu_e u_e}{H^2}$，则为

$$\frac{\nu_0}{2}\varphi_2\left[\frac{H}{\partial y}\frac{\partial^2 U}{\partial Y\partial X}+\frac{H}{\partial z}\frac{\partial^2 U}{\partial Z\partial X}-\frac{1}{U}\left(\frac{H\partial U}{\partial y\partial Y}+\frac{H\partial U}{\partial z\partial Z}\right)\frac{\partial U}{\partial X}\right]$$

$$=-K_1\frac{\partial p_0}{\rho_0\partial X}+\nu_0\left\{2\frac{\partial^2 U}{\partial X^2}+(1+\varphi^{\frac{2}{3}})\left[\left(\frac{H}{\partial y}\right)^2\frac{\partial^2 U}{\partial Y^2}+\left(\frac{H}{\partial z}\right)^2\frac{\partial^2 U}{\partial Z^2}+\frac{H}{\partial y}\frac{\partial^2 U}{\partial Y\partial X}\right]\right.$$

$$\left.+(\varphi^{\frac{2}{3}}-1)\frac{H}{\partial z}\frac{\partial^2 U}{\partial Z\partial X}+K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{\partial y}\right)^2\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y\partial X}+\left(\frac{H}{\partial z}\right)^2\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z\partial X}\right]\right\} \tag{7-4-9}$$

式中：

$$K_1=\frac{p_e H}{u_e\nu_e\rho_e} \tag{7-4-10}$$

$$K_2=\frac{2t' u_e}{H} \tag{7-4-11}$$

$$U(X,\ Y,\ Z)\big|_{X=0,Y=0,Z=0}=0 \tag{7-4-12}$$

$$U(X,\ Y,\ Z)\big|_{X=L,Y=0,Z=1}=U_{max} \tag{7-4-13}$$

$$\left.\frac{\partial U}{\partial Y}\right|_{Y=1}=0 \tag{7-4-14}$$

$$\left.\frac{\partial U}{\partial Z}\right|_{Z=1}=0 \tag{7-4-15}$$

$$\frac{\partial^2 U}{\partial Y^2}<0 \tag{7-4-16}$$

$$\frac{\partial^2 U}{\partial Z^2}<0 \tag{7-4-17}$$

$$p_0(X,\ Y,\ Z)\big|_{X=0}=1 \tag{7-4-18}$$

7.4.2　速度分布

理想流体区运动微分方程为

$$u\frac{du}{dx}=-\frac{1}{\rho}\frac{dp}{dx} \tag{7-4-19}$$

将其无因次化，取$\frac{u}{u_e}=U_0$代入式(7-4-19)，则有

$$K_0U_0\frac{dU_0}{dX}=-\frac{1}{\rho_0}\frac{dp_0}{dX} \tag{7-4-20}$$

式中：

$$K_0=\frac{\rho_e u_e^2}{p_e} \tag{7-4-21}$$

用式(7-4-21)置换式(7-4-9)中的$-\frac{1}{\rho_0}\frac{dp_0}{dX}$，则式(7-4-9)变为

$$\varphi_2\frac{H}{\delta_z}\frac{\nu_0}{2}\left[\frac{\delta_z}{\delta_y}\frac{\partial^2 U}{\partial Y\partial X}+\frac{\partial^2 U}{\partial Z\partial X}-\frac{1}{U}\left(\frac{\delta_z}{\delta_y}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\frac{\partial U}{\partial X}\right]$$

$$=ReU_0\frac{\mathrm{d}U_0}{\mathrm{s}X}+\nu_0\left\{2\frac{\partial^2 U}{\partial X^2}+(1+\varphi^{\frac{2}{3}})\left[\left(\frac{H}{\delta_y}\right)^2\frac{\partial^2 U}{\partial Y^2}+\left(\frac{H}{\delta_z}\right)^2\frac{\partial^2 U}{\partial Z^2}+\frac{H}{\delta_y}\frac{\partial^2 U}{\partial Y\partial X}\right]\right.$$

$$\left.+(\varphi^{\frac{2}{3}}-1)\frac{H}{\delta_z}\frac{\partial^2 U}{\partial Z\partial X}+K_2\varphi^{\frac{2}{3}}\left(\frac{H}{\delta_z}\right)^2\left[\left(\frac{\delta_z}{\delta_y}\right)^2\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y\partial X}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z\partial X}\right]\right\} \tag{7-4-22}$$

式中：

$$Re=\frac{u_e H}{\nu_e} \tag{7-4-23}$$

根据边界条件，选黏性(实际)流区速度分布

$$U=U_{\max}\left(\frac{X}{L}\right)^{F(X)}(2Y-Y^2)(2Z-Z^2) \tag{7-4-24}$$

代入式(7-4-22)，确定 $F(X)$。理想流体速度 U_0 可由式(7-3-23)借用。

首先计算 $ReU_0\dfrac{\mathrm{d}U_0}{\mathrm{d}X}$，将式(7-3-23)代入，则有

$$ReU_0\frac{\mathrm{d}U_0}{\mathrm{d}X}=\frac{F(X)Re(1-U_{\max})(1+X)^{F(X)-1}}{[1-(1+L)^{F(X)}]} \tag{7-4-25}$$

将式(7-4-25)与式(7-4-24)代入式(7-4-22)，得

$$\varphi_2\frac{H}{\delta_z}\frac{\nu_0}{2}\left\{\frac{\delta_z}{\delta_y}U_{\max}F(X)\left(\frac{X}{L}\right)^{F(X)-1}(2Z-Z^2)2(1-Y)+U_{\max}F(X)\left(\frac{X}{L}\right)^{F(X)-1}\right.$$

$$2(1-Z)(2Y-Y^2)-\frac{1}{U_{\max}\left(\frac{X}{L}\right)^{F(X)-1}(2Y-Y^2)(2Z-Z^2)}\left[\frac{\delta_z}{\delta_y}U_{\max}\left(\frac{X}{L}\right)^{F(X)}2(1-Y)\right.$$

$$\left.\left.(2Z-Z^2)+U_{\max}\left(\frac{X}{L}\right)^{F(X)}2(1-Z)(2Y-Y^2)\right]U_{\max}+F(X)\left(\frac{X}{L}\right)^{F(X)-1}(2Y-Y^2)(2Z-Z^2)\right\}$$

$$=\frac{F(X)Re(1-U_{\max})(1+X)^{F(X)-1}}{1-(1+L)^{F(X)}}+\nu_0\left\{2U_{\max}(2Y-Y^2)(2Z-Z^2)F(X)(F(X)-1)\right.$$

$$\left(\frac{X}{L}\right)^{F(X)-2}+(1+\varphi^{\frac{2}{3}})\left[\left(\frac{H}{\delta_y}\right)^2U_{\max}\left(\frac{X}{L}\right)^{F(X)}(-2)(2Z-Z^2)+\left(\frac{H}{\delta_z}\right)^2U_{\max}\left(\frac{X}{L}\right)^{F(X)}\right.$$

$$\left.(-2)(2Y-Y^2)+\frac{H}{\delta_y}U_{\max}2(1-Y)(2Z-Z^2)F(X)\left(\frac{X}{L}\right)^{F(X)-1}\right]+(\varphi^{\frac{2}{3}}-1)\frac{H}{\delta_z}U_{\max}$$

$$2(1-Z)(2Y-Y^2)F(X)\left(\frac{X}{L}\right)^{F(X)-1}+4K_2\varphi^{\frac{2}{3}}\left(\frac{H}{\delta_z}\right)^2U_{\max}F(X)\left(\frac{X}{L}\right)^{F(X)}$$

$$\left.\left[\left(\frac{\delta_z}{\delta_y}\right)^2(1-Y)^2(2Z-Z^2)^2+(1-Z)^2(2Y-Y^2)^2\right]\right\} \tag{A}$$

速度分布形态由边界条件决定，而运动控制方程对速度分布起着强度作用。$F(X)$是待定常数，故选择理想与实际流体运动速度相同的点，即取 $Z=1$，$Y=1$，则式(A)简化为

$$0=\frac{ReF(X)(1-U_{max})(1+X)^{F(X)-1}}{1-(1+L)^{F(X)}}+\nu_0\left\{8U_{max}F(X)(F(X)-1)\left(\frac{X}{L}\right)^{F(X)-2}\right.$$
$$\left.-2(1+\varphi^{\frac{2}{3}})\left[\left(\frac{H}{\delta_y}\right)^2+\left(\frac{H}{\delta_z}\right)^2\right]U_{max}\left(\frac{X}{L}\right)^{F(X)}\right\} \tag{B}$$

式(B)中，$\delta_y=B$，$\delta_z=H$，可写成

$$2\nu_0(1+\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right]U_{max}\left(\frac{X}{L}\right)^{F(X)}$$
$$=\frac{ReF(X)(1-U_{max})(1+X)^{F(X)-1}}{1-(1+L)^{F(X)}}+8\nu_0U_{max}F(X)(F(X)-1)\left(\frac{X}{L}\right)^{F(X)-2} \tag{C}$$

根据计算需要，对式(C)中的$\left(\frac{X}{L}\right)$取对数

$$2\nu_0U_{max}(1+\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right]F(X)\ln\left(\frac{X}{L}\right)$$
$$=ReF(X)(1-U_{max})(F(X)-1)\ln(1+X)-\ln[1-(1+L)^{F(X)}]+8\nu_0U_{max}F(X)\cdot$$
$$(F(X)-1)(F(X)-2)\ln\left(\frac{X}{L}\right) \tag{D}$$

进一步展开式(C)

$$2\nu_0U_{max}(1+\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right]F(X)(\ln X-\ln L)$$
$$=ReF(X)(1-U_{max})(F(X)-1)(\ln 1+\ln X)-\ln 1-\ln[-(1+L)^{F(X)}]+8\nu_0U_{max}F(X)$$
$$\cdot(F(X)-1)(F(X)-2)(\ln X-\ln L) \tag{E}$$

去掉 $\ln[-(1+L)^{F(X)}]$这一无意义项，$\ln 1=0$，$\ln L\ll Re$，$\nu_0\approx1$，则式(E)为

$$2U_{max}(1+\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right]F(X)\ln X$$
$$=ReF(X)(1-U_{max})(F(X)-1)\ln X+8U_{max}F(X)(F(X)-1)(F(X)-2)\ln X \tag{F}$$

将式(F)除以 $F(X)\ln X$，则为

$$2U_{max}(1+\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right]=Re(1-U_{max})(F(X)-1)+8U_{max}(F(X)^2-3F(X)+2) \tag{G}$$

$$8U_{max}F(X)^2+[Re(1-U_{max})-24U_{max}]F(X)+16U_{max}-Re(1-U_{max})$$
$$-2U_{max}(1+\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right]=0 \tag{H}$$

经过定性计算分析，只有取 $F(X)$一次式合理，然后进行量级比较，Re 远大于其他量值。故

$$F(X)=\frac{Re(1-U_{max})}{Re(1-U_{max})}=1 \tag{I}$$

理想流体：

$$U_0=\frac{1}{1-(1+L)}[(1-U_{max})(1+X)+U_{max}-(1+L)]$$

$$=\frac{-1}{L}[X(1-U_{\max})-L]=1-\frac{X}{L}(1-U_{\max}) \tag{7-4-26}$$

黏性(实际)流区速度分布：

$$U=U_{\max}\left(\frac{X}{L}\right)(2Y-Y^2)(2Z-Z^2) \tag{7-4-27}$$

应用速度公式时，首先要知道进口段无因次长度 L，对实际流区，速度分布计算尚需 δ_y，δ_z。

7.4.3 进口段长度

由于进口段长 l 处断面中心为最大速度 $u_{\max}$，而进口段中心区为理想流体，其速度沿程变化与压力变化相对应。而压力变化，或是让压力沿程下降，是因黏性流区剪应力做功消耗流体能量形成的，建立它们的平衡关系，就可以找到进口段长度 l。

在湍流情况下，其剪应力作为处理问题的思维方法，可以把它分为两种情况来考虑。第一种情况，单纯考虑其黏性剪应力；第二种情况，既考虑其黏性剪应力，又把其湍流附加剪应力纳入进来。

先按第一种情况推导其进口段长度计算公式。按黏性剪应力公式为

$$\tau_x=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) \tag{a}$$

式(a)也可以写成下式：

$$\tau_x=\mu u_e\left(\frac{1}{\delta_z}\frac{\partial U}{\partial Z}+\frac{1}{\delta_y}\frac{\partial U}{\partial Y}\right) \tag{b}$$

由式(7-4-19)

$$\left.\frac{\partial U}{\partial Z}\right|_{Z=0}=2U_{\max}\left(\frac{X}{L}\right)(2Y-Y^2)(1-Z) \tag{c}$$

$$\left.\frac{\partial U}{\partial Y}\right|_{Y=0}=2U_{\max}\left(\frac{X}{L}\right)(2Z-Z^2)(1-Y) \tag{d}$$

将式(c)与式(d)代入式(b)

$$\tau_x=2\mu U_{\max}\left(\frac{X}{L}\right)\left[\frac{1}{\delta_z}(2Y-Y^2)+\frac{1}{\delta_y}(2Z-Z^2)\right] \tag{7-4-28}$$

将剪应力做功对 X，Y，Z 积分，则得剪应力在进口段消耗的总能量

$$\begin{aligned}E_f &= 2\mu U_{\max}\int_0^L X\mathrm{d}X\int_0^1 L\left(\frac{X}{L}\right)\mathrm{d}\left(\frac{X}{L}\right)\left[\int_0^1\frac{1}{\delta_z}(2Y-Y^2)\mathrm{d}Y+\int_0^1\frac{1}{\delta_y}(2Z-Z^2)\mathrm{d}Z\right]\\ &=2\mu u_e U_{\max}\frac{L^3}{2}\left(\frac{1}{\delta_z}\frac{2}{3}+\frac{1}{\delta_y}\frac{2}{3}\right)=\frac{2}{3}\mu u_{\max}L^3\left(\frac{1}{H}+\frac{1}{B}\right)\end{aligned} \tag{7-4-29}$$

理想流体运动区适用伯努利方程，则

$$\frac{p_e}{\rho}+\frac{u_e^2}{2}=\frac{p_e}{\rho}+\frac{u_{\max}^2}{2} \tag{7-4-30}$$

$$\frac{p_e}{\rho}-\frac{p_e}{\rho}=\frac{u_{\max}^2}{2}-\frac{u_e^2}{2}$$

$$p_e - p_e = \rho\left(\frac{u_{max}^2 - u_e^2}{2}\right) = \rho\,\frac{u_{max}^2 - v_0^2}{2}$$

$$\Delta p = \frac{\rho(u_{max}^2 - v_0^2)}{2} \tag{7-4-31}$$

式(7-4-31)与式(7-4-29)相等，因为这个压力降就是由黏性流体运动区剪应力做功所消耗的能量，故得

$$L = \sqrt[3]{\frac{3(u_{max}^2 - v_0^2)}{\nu u_{max}\left(\frac{1}{H} - \frac{1}{B}\right)}} \tag{7-4-32}$$

$$l = \left[3\sqrt{\frac{3(u_{max}^2 - v_0^2)}{4\nu u_{max}\left(\frac{1}{H} - \frac{1}{B}\right)}}\right]H \tag{7-4-33}$$

第二种情况：同时考虑黏性剪应力与湍流附加剪应力。

$$\widetilde{\Omega}_x = \Omega_x + \Omega'_x \tag{7-4-34}$$

$$\Omega'_x = \Omega'_{yx} + \Omega'_{zx} \tag{7-4-35}$$

由式(3-14-1)得湍流附加剪应力为

$$\Omega'_x = \varphi^{\frac{2}{3}}\mu t\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] \tag{7-4-36}$$

$$\left(\frac{\partial u}{\partial y}\right)^2 = \frac{u_e^2}{\delta_y^2}\left(\frac{\partial U}{\partial Y}\right)^2 \tag{e}$$

$$\left(\frac{\partial u}{\partial z}\right)^2 = \frac{u_e^2}{\delta_z^2}\left(\frac{\partial U}{\partial Z}\right)^2 \tag{f}$$

将式(e)与式(f)代入式(7-4-36)

$$\widetilde{\Omega}_x = \varphi^{\frac{2}{3}}\mu t u_e^2\left[\frac{1}{\delta_y^2}\left(\frac{\partial U}{\partial Y}\bigg|_{Y=0}\right)^2 + \frac{1}{\delta_z^2}\left(\frac{\partial U}{\partial Z}\bigg|_{Z=0}\right)^2\right] \tag{g}$$

$$\widetilde{\Omega}_x = 2\varphi^{\frac{2}{3}}\mu t u_{max}^2\left(\frac{X}{L}\right)^2\left[\frac{1}{\delta_y^2}(2Z - Z^2)^2 + \frac{1}{\delta_z^2}(2Y - Y^2)^2\right] \tag{h}$$

湍流附加剪应力在黏性流区所做的总功为

$$\begin{aligned}
\widetilde{E}_x &= 2\varphi^{\frac{2}{3}}\mu t u_{max}^2\int_0^1 X\mathrm{d}X\int_0^1 L\left(\frac{X}{L}\right)^2\mathrm{d}\left(\frac{X}{L}\right)\left[\int_0^1\frac{1}{\delta_y^2}(2Z - Z^2)\mathrm{d}Z - \int_0^1\frac{1}{\delta_z^2}(2Y - Y^2)\mathrm{d}Y\right] \\
&= 2\varphi^{\frac{2}{3}}\mu t u_{max}^2\frac{L^2}{2}\frac{L}{3}\left(\frac{1}{\delta_y^2}\frac{8}{15} + \frac{1}{\delta_z^2}\frac{8}{15}\right) = 2\varphi^{\frac{2}{3}}\mu t u_{max}^2\frac{L^3}{3}\frac{8}{15}\left(\frac{1}{\delta_y^2} + \frac{1}{\delta_z^2}\right) \\
&= \frac{16}{45}\mu t\varphi^{\frac{2}{3}}u_{max}^2\left(\frac{1}{B^2} + \frac{1}{H^2}\right)L^3
\end{aligned} \tag{7-4-37}$$

由此得剪应力做的功为

$$E = E_x + \widetilde{E}_x = \left[\frac{2}{3}\mu u_{max}\left(\frac{1}{H} - \frac{1}{B}\right) + \frac{16}{45}\varphi^{\frac{2}{3}}\mu t u_{max}^2\left(\frac{1}{B^2} + \frac{1}{H^2}\right)\right]L^3 \tag{7-4-38}$$

式(7-4-31)与式(7-4-38)相等，得进口段无因次长度 L

$$L=\left\{\frac{0.5(u_{\max}^2-v_0^2)}{\nu\left[0.666u_{\max}\left(\frac{1}{H}-\frac{1}{B}\right)+0.356\varphi^{\frac{2}{3}}tu_{\max}^2\left(\frac{1}{B^2}+\frac{1}{H^2}\right)\right]}\right\}^{\frac{1}{3}} \tag{7-4-39}$$

有因次进口段长度 l

$$l=\left\{\frac{0.5(u_{\max}^2-v_0^2)}{\nu\left[0.666u_{\max}\left(\frac{1}{H}-\frac{1}{B}\right)+0.356\varphi^{\frac{2}{3}}tu_{\max}^2\left(\frac{1}{B^2}+\frac{1}{H^2}\right)\right]}\right\}^{\frac{1}{3}}H \tag{7-4-40}$$

7.4.4 理想与实际流体运动分界线 $\delta(x, y, z)$

根据矩形断面，依流量守恒原则，建立理想与实际流体运动分界线 $\delta(x, y, z)$的计算公式。

总流量

$$Q_{总}=u_eBH=v_0BH \tag{i}$$

理想流

$$Q_{理}=u_e\left[1-\frac{X}{L}(1-U_{\max})\right](H-\delta_z)(B-\delta_y) \tag{j}$$

黏性流

$$Q_{黏}=u_{\max}\delta_z\delta_y\left(\frac{X}{L}\right)\int_0^1\int_0^1(2Y-Y^2)(2Z-Z^2)\mathrm{d}Y\mathrm{d}Z \tag{k}$$

按流量守恒原则

$$u_eBH=u_e\left[1-\frac{X}{L}(1-U_{\max})\right][HB-(\delta_zB+\delta_yH)+\delta_z\delta_y]+\frac{4}{9}u_{\max}\frac{X}{L}\delta_z\delta_y \tag{l}$$

$$1=\left[1-\frac{X}{L}(1-U_{\max})\right]\left[1-\left(\frac{\delta_z}{H}+\frac{\delta_y}{B}\right)+\frac{\delta_z}{H}\frac{\delta_y}{B}\right]+\frac{4}{9}U_{\max}\frac{X}{L}\frac{\delta_z}{H}\frac{\delta_y}{B} \tag{l$'$}$$

令

$$\frac{\delta_z}{H}=\frac{\delta_y}{B} \tag{7-4-41}$$

将上式代入式(l)′中，则有

$$1=\left[1-\frac{X}{L}(1-U_{\max})\right]\left[1-2\frac{\delta_z}{H}+\left(\frac{\delta_z}{H}\right)^2\right]+\frac{4}{9}U_{\max}\frac{X}{L}\left(\frac{\delta_z}{H}\right)^2 \tag{l$''$}$$

将上式写成

$$\left\{\left[1-\frac{X}{L}(1-U_{\max})\right]+\frac{4}{9}U_{\max}\frac{X}{L}\right\}\left(\frac{\delta_z}{H}\right)^2-2\left[1-\frac{X}{L}(1-U_{\max})\right]\frac{\delta_z}{H}-\frac{X}{L}(1-U_{\max})=0 \tag{l$'''$}$$

近似处理

$$\frac{\delta_z}{H}=\frac{\frac{X}{L}(1-U_{\max})}{2\left[1-\frac{X}{L}(1-U_{\max})\right]} \tag{7-4-42}$$

根据边界条件，它应改为

$$\delta_z=\frac{2U_{\max}}{U_{\max}-1}\left\{\frac{\frac{X}{L}(U_{\max}-1)}{2\left[1-\frac{X}{L}(1-U_{\max})\right]}\right\}H \tag{7-4-43}$$

$$\delta_y=\frac{2U_{\max}}{U_{\max}-1}\left\{\frac{\frac{X}{L}(U_{\max}-1)}{2\left[1-\frac{X}{L}(1-U_{\max})\right]}\right\}B \tag{7-4-44}$$

7.4.5 压力变化

由于压力变化与断面无关，它只是距离的函数。因此可直接按理想流体区微分方程求解它，即

$$K_0U_0\frac{\mathrm{d}U_0}{\mathrm{d}X}=-\frac{1}{\rho_0}\frac{\mathrm{d}p_0}{\mathrm{d}X} \tag{7-4-20$'$}$$

由式(7-4-21)计算

$$\frac{\mathrm{d}U_0}{\mathrm{d}X}=\left[1-\frac{X}{L}(1-U_{\max})\right]\left[-\frac{(1-U_{\max})}{L}\right] \tag{m}$$

将式(m)代入上式

$$\frac{K_0}{L}\left[1-\frac{X}{L}(1-U_{\max})\right](1-U_{\max})=-\frac{1}{\rho_0}\frac{\mathrm{d}p_0}{\mathrm{d}X} \tag{m$'$}$$

对式(m)′积分

$$\frac{\rho_0K_0}{L}(1-U_{\max})X-\frac{\rho_0K_0}{2L^2}(1-U_{\max})^2X^2+c=p_0 \tag{m$''$}$$

当 $X=0$ 时，$p_0=1$，则有

$$p_0=1+\frac{\rho_0K_0}{L}(1-U_{\max})X-\frac{\rho_0K_0}{2L^2}(1-U_{\max})^2X^2 \tag{7-4-45}$$

变为有因次式

$$p=p_e+p_e\left[\frac{\rho_0K_0}{L}(1-U_{\max})X-\frac{\rho_0K_0}{2L^2}(1-U_{\max})^2X^2\right] \tag{7-4-46}$$

7.4.6 黏性流区断面平均速度与位置

$$[v_{平}]=\frac{[Q]}{[A]}=\frac{U_{\max}\left(\frac{X}{L}\right)\int_0^1\int_0^1(2Y-Y^2)(2Z-Z^2)\mathrm{d}Y\mathrm{d}Z}{|\,x\,|}=\frac{4}{9}U_{\max}\left(\frac{X}{L}\right) \tag{7-4-47}$$

$$v_{平}=\frac{4}{9}U_{\max}\left(\frac{X}{L}\right) \tag{7-4-48}$$

式(7-4-47)是黏性流区断面上无因次平均速度，式(7-4-48)是对应的有因次速度。它的位置可以利用无因次速度代入速度公式

$$\frac{4}{9}U_{\max}\frac{X}{L}=U_{\max}\left(\frac{X}{L}\right)(2Y_B-Y_B^2)(2Z_B-Z_B^2)$$

$$\frac{4}{9}=(2Y_B-Y_B^2)(2Z_B-Z_B^2) \tag{7-4-49}$$

要解出上式中断面上无因次速度位置 Y_B，Z_B，还必须有另外一个独立方程。通过对圆形断面对应情况分析，取

$$\frac{\delta_z}{H}=\frac{\delta_y}{B} \tag{7-4-50}$$

坐标选定及原点位置如图 7-12 所示，而理想与实际流体分界线在 z 方向与 δ_z 与坐标方向一致，而 δ_y 则与坐标 y 方向相反。无论 δ_y，δ_z 都是从壁面算起的，如图 7-13 与图 7-14 以及图 7-15 所示。结合图示，解出 Y_B，Z_B，将式(7-4-50)写成

$$\begin{aligned}\frac{4}{9}&=\left[2\frac{(B-y_B)}{\delta_y}-\frac{(B-y_B)^2}{\delta_y^2}\right]\left[2\frac{z_B}{\delta_z}-\left(\frac{z_B}{\delta_z}\right)^2\right]=\left[2\frac{\delta_{yB}}{\delta_y}-\frac{\delta_{yB}^2}{\delta_y^2}\right]\left[2\frac{z_B}{\delta_z}-\left(\frac{z_B}{\delta_z}\right)^2\right]\\&=\left[2\frac{\frac{B}{H}\delta_{zB}}{\frac{B}{H}\delta_z}-\frac{\left(\frac{B}{H}\delta_{zB}\right)^2}{\left(\frac{B}{H}\right)^2\delta_z^2}\right]\left[2\frac{z_B}{\delta_z}-\left(\frac{z_B}{\delta_z}\right)^2\right]=\left[2\frac{\delta_{zB}}{\delta_z}-\left(\frac{\delta_{zB}}{\delta_z}\right)^2\right]^2\\&=(2Z_B-2Z_B^2)^2\end{aligned} \tag{7-4-51}$$

由上式解出

$$Z_B^2-2Z_B+\frac{2}{3}=0 \tag{7-4-52}$$

由此得

$$Z_B=0.425 \tag{7-4-53}$$

$$z_B=0.425\delta_z \tag{7-4-54}$$

同理

$$Y_B=0.425 \tag{7-4-55}$$

$$y'_B=\delta_{yB}=0.425\delta_y \tag{7-4-56}$$

7.4.7 边层流界面上涡旋强度

由式(7-3-39)，结合本问题，则有

$$\begin{aligned}\omega_y\Big|_{y=B-y_B=\delta_{yB}}&=\frac{u_{\max}}{\delta_y}\left(\frac{X}{L}\right)(2Z_B-Z_B^2)(1-Y_B)\\&=\frac{u_{\max}}{\delta_y}\left(\frac{X}{L}\right)(2\times0.425-0.425^2)(1-0.425)\end{aligned}$$

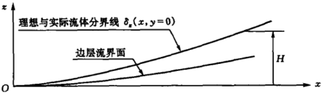


图 7-14　边层流界面与理想与实际流体分界面示意图

$$=0.385\frac{u_{max}}{\delta_y}\left(\frac{X}{L}\right) \tag{7-4-57}$$

同理

$$\omega_z\big|_{z=\delta_{zB}}=0.385\frac{u_{max}}{\delta_z}\left(\frac{X}{L}\right) \tag{7-4-58}$$

7.4.8　边层流界面涡旋横向分度

依式(7-41)，结合本问题

$$u_y^*=12\nu t\left(\omega_{y=y'_B=\delta_{yB}}\right)^2 v_{平}^{-1} \tag{7-4-59}$$

$$u_z^*=12\nu t\left(\omega_{z=\delta_{zB}}\right)^2 v_{平}^{-1} \tag{7-4-60}$$

7.4.9　涡旋体积分数 φ

依涡旋体积分数的定义，可得

$$\varphi=\frac{[秒]\pi(Hu_y^*+Bu_z^*)}{48\delta_y\delta_z} \tag{7-4-61}$$

7.4.10　应用举例

例 7-3　一矩形断面水管道，高为 $2H=0.64$m，宽 $2B=0.44$m，断面平均速度 $v_0=1$m/s，在进口管段，流体运动得到充分发展后，管中心最大速度 $u_{max}=1.85$m/s，进口压力为 2.2 个工程大气压，水的动力黏性系数 $\mu=1.005\text{N·s/m}^2$，运动黏性系数 $\nu=1.141\times10^{-6}\text{m}^2/\text{s}$，确定其进口段长度 l、理想与实际流体运动分界面 $\delta(x,\ y,\ z)$。

解　矩形管道水力半径：

$$R=\frac{A}{x}=\frac{AB}{A+B}=\frac{0.282}{0.22+0.32}=0.522\text{m}$$

管道流动流态，计算其雷诺数 Re：

$$Re=\frac{v_0R}{\nu}=\frac{1\times0.522\times10^6}{1.141}=4.57\times10^5$$

应用式(7-4-34)计算进口段长度 l：

$$l=\left[\frac{3(u_{\max}^2-v_0^2)}{4\nu u_{\max}\left(\frac{1}{H}+\frac{1}{B}\right)}\right]^{\frac{1}{3}}H=\left[\frac{3(1.85^2-1^2)\times 10^6}{4\times 1.141\times 1.85\left(\frac{1}{0.32}+\frac{1}{0.22}\right)}\right]^{\frac{1}{3}}0.32$$

$$=\left[\frac{2.423\times 10^6\times 0.75}{1.141\times 1.85(4.545+3.125)}\right]^{\frac{1}{3}}0.32=\left(\frac{1.819\times 10^6}{16.19}\right)^{\frac{1}{3}}0.32$$

$$=48.251\times 0.32=15.44\text{m}$$

$L=48.251$，计算理想与实际流体运动分界面，依式(7-4-43)与式(7-4-44)，例如 $X=10$，代入式(7-4-43)与式(7-4-44)

$$\delta_z=\frac{2U_{\max}}{U_{\max}-1}\left\{\frac{\frac{X}{L}(U_{\max}-1)}{2\left[1-\frac{X}{L}(1-U_{\max})\right]}\right\}H=\frac{2\times 1.85}{1.85-1}\left\{\frac{\frac{10}{48.251}(1.85-1)}{2\left[1-\frac{10}{48.251}(1-1.85)\right]}\right\}0.32$$

$$=\frac{3.70}{0.85}\left\{\frac{0.207\times 0.85}{2\times 1.176}\right\}0.32=4.353\times 0.075\times 0.32=0.104\text{m}$$

$$\delta_{zB}=0.425\times\delta_z=0.425\times 0.104=0.044\text{m}$$

同理，其他计算均列入表 7-14 中。

$$\delta_y=4.353\times 0.075\times B=0.0326\times 0.22=0.0712\text{m}$$

$$\delta_{yB}=0.425\times\delta_y=0.425\times 0.712=0.0303\text{m}$$

其他计算结果均列表 7-14 内。

表 7-14　　理想与实际流体分界面、边层流界面

X	10	20	30	40	48.251
$\frac{X}{L}$	0.207	0.414	0.622	0.829	1.000
$\delta_z(x)$/m	0.104	0.181	0.241	0.288	0.320
$\delta_{zB}(x)$/m	0.044	0.077	0.102	0.122	0.136
$\delta_y(x)$/m	0.071	0.125	0.166	0.198	0.220
$\delta_{yB}(x)$/m	0.030	0.053	0.071	0.084	0.094

将表 7-14 中理想与实际流体分界线与边层流界面绘成曲线（图 7-15）。

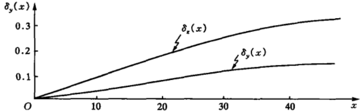


图 7-15　$y=0$ 理想与实际流体分界线以及边层流分界线

例 7-4　继续例 7-3，续作 $X=30$，$Z=70$ 时断面上速度分布以及进口段终点，即流动得到充分发展处的速度。

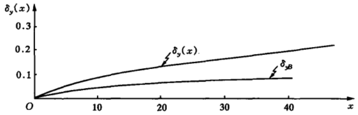


图 7-16　$z=H$，理想与实际流体分界线，边层流分界线

首先计算 $X=30$ 处断面上速度分布，由式(7-4-27)′

$$u=u_{\max}\left(\frac{X}{L}\right)(2Y-Y^2)(2Z-Z^2) \tag{7-4-27$'$}$$

得

$$\begin{aligned} u &= 1.85\left(\frac{30}{48.251}\right)(2Y-Y^2)(2Z-Z^2)=1.85\times0.622(2Y-Y^2)(2Z-Z^2) \\ &= 1.151(2Y-Y^2)(2Z-Z^2) \end{aligned} \tag{a}$$

式(a)是 $X=30$ 处断面上速度分布。计算必须分为 y 方向与 z 方向来计算，因为绘速度分布图时要分 xOz 平面与 xOy 平面。

对 $X=30$ 处，xOz 平面上速度分布计算公式为 $Y=1$，则有

$$u_z=1.151(2Z-Z^2) \tag{b}$$

$X=30$，$\delta_z(x)=0.241$，则式(b)为

$$u_z=1.151\left[2\times\frac{z}{0.241}-\left(\frac{z}{0.241}\right)^2\right] \tag{c}$$

式(c)是具体计算 $X=30$，xOz 断面速度公式。设 $z=0.05\text{m}$，则

$$u_z=1.151\left[2\,\frac{0.05}{0.241}-\left(\frac{0.05}{0.241}\right)^2\right]=1.151(2\times0.207-0.043)=1.151\times0.372=0.428\text{m/s}$$

$z=0.1\text{m}$

$$u_z=1.151\left[2\,\frac{0.1}{0.241}-\left(\frac{0.1}{0.241}\right)^2\right]=1.151(2\times0.415-0.172)=1.151\times0.658=0.757\text{m/s}$$

$z=0.15\text{m}$

$$u_z=1.151\left[2\,\frac{0.15}{0.241}-\left(\frac{0.15}{0.241}\right)^2\right]=1.151(2\times0.622-0.387)=1.151\times0.857=0.986\text{m/s}$$

$z=0.2\text{m}$　　　　$u_z=1.108\text{m/s}$

$z=0.241\text{m}$　　　　$u_z=1.151\text{m/s}$

其他计算方法相同，所有结果均列入表 7-15 中。

表 7-15　　过 $Y=1$ 的 xOz 平面各断面速度分布

X	30		40		48.251	
$\frac{X}{L}$	0.622		0.829		1.000	
$\delta_z(x)/\mathrm{m}$	0.241		0.288		0.32	
结果	z/m	$u_x/(\mathrm{m/s})$	z/m	$u_x/(\mathrm{m/s})$	z/m	$u_x/(\mathrm{m/s})$
	0.05	0.428	0.05	0.488	0.05	0.533
	0.10	0.757	0.10	0.879	0.10	0.977
	0.15	0.986	0.15	1.188	0.15	1.328
	0.20	1.118	0.20	1.390	0.20	1.589
	0.241	1.151	0.25	1.506	0.25	1.761
			0.288	1.534	0.30	1.844
					0.32	1.850

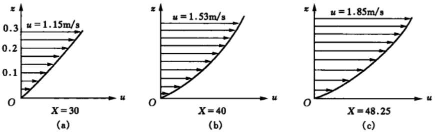


图 7-17　xOz 平面各断面速度分布

xOy 平面上 $Z=1$，各断面速度分布计算的基本公式

$$u_y = 1.85\left(\frac{X}{L}\right)(2Y - Y^2) \tag{d}$$

当 $X=40$ 时，该断面速度计算公式为

$$u_y = 1.85\left(\frac{40}{18.251}\right)(2Y - Y^2) = 1.85\times 0.829(2Y - Y^2) = 1.534(2Y - Y^2) \tag{e}$$

例如 $y=0.05\mathrm{m}$，其速度为

$$u_y = 1.534\left[2\times\frac{0.05}{0.198} - \left(\frac{0.05}{0.198}\right)^2\right] = 1.534(2\times 0.252 - 0.064) = 1.534\times 0.189 = 0.692\mathrm{m/s}$$

$y=0.1\mathrm{m}$，则其速度为

$$u_y = 1.534\left[2\times\frac{0.1}{0.198} - \left(\frac{0.1}{0.198}\right)^2\right] = 1.534(2\times 0.505 - 0.255) = 1.534\times 0.755 = 1.058\mathrm{m/s}$$

$y=0.15\mathrm{m}$，则其速度为

$$u_y = 1.534\left[2\times\frac{0.15}{0.198} - \left(\frac{0.15}{0.198}\right)^2\right] = 1.534(2\times 0.758 - 0.574) = 1.534\times 0.942 = 1.445\mathrm{m/s}$$

其他计算方法相同，结果均列入表 7-16。

表 7-16　　xOy 平面各断面速度分布($Z=1$)

X	30		40		48.251	
$\frac{X}{L}$	0.622		0.829		1.000	
$\delta_y(x)/\mathrm{m}$	0.166		0.198		0.220	
结果	y/m	$u_y/(\mathrm{m/s})$	y/m	$u_y/(\mathrm{m/s})$	y/m	$u_y/(\mathrm{m/s})$
	0.05	0.588	0.05	0.676	0.05	0.744
	0.10	0.968	0.10	1.158	0.10	1.301
	0.15	1.141	0.15	1.445	0.15	1.475
	0.166	1.151	0.198	1.534	0.20	1.835
					0.22	1.850

将表 7-16 中各断面速度绘出（图 7-18）。

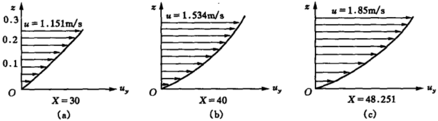


图 7-18　xOy 平面各断面上速度分布

例 7-5　继作例 7-3，求 $X=30$，$X=40$，$X=48.251$ 各个断面上的断面平均速度 v_0，涡旋强度 ω_y，ω_z，涡旋横向运动速度 u_y^*，u_z^*，准定常时间 t_K，涡旋直径 d_{sy}，d_{sz}，涡旋体积分数 φ。

解　计算断面平均速度 v_0，应用式(7-4-49)，当 $X=30$ 时

$$v_0=\frac{4}{9}u_{\max}\left(\frac{X}{L}\right)=0.444\times1.85\times0.622=0.511\mathrm{m/s}$$

其他计算方法相同，结果均列入表 7-17 中。

计算涡旋强度 ω_y，ω_z，应用式(7-4-58)与式(7-4-59)。例如当 $X=30$ 时

$$\omega_y=0.385\frac{u_{\max}}{\delta_y}\left(\frac{X}{L}\right)=0.385\times\frac{1.85}{\delta_y}\times0.622=\frac{0.712\times0.622}{0.166}=\frac{0.443}{0.166}=2.668\mathrm{rad/s}$$

将弧度成度：

$$\omega_y^0=2.668\times53.2=142°/\mathrm{s}$$

应用式(7-4-59)计算 ω_z：

$$\omega_z=0.385\times\frac{1.85}{\delta_z}\times0.622=\frac{0.443}{0.241}=1.838\mathrm{rad/s}$$

其他同法计算结果列入表 7-17 内。

计算涡旋横向运动速度 u_y^*，u_z^*。由式(7-4-60)将其改写成

表 7-17　　例 7-5 问题计算结果

X	30		40		48.251	
$\frac{X}{L}$	0.622		0.829		1.00	
v_0/(m/s)	0.511		0.681		0.821	
理想与实际流体分界/m	δ_y	δ_z	δ_y	δ_z	δ_y	δ_z
	0.166	0.241	0.198	0.288	0.220	0.320
涡旋强度，弧度与角度	ω_y	ω_z	ω_y	ω_z	ω_y	ω_z
	2.668 142°	1.838 97.78°	2.982 158.8°	2.049 109°	3.236 172°	2.225 118°
边层流距 δ_y，δ_z/m	$\delta_y-\delta_{yB}$	$\delta_z-\delta_{zB}$	$\delta_y-\delta_{yB}$	$\delta_z-\delta_{zB}$	$\delta_y-\delta_{yB}$	$\delta_z-\delta_{zB}$
	0.095	0.139	0.114	0.166	0.126	0.184
涡旋横向速度/(m/s)	u_y^*	u_z^*	u_y^*	u_z^*	u_y^*	u_z^*
	0.226	0.189	0.241	0.199	0.249	0.206
准定常时间/s	t_y	t_z	t_y	t_z	t_y	t_z
	0.42	0.74	0.48	0.83	0.506	0.903
涡旋直径/m	d_y	d_z	d_y	d_z	d_y	d_z
	0.004	0.005	0.0039	0.0047	0.0038	0.0043
涡旋体积分数	0.057		0.051		0.037	

$$u_y^* = \sqrt{\frac{12\nu(\delta_y-\delta_{yB})\omega_y^2}{v_0}} \tag{7-4-59$'$}$$

式(7-4-60)改为

$$u_z^* = \sqrt{\frac{12\nu(\delta_z-\delta_{zB})\omega_z^2}{v_0}} \tag{7-4-60$'$}$$

例如当 $X=30$，计算 u_y^*，u_z^* 如下：

$$u_y^* = \sqrt{\frac{12\times1.141\times10^{-6}(0.166-0.071)142^2}{0.511}} = \sqrt{\frac{0.0262}{0.511}} = \sqrt{0.0512} = 0.226\text{m/s}$$

$$u_z^* = \sqrt{\frac{12\times1.141\times10^{-6}(0.241-0.102)97.8^2}{0.511}} = \sqrt{\frac{0.0182}{0.511}} = \sqrt{0.036} = 0.189\text{m/s}$$

其他计算结果均列入表 7-17 内。

计算准定常时间 t_y，t_z，当 $X=30$ 时，由下式计算：

$$t_y = \frac{\delta_y-\delta_{yB}}{u_y^*} = \frac{0.166-0.071}{0.226} = 0.42\text{s}$$

$$t_z = \frac{\delta_z-\delta_{zB}}{u_z^*} = \frac{0.241-0.102}{0.189} = 0.735\text{s}$$

同理，其他计算结果均列入表 7-17 内。

计算涡旋直径 d_y，d_z，当 $X=30$ 时，由下式计算：

$$d_y=\frac{2}{100}\sqrt{\frac{1.141}{\omega_y\times 10}}=0.02\sqrt{\frac{1.141}{2.668\times 10}}=0.004\text{m}$$

$$d_z=\frac{2}{100}\sqrt{\frac{1.141}{\omega_z\times 10}}=0.02\sqrt{\frac{1.141}{1.838\times 10}}=0.005\text{m}$$

其他计算结果均列入表 7-17 内。

计算涡旋体积分数 φ，由式(7-4-62)进行。

$$\varphi=\frac{[\text{秒}](Hu_y^*+Bu_z^*)}{48\delta_y\delta_z} \tag{7-4-61$'$}$$

例如 $X=30$，则有

$$\varphi=\frac{0.32\times 0.226+0.22\times 0.189}{48\times 0.166\times 0.241}=\frac{0.072+0.042}{1.92}=\frac{0.114}{1.92}=0.059\approx 0.06$$

又如，当 $X=40$ 时

$$\varphi=\frac{0.32\times 0.241+0.22\times 0.199}{48\times 0.198\times 0.288}=\frac{0.21}{2.737}=0.044$$

以上是应用式(7-4-34)对例 7-3 问题计算出其进口长度 l。然后依此，进行一系列的计算得出对应的各种运动参数值，其中涡旋体积分数 φ 与准定常时间 t，还为应用式(7-4-40)得到方便。

式(7-4-34)是单纯考虑黏性剪应力得出的，从理论上讲，它是不完备的。只有同时纳入黏性剪应力与湍流附加剪应力得出的公式式(7-4-40)才是最合理。现在应用它对例 7-3 题进行计算诸多问题。

例 7-6　对例 7-3 题应用式(7-4-40)进行计算。

解　首先应用式(7-4-40)计算其进口段长度：

$$L=\left\{\frac{0.5(u_{\max}^2-v_0^2)}{\nu\left[0.666u_{\max}\left(\frac{1}{H}-\frac{1}{B}\right)+0.356\varphi^{\frac{2}{3}}tu_{\max}^2\left(\frac{1}{H^2}+\frac{1}{B^2}\right)\right]}\right\}^{\frac{1}{3}}$$

$$=\sqrt[3]{\frac{0.5(1.85^2-1^2)\times 10^6}{1.141\left\{0.666\times 1.85\left(\frac{1}{0.32}+\frac{1}{0.22}\right)+0.356\times 0.04\times 0.6(1.85)^2\left[\left(\frac{1}{0.32}\right)^2+\left(\frac{1}{0.22}\right)^2\right]\right\}}}$$

$$=\sqrt[3]{\frac{1.211\times 10^6}{1.141[1.332(3.125+4.545)+0.0085\times 3.423(9.766+20.65)]}}$$

$$=\sqrt[3]{\frac{1.211\times 10^6}{1.141(1.232\times 7.67+0.029\times 30.423)}}=\sqrt[3]{\frac{1.211\times 10^6}{1.141(9.449+0.885)}}$$

$$=\sqrt[3]{\frac{1.211\times 10^6}{1.141\times 10.334}}=\sqrt[3]{\frac{1.211\times 10^6}{11.792}}=46.875\text{m}$$

进口段长度：

$$l=LH=46.875\times 0.32=14.97\text{m}$$

计算理想与实际流体运动分界面，即确定 δ_z，δ_y，依式(7-4-44)与式(7-4-45)。例如当 $X=10$ 时，则有

$$\delta_z=\frac{2U_{max}}{U_{max}-1}\left\{\frac{\frac{X}{L}(U_{max}-1)}{2\left[1-\frac{X}{L}(1-U_{max})\right]}\right\}H=\frac{2\times1.85}{1.85-1}\left\{\frac{\frac{10}{46.875}(1.85-1)}{2\left[1-\frac{10}{46.875}(1-1.85)\right]}\right\}0.32$$

$$=\frac{3.70}{0.85}\left\{\frac{0.213\times0.85}{2[1-0.213\times(-0.85)]}\right\}0.32=4.353\left\{\frac{0.181}{2\times1.181}\right\}0.32$$

$$=4.353\times0.026\times0.32=0.107\text{m}$$

$$\delta_{zB}=0.425\delta_z=0.425\times0.107=0.045\text{m}$$

$$\delta_y=\frac{2U_{max}}{U_{max}-1}\left\{\frac{\frac{X}{L}(U_{max}-1)}{2\left[1-\frac{X}{L}(1-U_{max})\right]}\right\}B=4.353\times0.076\times0.22=0.331\times0.22=0.073\text{m}$$

$$\delta_{yB}=0.425\delta_y=0.425\times0.073=0.031\text{m}$$

其他计算结果均列入表 7-18 内。

表 7-18　　理想与实际流体分界面，边层流界面

X	10	20	30	40	46.875
$\frac{X}{L}$	0.213	0.185	0.245	0.293	0.320
$\delta_z(x)$/m	0.107	0.185	0.245	0.293	0.320
$\delta_{zB}(x)$/m	0.045	0.787	0.104	0.124	0.136
$\delta_y(x)$/m	0.073	0.127	0.168	0.201	0.220
$\delta_{yB}(x)$/m	0.031	0.540	0.072	0.086	0.094

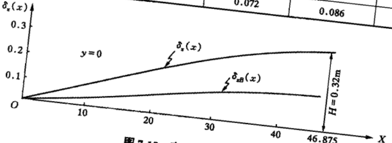


图 7-19　表 7-18 数据图

例 7-7　续作例 7-3，绘 $X=30$，$X=40$，$X=46.875$ 处断面速度分布。

由式(7-4-28)，当 $X=30$ 时，则为

$$u=1.85\frac{30}{46.875}(2Y-Y^2)(2Z-Z^2)=1.85\times0.64(2Y-Y^2)(2Z-Z^2)$$

$$=1.184(2Y-Y^2)(2Z-Z^2)$$

当 $Y=1$ 时，在 xOz 平面上

$$u_z=1.184(2Z-Z^2) \tag{f}$$

$$u_z = 1.184\left[2\frac{z}{0.245} - \left(\frac{z}{0.245}\right)^2\right] \tag{g}$$

当 $\delta_z = 0.245\text{m}$ 时，$z = 0.05\text{m}$

$$u_z = 1.184\left[2\frac{0.05}{0.245} - \left(\frac{0.05}{0.245}\right)^2\right] = 1.184(2\times0.204 - 0.042) = 0.422\text{m/s}$$

$z = 0.1\text{m}$

$$u_z = 1.184\left[2\frac{0.1}{0.245} - \left(\frac{0.1}{0.245}\right)^2\right] = 1.184(2\times0.408 - 0.167) = 1.184\times0.649 = 0.768\text{m/s}$$

$z = 0.15\text{m}$

$$u_z = 1.184\left[2\frac{0.15}{0.245} - \left(\frac{0.15}{0.245}\right)^2\right] = 1.184(2\times0.612 - 0.375) = 1.184\times0.849 = 1.005\text{m/s}$$

$z = 0.2\text{m}$

$$u_z = 1.184\left[2\frac{0.2}{0.245} - \left(\frac{0.2}{0.245}\right)^2\right] = 1.184(2\times0.816 - 0.666) = 1.184\times0.966 = 1.148\text{m/s}$$

$z = 0.245\text{m}$，$u_z = 1.184\text{m/s}$。

当 $X = 40$，$X = 46.875$ 时，以同法计算，其结果一同列入表 7-19 内。

表 7-19　　$Y = 1$，xOz 平面内各断面速度分布

X	30		40		46.875	
$\frac{X}{L}$	0.64		0.853		1.000	
$\delta_z(x)$/m	0.245		0.293		0.32	
结果	z/m	u_z/(m/s)	z/m	u_z/(m/s)	z/m	u_z/(m/s)
	0.05	0.422	0.05	0.494	0.05	0.533
	0.10	0.768	0.10	0.893	0.10	0.977
	0.15	1.005	0.15	1.228	0.15	1.328
	0.20	1.144	0.20	1.345	0.20	1.589
	0.245	1.184	0.25	1.543	0.25	1.761
			0.293	1.578	0.30	1.844
					0.32	1.850

当 $Z = 1$ 时，在 xOy 平面上的计算公式为

$$u_y = 1.184\left[2\frac{y}{\delta_y} - \left(\frac{y}{\delta_y}\right)^2\right] \tag{h}$$

当 $\delta_y = 0.168\text{m}$ 时

$$u_y = 1.184\left[2\frac{y}{0.168} - \left(\frac{y}{0.168}\right)^2\right] \tag{i}$$

当 $y = 0.05\text{m}$ 时

$$u_y = 1.184\left[2\frac{0.05}{0.168} - \left(\frac{0.05}{0.168}\right)^2\right] = 1.184(2\times0.298 - 0.089) = 1.184\times0.507 = 0.600\text{m/s}$$

$y = 0.1\text{m}$

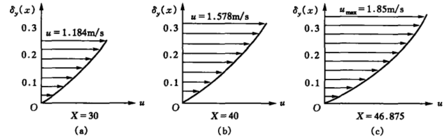


图 7-20　$Y=1$，xOz 平面上各断面速度分布

$$u_y=1.184\left[2\frac{0.1}{0.168}-\left(\frac{0.1}{0.168}\right)^2\right]=1.184(2\times0.592-0.354)=1.184\times0.83=0.983\text{m/s}$$

$y=0.15$m

$$u_y=1.184\left[2\frac{0.15}{0.168}-\left(\frac{0.15}{0.168}\right)^2\right]=1.184(2\times0.893-0.791)=1.184\times0.989=1.171\text{m/s}$$

$y=0.168$m，$u_y=1.184$m/s。其他计算结果均列入表 7-20 内。

表 7-20　　$Z=1$，xOy 平面上各断面速度分布

X	30		40		46.875	
$\frac{X}{L}$	0.64		0.853		1.000	
$\delta_y(x)$/m	0.168		0.201		0.220	
结果	y/m	u_y/(m/s)	y/m	u_y/(m/s)	y/m	u_y/(m/s)
	0.05	0.600	0.05	0.688	0.05	0.744
	0.10	0.983	0.10	1.180	0.10	1.300
	0.15	1.171	0.15	1.475	0.15	1.526
	0.168	1.184	0.20	1.576	0.20	1.840
			0.201	1.578	0.22	1.850

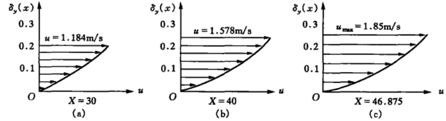


图 7-21　$Z=1$，xOy 平面上各断面速度分布

例 7-8　续作例 7-3，求 $X=30$，$X=40$，$X=46.875$ 各个断面上平均速度 v_0，涡旋强度 ω_y，ω_z，涡旋横向运动速度 u_y^*，u_z^*，准定常流时间 t_y，t_z，涡旋直径 d_y，d_z，涡旋体积分数 φ。

解　计算断面平均速度 v_0，应用式(7-4-49)，当 $X=30$ 时

$$v_0=\frac{4}{9}u_{\max}\left(\frac{X}{L}\right)=0.444\times1.85\times0.64=0.526\text{m/s}$$

其他计算方法相同，结果均列入表 7-21 中。

计算涡旋强度 ω_y，ω_z，应用式(7-4-57)与式(7-4-58)。当 $X=30$ 时

$$\omega_y=0.385\times\frac{u_{\max}}{\delta_y}\left(\frac{X}{L}\right)=0.385\times\frac{1.85}{0.168}\times\frac{30}{46.875}=0.385\times11.012\times0.64=2.713\text{rad/s}$$

对应的角速度

$$\omega_y^0=2.713\times53.2=144.349°/\text{s}$$

$$\omega_z=0.385\times\frac{u_{\max}}{\delta_z}\left(\frac{X}{L}\right)=0.385\times\frac{1.85}{0.245}\times0.64=0.385\times7.551\times0.64=1.86\text{rad/s}$$

对应的角速度 ω_y^0：

$$\omega_y^0=1.86\times53.2=98.952°/\text{s}$$

其他计算结果均列入表 7-21 内。

计算涡旋横向运动速度 u_y^*，u_z^*。由式(7-4-60)，将其改写成

$$u_y^*=\sqrt{\frac{12\nu(\delta_y-\delta_{yB})\omega_y^2}{v_0}}\tag{7-4-59)''$$

$$u_z^*=\sqrt{\frac{12\nu(\delta_z-\delta_{zB})\omega_z^2}{v_0}}\tag{7-4-60)''$$

例如当 $X=30$，计算 u_y^*，u_z^*：

$$u_y^*=\sqrt{\frac{12\times1.141\times10^{-6}\times(0.168-0.072)\times144^2}{0.526}}=\sqrt{\frac{13.692\times10^{-6}\times0.086\times20736}{0.526}}$$

$$=\sqrt{\frac{0.0244}{0.526}}=\sqrt{0.046}=0.215\text{m/s}$$

$$u_z^*=\sqrt{\frac{13.692\times10^{-6}\times(0.245-0.104)\times99^2}{0.526}}=\sqrt{\frac{13.692\times10^{-6}\times0.141\times9801}{0.526}}$$

$$=\sqrt{\frac{0.018921}{0.526}}=\sqrt{0.0359}=0.1896\approx0.19\text{m/s}$$

其他计算结果均列入表 7-21 内。

计算准定常流时间 t_y，t_z。当 $X=30$ 时，由下式计算：

$$t_y=\frac{\delta_y-\delta_{yB}}{u_y^*}=\frac{0.168-0.072}{0.215}=0.4\text{s}$$

$$t_z=\frac{\delta_z-\delta_{zB}}{u_z^*}=\frac{0.245-0.104}{0.19}=0.74\text{s}$$

其他计算结果均列入表 7-21 内。

计算涡旋直径 d_y，d_z。当 $X=30$ 时

$$d_y=\frac{2}{100}\sqrt{\frac{1.141}{\omega_y\times 10}}=0.02\sqrt{\frac{1.141}{2.713\times 10}}=0.0041\text{m}$$

$$d_z=\frac{2}{100}\sqrt{\frac{1.141}{\omega_z\times 10}}=0.02\sqrt{\frac{1.141}{1.86\times 10}}=0.005\text{m}$$

其他计算结果均列入表 7-21 内。

计算涡旋体积分数 φ。由式(7-4-61)′

$$\varphi=\frac{[秒](Hu_y^*+Bu_z^*)}{48\delta_y\delta_z}$$

例如 $X=30$

$$\varphi=\frac{0.32\times 0.215+0.22\times 0.19}{48\times 0.168\times 0.245}=\frac{0.043+0.0418}{1.976}=\frac{0.085}{1.976}=0.043$$

其他计算结果均列入表 7-21 内。

表 7-21 例 7-8 计算结果

X	30		40		46.875	
$\frac{X}{L}$	0.64		0.853		1.0	
v_0/(m/s)	0.526		0.700		0.821	
理想与实际流体分界/m	δ_y	δ_z	δ_y	δ_z	δ_y	δ_z
	0.168	0.245	0.201	0.293	0.22	0.32
涡旋强度弧度与角度	ω_y	ω_z	ω_y	ω_z	ω_y	ω_z
	2.713 144°	1.86 99°	3.023 160.8°	2.073 110.3°	3.913 208.2°	2.226 118.4°
边层流距 δ_y, δ_z/m	$\delta_y-\delta_{yB}$	$\delta_z-\delta_{zB}$	$\delta_y-\delta_{yB}$	$\delta_z-\delta_{zB}$	$\delta_y-\delta_{yB}$	$\delta_z-\delta_{zB}$
	0.086	0.141	0.115	0.169	0.126	0.184
涡旋横向速度/(m/s)	u_y^*	u_z^*	u_y^*	u_z^*	u_y^*	u_z^*
	0.215	0.19	0.241	0.200	0.302	0.207
准定常流时间/s	t_y	t_z	t_y	t_z	t_y	t_z
	0.4	0.74	0.477	0.845	0.417	0.889
涡旋直径/m	d_y	d_z	d_y	d_z	d_y	d_z
	0.004	0.005	0.0039	0.0047	0.0034	0.0045
涡旋体积分数	0.043		0.043		0.03	

第 8 章　可压缩湍流管道流动

本章在定常流、忽略质量力与分子能量条件下，讨论可压缩湍流管道流动。主要讨论可压缩流体是否存在进口段问题，以及等质量流与变质量流管段的流动问题。

8.1　可压缩湍流圆形管道进口段分析

以前，管道进口段的定义为：流体以均匀速度进入管道，在壁面摩擦力的作用下，断面速度分布沿流程逐渐变化，均匀的分布速度逐渐加大，范围逐渐缩小，受壁面影响，实际流体速度分布沿程逐渐加大，直到断面上速度分布均为实际流而理想流体消失，这段管道称之为进口段。

由于研究的是有温度变化可压缩流体的运动，它在进口段应按气体状态方程变化。可压缩流体进口段的流动情况与坐标系、原点的选定如图 8-1 所示。

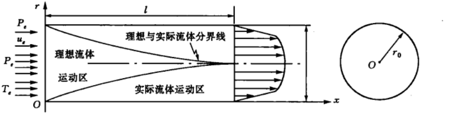


图 8-1　圆管道进口段流动示意图

研究是以实际流体运动为对象的，而把理想流体运动作为边界条件加以利用。只要找到实际流体运动速度分布，则理想流体运动速度分布随之而解决。

8.1.1　运动控制方程与边界条件

结合所研究问题，首先选定物理方程与边界条件，然后将其转化为无因次方程与边界条件，每种情况又分为实际流体与理想流体加以列出。

8.1.1.1　物理方程与边界条件

现分实际流体运动区与理想流体运动区分别加以介绍。

(1) 实际流体运动区

控制实际流体运动的方程有 4 个：动量方程，能量方程，连续性方程，气体动态方程。

① 动量微分方程。依式(4-5-16)，结合本问题为

$$\frac{\nu}{2}\frac{\partial p}{\rho\partial x}\frac{\partial u}{\partial y}+u\frac{\partial u}{\partial x}+\frac{\nu}{2}\left(\frac{1}{u}\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left[\frac{4}{3}\frac{\partial^2 u}{\partial x^2}+(1-\varphi^{\frac{2}{3}})\frac{\partial^2 u}{\partial y^2}+6\varphi^{\frac{2}{3}}t\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right] \tag{8-1-1}$$

② 能量微分方程。依式(4-5-41)，结合本问题，忽略分子能量 e，认为涡旋温度与连续相一样，则为

$$u^2\frac{\partial u}{\partial x}-\frac{u^3}{2}\frac{\partial \rho}{\partial x}+\frac{\varphi}{2}\nu u\frac{\partial^2 u}{\partial x\partial y}+\frac{\nu}{2}\frac{\partial u}{\partial y}\left(\frac{u}{\rho}\frac{\partial \rho}{\partial x}+\frac{\partial u}{\partial x}\right)$$
$$=\frac{\lambda}{\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+u\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}\right)-\frac{1}{2}\varphi^{\frac{2}{3}}\nu\frac{\partial u}{u\partial y}\left[-\frac{\partial p}{\rho\partial x}+\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}+2\nu t\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right] \tag{8-1-2}$$

③ 连续性方程：

$$\rho_e u_e=\rho u=\text{常数} \tag{8-1-3}$$

$$\frac{\partial p}{\rho\partial x}=-\frac{1}{u}\frac{\partial u}{\partial x} \tag{8-1-4}$$

④ 气体状态方程：

$$\frac{p}{\rho}=RT \tag{8-1-5}$$

$$\frac{\partial p}{\partial x}=R\left(T\frac{\partial p}{\partial x}+\rho\frac{\partial T}{\partial x}\right) \tag{8-1-6}$$

边界条件：

$$u(x,\ y)\big|_{x=0}=0 \tag{8-1-7}$$

$$u(x,\ y)\big|_{y=\delta(x)}=u(x) \tag{8-1-8}$$

$$\left.\frac{\partial u(x,\ y)}{\partial y}\right|_{y=\delta(x)}=0 \tag{8-1-9}$$

$$\frac{\partial^2 u(x,\ y)}{\partial y^2}<0 \tag{8-1-10}$$

$$p(x,\ y)\big|_{x=0}=p_e \tag{8-1-11}$$

$$T(x,\ y)\big|_{x=0}=T_e \tag{8-1-12}$$

$$\rho(x,\ y)\big|_{x=0}=\rho_e \tag{8-1-13}$$

(2) 理想流区的控制方程与边界条件

动量微分方程：

$$u_0\frac{\mathrm{d}u_0}{\mathrm{d}x}=-\frac{\mathrm{d}p}{\rho\mathrm{d}x} \tag{8-1-14}$$

$$u_0(x)\big|_{x=0}=u_e \tag{8-1-15}$$

8.1.1.2　无因次运动控制方程与边界条件

为了运用数学分析，必须将物理方程与边界条件转化为无因次方程与边界条件。为此取

$$\frac{u}{u_e}=U,\ \frac{y}{\delta(x)}=Y,\ \frac{x}{r_0}=X,\ \frac{\nu}{\nu_e}=\nu_0,\ \frac{p}{p_e}=p_0,\ \frac{\rho}{\rho_e}=\rho_0,\ \frac{T}{T_e}=T_0$$

式中：u_0——管道进口处断面上均匀速度；

$\delta(x)$——理想流体与实际流体分界线；

r_0——管道半径；

p_e——管道进口处气体压力；

ρ_e——管道进口处气体密度；

T_e——管道进口处气体温度。

将以上比值代入物理方程与边界条件中去。首先代入式(8-1-1)，然后代入能量方程，则有：

$$\frac{\nu_e}{2}\frac{u_e}{r_0}\frac{\nu_e}{\delta(x)}\frac{\partial\rho}{\partial X}\frac{\partial U}{\partial Y}+\frac{u_e^2}{r_0}U\frac{\partial U}{\partial X}+\frac{\nu_e}{2}\frac{u_e}{\delta(x)}\frac{\nu_0}{r_0}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}$$

$$=-\frac{p_0}{\rho_0 r_0}\frac{\partial p_0}{\rho_0\partial X}+\frac{4}{3}\frac{\nu_e u_e}{r_0^2}\nu_0\frac{\partial^2 U}{\partial X^2}+\frac{\nu_e u_e(1-\varphi^{\frac{2}{3}})}{\delta(x)^2}\nu_0\frac{\partial^2 U}{\partial Y^2}+\frac{6\nu_e u_e^2}{\delta(x)^3}t\nu_0\varphi^{\frac{2}{3}}\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\qquad(8\text{-}1\text{-}1)'$$

除以$\frac{\nu_e u_e}{r_0^2}$，则有

$$\frac{1}{2}\frac{r_0}{\delta(x)}\frac{\nu_0}{\rho_0}\frac{\partial\rho_0}{\partial X}\frac{\partial U}{\partial Y}+ReU\frac{\partial U}{\partial X}+\frac{r_0}{\delta(x)}\nu_0\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}$$

$$=-K_1\frac{1}{\rho_0}\frac{\partial p_0}{\partial X}+\nu_0\left[\frac{4}{3}\frac{\partial^2 U}{\partial X^2}+(1-\varphi^{\frac{2}{3}})\left(\frac{r_0}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial Y^2}+K_2\varphi^{\frac{2}{3}}\left(\frac{r_0}{\delta(x)}\right)^2\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right]$$

(8-1-16)

式中：

$$Re=\frac{u_e r_0}{\nu_e}\qquad(8\text{-}1\text{-}17)$$

$$K_1=\frac{p_0 r_0}{\rho_e\nu_e u_e}\qquad(8\text{-}1\text{-}18)$$

$$K_2=\frac{6tu_e}{\delta(x)}\qquad(8\text{-}1\text{-}19)$$

无因次能量方程：由式(8-1-2)可得

$$\frac{u_e^3}{r_0}U^2\frac{\partial U}{\partial X}-\frac{u_e^3}{2r_0}\frac{U^3}{\rho_0}\frac{\partial\rho_0}{\partial X}+\frac{\varphi}{2}\frac{\nu_0}{r_0}\frac{u_e^2}{\delta(x)}\nu_0 U\frac{\partial^2 U}{\partial X\partial Y}+\frac{1}{2}\frac{\nu_e}{r_0}\frac{u_e^2}{\delta(x)}\nu_0\frac{\partial U}{\partial Y}\left(\frac{U}{\rho_0}\frac{\partial p_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$=\frac{\lambda_e T_e\lambda_0}{\rho_e r_0\delta(x)\rho_0}\left(\frac{\partial^2 T}{\partial X^2}+\frac{\partial^2 T}{\partial Y^2}\right)+U\left(-\frac{p_e}{\rho_e}\frac{u_e}{r_0}\frac{\partial p_0}{\rho_0\partial X}+\frac{\nu_e u_e^2}{r_0\delta(x)}\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}\right)$$

$$-\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0}{U}\frac{\partial U}{\partial Y}\left[-\frac{\nu_e p_e}{r_0\delta(x)\rho_e}\frac{\partial p_0}{\rho_0\partial X}+\frac{4}{3}\frac{\nu_e u_e}{\delta(x)r_0^2}\nu_0\frac{\partial^2 U}{\partial X^2}+\frac{2\nu_e^2 tu_e^2}{\delta(x)r_0^3}\nu_0\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right]\qquad(8\text{-}1\text{-}2)'$$

全式除以$\frac{\nu_e u_e^2}{r_0^2}$，则有

$$ReU^3\frac{\partial U}{\partial X}-\frac{1}{2}ReU^3\frac{\partial\rho_0}{\rho_0\partial X}+\frac{\varphi}{2}\nu_0\frac{r_0}{\delta(x)}U\frac{\partial U}{\partial X}\frac{\partial U}{\partial Y}-\frac{1}{2}\frac{r_0}{\delta(x)}\nu_0\frac{\partial U}{\partial Y}\left(\frac{U}{\rho_0}\frac{\partial\rho_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$= \frac{K_3\lambda_0}{\rho_0}\frac{r_0}{\delta(x)}\left(\frac{\partial^2 T_0}{\partial X^2}+\frac{\partial^2 T_0}{\partial Y^2}\right)+U\left(-K_4\frac{\partial p_0}{\rho_0\partial X}+\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}\right)$$

$$-\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0\partial U}{U\partial Y}\left(-K_5\frac{r_0}{\delta(x)}\frac{\partial p_0}{\rho_0\partial X}+\frac{4}{3}\frac{\nu_0}{R'_e}\frac{\partial^2 U}{\partial X^2}+K_6\nu_0\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right) \tag{8-1-20}$$

式中：

$$Re=\frac{u_e r_0}{\nu_e} \tag{8-1-21}$$

$$Re'=\frac{\delta(x)u_e}{\nu_e} \tag{8-1-22}$$

$$K_3=\frac{\lambda_e T_e}{\rho_e\nu_e u_e^2} \tag{8-1-23}$$

$$K_4=\frac{p_e r_0}{\nu_e u_e\rho_e} \tag{8-1-24}$$

$$K_5=\frac{p_e}{\rho_e u_e^2} \tag{8-1-25}$$

$$K_6=\frac{2t\nu_e}{\delta(x)r_0} \tag{8-1-26}$$

无因次气体状态方程：

$$\frac{p_0}{\rho_0}=R_0T_0 \tag{8-1-27}$$

$$\frac{\partial p_0}{\partial X}=T_0\frac{\partial\rho_0}{\partial X}+\rho_0\frac{\partial T_0}{\partial X} \tag{8-1-28}$$

无因次质量守恒方程：

$$1=\rho_0 U \tag{8-1-29}$$

$$\frac{1}{\rho_0}\frac{\partial\rho}{\partial X}=\frac{1}{U}\frac{\partial U}{\partial X} \tag{8-1-30}$$

无因次理想流体微分方程与边界条件：

$$U_0\frac{\mathrm{d}U_0}{\mathrm{d}X}=-K_5\frac{1}{\rho_0}\frac{\mathrm{d}p_0}{\mathrm{d}X} \tag{8-1-31}$$

$$U_0(X)\big|_{X=0}=1 \tag{8-1-32}$$

无因次实际流体运动边界条件：

$$U(X,\ Y)\big|_{X=0}=0 \tag{8-1-7$'$}$$

$$U(X,\ Y)\big|_{Y=1}=U(X) \tag{8-1-8$'$}$$

$$\left.\frac{\partial U(X,\ Y)}{\partial Y}\right|_{Y=1}=0 \tag{8-1-9$'$}$$

$$\frac{\partial^2 U(X,\ Y)}{\partial Y^2}<0 \tag{8-1-10$'$}$$

无因次理想流体运动边界条件：

$$U_0(X)\big|_{X=0}=1 \tag{8-1-15$'$}$$

无因次理想与实际流体共用边界条件：

$$P_0(X,\ Y)\big|_{X=0}=1 \tag{8-1-11$'$}$$

$$T_0(X,\ Y)\big|_{X=0}=1 \tag{8-1-12$'$}$$

$$\rho_0(X,\ Y)\big|_{X=0}=1 \tag{8-1-13$'$}$$

8.1.1.3　无因次综合运动控制方程

利用质量守恒，将式(8-1-16)式(8-1-20)中的密度转化速度，利用气体状态方程，将压力转化为速度与温度。动量方程为

$$\begin{aligned}&-\frac{1}{2}\frac{r_0}{\delta(x)}\frac{\partial U}{\partial X}\frac{\partial U}{\partial Y}+ReU\frac{\partial U}{\partial X}+\frac{r_0}{\delta(x)}\nu_0\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}\\&=K_1U\left(\frac{T_0}{U^2}\frac{\partial U}{\partial X}-\frac{1}{U}\frac{\partial T_0}{\partial X}\right)+\nu_0\left[\frac{4}{3}\frac{\partial^2 U}{\partial X^2}+(1-\varphi^{\frac{2}{3}})\left(\frac{r_0}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial Y^2}+K_2\varphi^{\frac{2}{3}}\left(\frac{r_0}{\delta(x)}\right)^2\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right]\end{aligned} \tag{8-1-16$'$}$$

能量方程为

$$\begin{aligned}&ReU^2\left(U+\frac{1}{2}\right)\frac{\partial U}{\partial X}+\frac{\varphi}{2}\nu_0\frac{r_0}{\delta(x)}U\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}\\&=K_3\frac{\lambda_0}{\rho_0}\frac{r_0}{\delta(x)}\left(\frac{\partial^2 T_0}{\partial X^2}+\frac{\partial^2 T_0}{\partial Y^2}\right)+\left(\frac{1}{2}\varphi^{\frac{2}{3}}K_5\nu_0\frac{\partial U}{\partial Y}\frac{r_0}{\delta(x)}-K_4U\right)\left(\frac{1}{U}\frac{\partial T_0}{\partial X}-\frac{T_0}{U^2}\frac{\partial U}{\partial X}\right)\\&\quad+\frac{4}{3}\nu_0U\frac{\partial^2 U}{\partial X^2}-\frac{1}{2}\varphi^{\frac{2}{3}}\frac{\nu_0}{U}\frac{\partial U}{\partial Y}\left(\frac{4}{3}\frac{\nu_0}{R'_e}\frac{\partial^2 U}{\partial X^2}+\frac{r_0}{\delta(x)}U\frac{\partial U}{\partial X}+K_6\nu_0\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right)\end{aligned} \tag{8-1-20$'$}$$

上两式是可压缩湍流进口段实际流区流体运动无因次控制微分方程，是求速度分布的基础。

8.1.2　可压缩湍流管道无进口段

不可压缩流体湍流管道进口段已在第 7 章讨论过，可压缩湍流管道是否存在进口段，本书无实验实测条件加以验证，只好以理论分析证明它不存在。在 8.1.1 节设想存在进口段，按其设想分析理想与实际流体分界线 $\delta(x)$ 的公式，分析结果 $\delta(x)$ 不存在。

8.1.2.1　进口段速度分布

(1) 理想流体运动区

根据边界条件为

$$U_0=(1+X)^{\alpha} \tag{8-1-33}$$

(2) 实际流体运动区

根据边界条件为

$$U=X^{\alpha}(2Y-Y^2) \tag{8-1-34}$$

8.1.2.2 利用质量流守恒推导 $\delta(x)$

(1) 总质量流

$$Q_{总}=\rho_e v_e \pi r_0^2 \tag{1}$$

(2) 理想流体运动区质量流

$$Q_{理}=\rho\pi[r_0-\delta(x)]v_e(1+X)^\alpha \tag{2}$$

(3) 实际流区质量流

$$Q_{实}=\rho 2\pi r_0 v_e X^\alpha\int_0^{\delta(x)}(2Y-Y^2)\mathrm{d}y=2\pi\rho r_0 v_e X^\alpha\int_0^{\delta(x)}\left[2\left(\frac{y}{\delta(x)}\right)-\frac{y^2}{\delta(x)^2}\right]\mathrm{d}y$$

$$=2\rho\pi r_0 v_e X^\alpha\left(\delta(x)-\frac{1}{3}\delta(x)\right)=\frac{4}{3}\rho\pi r_0 v_e X^\alpha\delta(x) \tag{3}$$

根据质量流守恒，则有

$$\rho_e v_e\pi r_0^2=\rho\pi[r_0^2-2\delta(x)r_0+\delta(x)^2]v_e(1+X)^\alpha+\frac{4}{3}\rho\pi r_0 v_e X^\alpha\delta(x) \tag{4}$$

将上式化简

$$1=\frac{\rho}{\rho_e}\left\{\left[1-2\frac{\delta(x)}{r_0}+\left(\frac{\delta(x)}{r_0}\right)^2\right](1+X)^\alpha+\frac{4}{3}X^\alpha\frac{\delta(x)}{r_0}\right\}$$

$$1=\rho_0\left\{\left[1-2\frac{\delta(x)}{r_0}+\left(\frac{\delta(x)}{r_0}\right)^2\right](1+X)^\alpha+\frac{4}{3}X^\alpha\frac{\delta(x)}{r_0}\right\}$$

$$1=\frac{1}{U}\left\{\left[1-2\frac{\delta(x)}{r_0}+\left(\frac{\delta(x)}{r_0}\right)^2\right](1+X)^\alpha+\frac{4}{3}X^\alpha\frac{\delta(x)}{r_0}\right\}$$

$$1=\left[1-2\frac{\delta(x)}{r_0}+\left(\frac{\delta(x)}{r_0}\right)^2\right]\frac{(1+X)^\alpha}{v_0}+\frac{4}{3}\frac{X^\alpha\delta(x)}{U\quad r_0}$$

$$1=\left[1-2\frac{\delta(x)}{r_0}+\left(\frac{\delta(x)}{r_0}\right)^2\right]\frac{(1+X)^\alpha}{(1+X)^\alpha}+\frac{4}{3}\frac{X^\alpha}{X^\alpha(2Y-Y^2)}\frac{\delta(x)}{r_0}$$

$$1=1-2\frac{\delta(x)}{r_0}+\left(\frac{\delta(x)}{r_0}\right)^2+\frac{4}{3}\frac{1}{\frac{2}{3}}\frac{\delta(x)}{r_0}$$

$$1=1+\left(\frac{\delta(x)}{r_0}\right)^2,\quad \delta(x)=0$$

$\delta(x)=0$，说明理想与实际流体运动分界线不存在。当然，进口段也就不存在。

8.1.2.3 利用综合无因次动量方程推导 $\delta(x)$

将式(8-1-33)与式(8-1-34)代入式(8-1-16)′，分项计算

$$\frac{1}{2}\frac{r_0}{\delta(x)}\frac{v_0}{U}\frac{\partial U}{\partial X}\frac{\partial U}{\partial Y}=-\frac{1}{2}\frac{r_0}{\delta(x)}\frac{\nu_0\alpha X^{\alpha-1}(2Y-Y^2)X^\alpha 2(1-Y)}{X^\alpha(2Y-Y^2)}$$

$$=-\frac{r_0}{\delta(x)}\nu_0\alpha X^{\alpha-1}(1-Y)=-\frac{r_0}{2\delta(x)}\alpha X^{\alpha-1} \tag{1}$$

$$ReU\frac{\partial U}{\partial X}=ReX^\alpha(2Y-Y^2)^2\alpha X^{\alpha-1}=0.533Re\alpha X^{2\alpha-1} \tag{2}$$

$$\nu_0 \frac{r_0}{\delta(x)} \frac{\partial U}{\partial Y} \frac{\partial U}{\partial X} = 2\nu_0 \frac{r_0}{\delta(x)} \alpha(1-Y)(2Y-Y^2)X^{2\alpha-1} = 0.5\nu_0 \frac{r_0}{\delta(x)} \alpha X^{2\alpha-1} \tag{3}$$

$$ReU_0 \frac{\mathrm{d}U_0}{\mathrm{d}X} = Re(1+X)^\alpha \alpha(1+X)^{\alpha-1} = Re\alpha(1+X)^{2\alpha-1} \tag{4}$$

$$\frac{4}{3}\nu_0 \frac{\partial^2 U}{\partial X^2} = \frac{4}{3}\nu_0(2Y-Y^2)\alpha(\alpha-1)X^{\alpha-2} \tag{5}$$

$$\nu_0(1-\varphi^{\frac{2}{3}})\left(\frac{r_0}{\delta(x)}\right)^2 \frac{\partial^2 U}{\partial Y^2} = \nu_0(1-\varphi^{\frac{2}{3}})\left(\frac{r_0}{\delta(x)}\right)^2 X^\alpha(-2) = -2(1-\varphi^{\frac{2}{3}})\left(\frac{r_0}{\delta(x)}\right)^2 X^\alpha \tag{6}$$

$$\nu_0 K_2 \varphi^{\frac{2}{3}}\left(\frac{r_0}{\delta(x)}\right)^2 \frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} = -4\varphi^{\frac{2}{3}}\nu_0 K_2\left(\frac{r_0}{\delta(x)}\right)^2(1-Y)X^{2\alpha} = -2\varphi^{\frac{2}{3}}\nu_0 K_2\left(\frac{r_0}{\delta(x)}\right)^2 X^{2\alpha} \tag{7}$$

将式(1)～式(7)代回式(8-1-16)′，得

$$\begin{aligned}&-\frac{1}{2}\frac{r_0}{\delta(x)}\nu_0 \alpha X^{\alpha-1} + 0.533\alpha Re X^{2\alpha-1} + 0.5\frac{r_0}{\delta(x)}\alpha X^{2\alpha-1}\\ &= Re\alpha(1+X)^{2\alpha-1} + 0.889\nu_0\alpha(\alpha-1)^{\alpha-2} - 2\nu_0(1-\varphi^{\frac{2}{3}})\left(\frac{r_0}{\delta(x)}\right)^2 X^\alpha - 2\varphi^{\frac{2}{3}}\nu_0 K_2\left(\frac{r_0}{\delta(x)}\right)^2 X^{2\alpha}\end{aligned} \tag{8-1-35}$$

因为 $\nu_0 \approx 1$，将上式整理成

$$\begin{aligned}&\left[2(1-\varphi^{\frac{2}{3}})X^\alpha + 2\varphi^{\frac{2}{3}}K_2 X^{2\alpha}\right]\left(\frac{r_0}{\delta(x)}\right)^2 + 0.5(X^{2\alpha-1}-X^{\alpha-1})\frac{r_0}{\delta(x)} - 0.467\alpha Re X^{2\alpha-1}\\ &-0.889\alpha(\alpha-1)X^{\alpha-2} = 0\end{aligned} \tag{8-1-36}$$

令

$$a = 2(1-\varphi^{\frac{2}{3}})X^\alpha + 2\varphi^{\frac{2}{3}}K_2 X^{2\alpha} \tag{8-1-37}$$

$$b = 0.5(X^{2\alpha-1} - X^{\alpha-1}) \tag{8-1-37}$$

$$c = -0.467\alpha Re X^{2\alpha-1} + 0.889\alpha(\alpha-1)X^{\alpha-2} \tag{8-1-39}$$

$$\frac{r_0}{\delta(x)} = \frac{-b+\sqrt{b^2-4ac}}{2a} \tag{8-1-40}$$

$$\delta(x) = \frac{2r_0 a}{-b+\sqrt{b^2-4ac}} \tag{8-1-41}$$

当 $X=0$ 时，$a=0$，$b=0$，$c=0$，则有

$$\delta(x) = \frac{2r_0 0}{-0+0} = \frac{0}{0} \tag{8-1-42}$$

说明 $\delta(x)$没有意义，即它不存在。以上是通过圆管道进行分析所得结果，矩形管道也会如此。所以可压缩湍流管道与不可压缩湍流管道不同，它没有理想与实际流体分界 $\delta(x)$。研究它应用新的思路去解决。

8.2 可压缩湍流管道进口段

可压缩与不可压缩湍流管道进口段的概念是不同的。前者都是实际流体运动；而后者，其中间为理想流体，外围为实际流体运动。

一般气体管道，分为等温与绝热两种情况，但其与气源连接时，由于气源与管道温度不同，总是存在一个过渡段，也就是进口段。在进口段，气体状态变化，是依气体状态方程。所以，应对其运动规律另作分析。

本节讨论圆形与矩形两种情况的进口段。

8.2.1 可压缩湍流圆管进口段

为进行数学分析，必须首先结合本问题选定坐标系与原点位置。见图 8-2 和图 8-3。

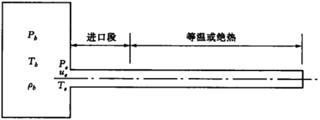


图 8-2　可压缩湍流进口段示意图

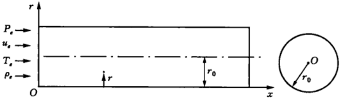


图 8-3　坐标系与原点选定示意图

8.2.1.1 运动控制方程与边界条件

(1) 动量方程

依式(4-5-16)，结合本节问题，得

$$\frac{\nu}{2}\frac{\partial\rho}{\rho\partial x}\frac{\partial u}{\partial r}+\frac{\partial u}{\partial x}+\frac{\nu}{2}\left(\frac{1}{u}\frac{\partial u}{\partial r}\right)\frac{\partial u}{\partial x}$$
$$=-\frac{\partial p}{\rho\partial x}+\nu\left[\frac{4}{3}\frac{\partial^2 u}{\partial x^2}+(1-\varphi^{\frac{2}{3}})\frac{\partial^2 u}{\partial r^2}+6\varphi^{\frac{2}{3}}t\ \frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}\right] \tag{8-2-1}$$

(2) 能量方程

由于在进口段气体运动有温度变化，所以，应引入能量方程。依式(4-5-4)忽略分子能 e，涡旋温度与连续相一样，结合本节问题为

$$u^2\frac{\partial u}{\partial x}-\frac{u^3}{2}\frac{\partial p}{\rho\partial x}+\frac{\varphi}{2}\nu u\frac{\partial^2 u}{\partial x\partial r}+\frac{1}{2}\nu\frac{\partial u}{\partial r}\left(\frac{u}{\rho}\frac{\partial p}{\partial x}+\frac{\partial u}{\partial x}\right)$$

$$=\frac{\lambda}{\rho}\left(\frac{\partial^T}{\partial x^2}+\frac{\partial^2 T}{\partial r^2}\right)+u\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}\right)-\frac{1}{2}\varphi^{\frac{2}{3}}\nu\frac{\partial u}{u\partial r}\left[-\frac{\partial p}{\rho\partial x}+\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}+2\nu t\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}\right] \tag{8-2-2}$$

(3) 气体状态方程

$$\frac{p}{\rho}=RT \tag{8-2-3}$$

$$\frac{\partial p}{\partial x}=R\left(T\frac{\partial \rho}{\partial x}+\rho\frac{\partial T}{\partial x}\right) \tag{8-2-4}$$

(4) 连续方程(质量守恒)

$$\rho_e u_e=\rho u=\text{常数} \tag{8-2-5}$$

$$\frac{\partial p}{\rho\partial x}=-\frac{1}{u}\frac{\partial u}{\partial x} \tag{8-2-6}$$

$$u(x,\ r)|_{x=0}=v_e=u_e \tag{8-2-7}$$

$$\left.\frac{\partial u(x,\ r)}{\partial r}\right|_{r=r_0}=0 \tag{8-2-8}$$

$$\frac{\partial^2 u(x,\ r)}{\partial r^2}<0 \tag{8-2-9}$$

$$p(x,\ r)|_{x=0}=p_e \tag{8-2-10}$$

$$T(x,\ r)|_{x=0}=T_e \tag{8-2-11}$$

$$\rho(x,\ r)|_{x=0}=\rho_e \tag{8-2-12}$$

8.2.1.2 无因次运动控制方程与边界条件

取

$$\frac{x}{r_0}=X,\ \frac{r}{r_0}=R,\ \frac{u}{u_e}=U,\ \frac{\rho}{\rho_e}=\rho_0,\ \frac{p}{p_e}=p_0,\ \frac{T}{T_e}=T_0,\ \frac{\nu}{\nu_e}=\nu_0$$

将以上比值代入物理运动方程与边界条件。

(1) 动量方程

$$\frac{1}{2}\frac{\nu_0}{\rho_0}\frac{\partial\rho_0}{\partial X}\frac{\partial U}{\partial R}+ReU\frac{\partial U}{\partial X}+\frac{\nu_0}{2}\frac{\partial U}{\partial R}\frac{\partial U}{\partial X}$$

$$=-K_1\frac{1}{\rho_0}\frac{\partial p_0}{\partial X}+\nu_0\left[\frac{4}{3}\frac{\partial^2 U}{\partial X^2}+(1-\varphi^{\frac{2}{3}})\frac{\partial^2 U}{\partial R^2}+K_2\varphi^{\frac{2}{3}}\frac{\partial U}{\partial R}\frac{\partial^2 U}{\partial R^2}\right] \tag{8-2-13}$$

(2) 能量方程

$$\frac{u_e^3}{r_0}U^3\frac{\partial U}{\partial X}-\frac{u_e^3}{2r_0}\frac{U^3}{\rho_0}\frac{\partial\rho_0}{\partial X}+\frac{\varphi}{2}\frac{\nu_0}{r_0^2}u_e^2\nu_0U\frac{\partial^2 U}{\partial X\partial R}+\frac{1}{2}\frac{\nu_e u_e^2}{r_0^2}\nu_0\frac{\partial U}{\partial R}\left(\frac{U}{\rho_0}\frac{\partial p_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$=\frac{\lambda_e T_e\lambda_0}{\rho_e r_0^2\rho_0}\left(\frac{\partial^2 T_0}{\partial X^2}+\frac{\partial^2 T_0}{\partial R^2}\right)+U\left(-\frac{p_e u_e}{\rho_e r_0}\frac{\partial p_0}{\rho_0\partial X}+\frac{\nu_e u_e^2}{r_0^2}\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}\right)$$

$$-\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0\partial U}{U\partial R}\left[-\frac{\nu_e p_e}{r_0^2\rho_e}\frac{\partial p_0}{\rho_0\partial X}+\frac{4}{3}\frac{\nu_e u_e}{r_0^3}\nu_0\frac{\partial^2 U}{\partial X^2}+\frac{2\nu_e^2 t u_e^2}{r_0^4}\nu_0\frac{\partial U\partial^2 U}{\partial R\,\partial R^2}\right] \quad (8\text{-}2\text{-}2)'$$

全式除以$\dfrac{\nu_e u_e^2}{r_0^2}$，得

$$ReU^3\frac{\partial U}{\partial X}-\frac{1}{2}ReU^3\frac{\partial\rho_0}{\rho_0\partial X}+\frac{\varphi}{2}\nu_0 U\frac{\partial^2 U}{\partial X\partial R}+\frac{\nu_0\partial U}{2\,\partial R}\left(\frac{U\partial\rho_0}{\rho_0\,\partial X}+\frac{\partial U}{\partial X}\right)$$
$$=\frac{K_3\lambda_0}{\rho_0}\left(\frac{\partial^2 T_0}{\partial X^2}+\frac{\partial^2 T_0}{\partial R^2}\right)+U\left(-K_4\frac{\partial p_0}{\rho_0\partial X}+\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}\right)$$
$$-\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0\partial U}{U\partial R}\left(-K_5\frac{\partial p_0}{\rho_0\partial X}+\frac{4}{3}\frac{\nu_0}{Re}\frac{\partial^2 U}{\partial X^2}+K_6\nu_0\frac{\partial^2 U\partial U}{\partial R^2\partial R}\right) \quad (8\text{-}2\text{-}14)$$

(3) 状态方程

$$\frac{p_0}{\rho_0}=R_0T_0 \quad (8\text{-}2\text{-}15)$$

$$\frac{\partial p_0}{\partial X}=T_0\frac{\partial\rho_0}{\partial X}+\rho_0\frac{\partial T_0}{\partial X} \quad (8\text{-}2\text{-}16)$$

(4) 质量守恒

由 $\rho_e u_e=\rho u$，则有

$$1=\rho_0 U \quad (8\text{-}2\text{-}17)$$

$$\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial X}=-\frac{1}{U}\frac{\partial U}{\partial X} \quad (8\text{-}2\text{-}18)$$

以上各式中：

$$Re=\frac{r_0u_e}{\nu_e} \quad (8\text{-}2\text{-}19)$$

$$K_1=\frac{p_e r_0}{\rho_e\nu_e u_e} \quad (8\text{-}2\text{-}20)$$

$$K_2=\frac{6tu_e}{r_0} \quad (8\text{-}2\text{-}21)$$

$$K_3=\frac{\lambda_e T_e}{\rho_e\nu_e u_e^2} \quad (8\text{-}2\text{-}22)$$

$$K_4=\frac{p_e r_0}{\rho_e u_e\nu_e} \quad (8\text{-}2\text{-}23)$$

$$K_5=\frac{p_e}{\rho_e u_e^2} \quad (8\text{-}2\text{-}24)$$

$$K_6=\frac{2\nu_e t}{r_0^2} \quad (8\text{-}2\text{-}25)$$

(5) 边界条件

$$U(X,\ R)|_{X=0}=1 \tag{8-2-7)'$$

$$\left.\frac{\partial U(X,\ R)}{\partial R}\right|_{R=1}=0 \tag{8-2-8)'$$

$$\frac{\partial^2 U(X,\ R)}{\partial R^2}<0 \tag{8-2-9)'$$

$$P_0(X,\ R)|_{X=0}=1 \tag{8-2-10)'$$

$$T_0(X,\ R)|_{X=0}=1 \tag{8-2-11)'$$

$$\rho_0(X,\ R)|_{X=0}=1 \tag{8-2-12)'$$

$$U(X,\ R)|_{R=0}=0 \tag{8-2-13)'$$

8.2.1.3 速度与温度分布

根据边界条件，选定速度与温度分布分别为

$$U=(1+X)^{\alpha}(2R-R^2) \tag{8-2-26}$$

$$T=(1+X)^{\beta} \tag{8-2-27}$$

两式必须首先确定参变常数 α，β。为此，将式(8-2-26)代入式(8-2-13)，式(8-2-27)代入式(8-2-14)，同时利用质量守恒，气态方程，将两式中压力 p_0 与密度 ρ_0 均转化成速度 U 与温度 T_0，则得两个各自只含 α 与 β 的方程，然后联立求解。式(8-2-26)代入式(8-2-13)，各项结果

$$\frac{1}{2}\frac{v_0}{\rho_0}\frac{\partial\rho_0}{\partial X}\frac{\partial U}{\partial R}=\frac{1}{2}\frac{\nu_0}{U}\frac{\partial U}{\partial X}\frac{\partial U}{\partial R}=\nu_0\alpha(1+X)^{\alpha-1}(1-R)$$

$$=\nu_0\alpha(1+X)^{\alpha-1}\int_0^1(1-R)\mathrm{d}R=\frac{1}{2}\nu_0\alpha(1+X)^{\alpha-1} \tag{1}$$

$$ReU\frac{\partial U}{\partial X}=Re(1+X)^{2\alpha-1}\alpha\int_0^1(2R-R^2)^2\mathrm{d}R=Re\alpha(1+X)^{2\alpha-1}\int_0^1(4R^2-4R^3+R^4)\mathrm{d}R$$

$$=\frac{17}{15}Re\alpha(1+X)^{\alpha-1} \tag{2}$$

$$\frac{\nu_0}{2}\frac{1}{U}\frac{\partial U}{\partial R}\frac{\partial U}{\partial X}=\frac{\nu_0}{2}\alpha(1+X)^{\alpha-1} \tag{3}$$

$$-K_1\frac{1}{\rho_0}\frac{\partial p_0}{\partial X}=-K_1U\frac{\partial p_0}{\partial X}=-K_1U\left(T_0\frac{\partial\rho_0}{\partial X}+\rho_0\frac{\partial T_0}{\partial X}\right)=-K_1U\left[T_0\frac{\partial}{\partial X}\left(\frac{1}{U}\right)+\frac{1}{U}\frac{\partial T_0}{\partial X}\right]$$

$$=K_1\left(\frac{T_0}{U}\frac{\partial U}{\partial X}-\frac{\partial T_0}{\partial X}\right)=K_1\left[\frac{\alpha(1+X)^{\beta}}{1+X}+\beta(1+X)^{\beta-1}\right] \tag{4}$$

$$\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}=\frac{8}{3}\nu_0\alpha(\alpha-1)(1+X)^{\alpha-2} \tag{5}$$

$$\nu_0(1-\varphi^{\frac{2}{3}})\frac{\partial^2 U}{\partial R^2}=-2\nu_0(1-R)(1-\varphi^{\frac{2}{3}})(1+X)^{\alpha} \tag{6}$$

$$\nu_0K_3\varphi^{\frac{2}{3}}\frac{\partial U}{\partial R}\frac{\partial^2 U}{\partial R^2}=-4K_2\varphi^{\frac{2}{3}}\nu_0(1-R)(1+X)^{2\alpha} \tag{7}$$

将式(1)～式(7)代回式(8-2-13)，取 $\nu_0\approx1$，则

$$\frac{1}{2}\alpha(1+X)^{\alpha-1}+\frac{17}{15}Re\alpha(1+X)^{\alpha-1}+\frac{1}{2}\alpha(1+X)^{\alpha-1}$$

$$=K_1[\alpha(1+X)^{\beta-1}+\beta(1+X)^{\beta-1}]+\frac{8}{3}(\alpha-1)\alpha(1+X)^{\alpha-2}-(1-\varphi^{\frac{2}{3}})(1+X)^{\alpha}$$

$$-2K_2\varphi^{\frac{2}{3}}(1+X)^{2\alpha} \tag{8-2-13)$'$($$

整理为

$$\left(1+\frac{17}{15}Re\right)\alpha(1+X)^{\alpha-1}$$

$$=K_1(\alpha+\beta)(1+X)^{\beta-1}+\frac{8}{3}\alpha(\alpha-1)(1+X)^{\alpha-1}-(1-\varphi^{\frac{2}{3}})(1+X)^{\alpha}-2K_2\varphi^{\frac{2}{3}}(1+X)^{2\alpha}$$

(8-2-13)″

对 $(1+X)$ 项取对数

$$\left(1+\frac{17}{15}Re\right)\alpha(\alpha-1)\ln(1+X)$$

$$=K_1(\alpha+\beta)(\beta-1)\ln(1+X)+\frac{8}{3}(\alpha-1)\alpha(\alpha-2)\ln(1+X)-(1-\varphi^{\frac{2}{3}})\alpha\ln(1+X)$$

$$-4K_2\varphi^{\frac{2}{3}}\alpha\ln(1+X)$$

(8-2-13)‴

$$\left(1+\frac{17}{15}Re\right)(\alpha-1)=K_1\frac{(\alpha+\beta)(\beta-1)}{\alpha}+\frac{8}{3}(\alpha-1)(\alpha-2)-(1-\varphi^{\frac{2}{3}})-4K_2\varphi^{\frac{2}{3}}$$

(8-2-13)⁗

整理化简为

$$\alpha^3-0.425Re\alpha^2+[0.425Re+0.375K_1(\beta-1)-1.5K_2\varphi^{\frac{2}{3}}]\alpha+0.375K_1\beta(\beta-1)=0 \tag{8-2-28}$$

将式(8-2-26)与式(8-2-27)代入式(8-2-14)，分别计算各项为

$$ReU^3\frac{\partial U}{\partial X}=\frac{16}{81}Re(1+X)^{4\alpha-1}\alpha \tag{1}$$

$$-\frac{1}{2}ReU^3\frac{\partial\rho_0}{\rho_0\partial X}=\frac{4}{27}Re(1+X)^{3\alpha-1}\alpha \tag{2}$$

$$\frac{1}{2}\varphi\nu_0U\frac{\partial^2U}{\partial X\partial R}=\frac{\varphi}{6}\nu_0\alpha(1+X)^{2\alpha-1} \tag{3}$$

$$\frac{\nu_0\partial U}{2\partial R}\left(\frac{U\partial\rho_0}{\rho_0\partial X}+\frac{\partial U}{\partial X}\right)=\frac{\nu_0}{2}2(1-R)(1+X)^{\alpha}\left(\frac{U\partial U}{U\partial X}+\frac{\partial U}{\partial X}\right)=0 \tag{4}$$

$$\frac{K_3\lambda_0}{\rho_0}\left(\frac{\partial^2T_0}{\partial X^2}+\frac{\partial^2T_0}{\partial R^2}\right)=K_3\lambda_0U\left(\frac{\partial^2T_0}{\partial X^2}+0\right)=K_3\lambda_0(1+X)^{\alpha}(2R-R^2)[\beta(\beta-1)(1+X)^{\beta-2}]$$

$$=\frac{2}{3}K_3\lambda_0(1+X)^{\alpha}\beta(\beta-1)(1+X)^{\beta-2} \tag{5}$$

$$U\left(-K_4\frac{\partial p_0}{\rho_0\partial X}\right)+K_5\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0\partial U}{U\partial R}\frac{\partial p_0}{\rho_0\partial X}$$

$$=\left(K_5\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0\partial U}{U\partial R}-K_4U\right)\frac{\partial p_0}{\rho_0\partial X}=\left(\frac{\varphi^{\frac{2}{3}}}{2}K_5\frac{\nu_0\partial U}{U\partial R}-K_4U\right)U\frac{\partial p_0}{\partial X}$$

$$=\left(\frac{\varphi^{\frac{2}{3}}}{2}K_5\nu_0\frac{\partial U}{\partial R}-K_4U^2\right)\frac{\partial p_0}{\partial X}=\left(\frac{\varphi^{\frac{2}{3}}}{2}K_5\nu_0\frac{\partial U}{\partial R}-K_4U^2\right)\left(\frac{T_0\partial\rho_0}{\partial X}+\rho_0\frac{\partial T_0}{\partial X}\right)$$

$$=\left(\frac{\varphi^{\frac{2}{3}}}{2}K_5\nu_0\frac{\partial U}{\partial R}-K_4U^2\right)\left[T_0\frac{\partial}{\partial X}\left(\frac{1}{U}\right)+\frac{1}{U}\frac{\partial T_0}{\partial X}\right]$$

$$=\left(\frac{\varphi^{\frac{2}{3}}}{2}K_5\nu_0\frac{\partial U}{\partial R}-K_4U^2\right)\left(\frac{1}{U}\frac{\partial T_0}{\partial X}-\frac{T_0\partial U}{U^2\partial X}\right)$$

$$=K_4T_0\frac{\partial U}{\partial X}-\frac{\nu_0}{2}\varphi^{\frac{2}{3}}K_5\frac{T_0\partial U\partial U}{U^2\partial R\partial X}+\frac{\nu_0}{2}\varphi^{\frac{2}{3}}K_5\frac{1}{U}\frac{\partial U}{\partial R}\frac{\partial T_0}{\partial X}-K_4U\frac{\partial T_0}{\partial X}$$

$$=K_4\left(T_0\frac{\partial U}{\partial X}-U\frac{\partial T_0}{\partial X}\right)+\frac{1}{2}\varphi^{\frac{2}{3}}\nu_0K_5\frac{1}{U}\frac{\partial U}{\partial R}\left(\frac{\partial T_0}{\partial X}-\frac{T_0\partial U}{U\partial X}\right)$$

$$=K_4[\alpha(2R-R^2)(1+X)^{\alpha+\beta-1}-(2R-R^2)\beta(1+X)^{\alpha+\beta-1}]$$

$$+\frac{1}{2}\frac{\nu_0\varphi^{\frac{2}{3}}K_5(1-R)}{2R-R^2}[\beta(1+X)^{\beta-1}-(1+X)^{\beta-1}]$$

$$=K_4(2R-R^2)(1+X)^{\alpha+\beta-1}(\alpha-\beta)+\frac{\nu_0\varphi^{\frac{2}{3}}K_5}{2R-R^2}(1+X)^{\beta-1}(\beta-1)$$

$$=3K_4(1+X)^{\alpha+\beta-1}(\alpha-\beta)+\frac{3}{2}\nu_0\varphi^{\frac{2}{3}}K_5(1+X)^{\beta-1}(\beta-1)\tag{6}$$

$$\frac{4}{3}\nu_0U\frac{\partial^2U}{\partial X^2}=\frac{4}{3}\nu_0(1+X)^{\alpha}(2R-R^2)^2\alpha(\alpha-1)(1+X)^{\alpha-2}=\frac{16}{27}\alpha'(\alpha-1)(1+X)^{\alpha-2}\tag{7}$$

$$-\frac{2}{3}\frac{\varphi^{\frac{2}{3}}\nu_0^2\partial U\partial^2U}{URe\,\partial R\,\partial X^2}=-\frac{2}{3}\varphi^{\frac{2}{3}}\nu_0^2\frac{1-R}{Re}\alpha(\alpha-1)(1+X)^{\alpha-2}=-\frac{2}{3}\frac{\varphi^{\frac{2}{3}}\nu_0^2}{Re}\alpha(\alpha-1)(1+X)^{\alpha-2}\tag{8}$$

$$\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0K_6}{U}\left(\frac{\partial U}{\partial R}\right)^2\frac{\partial^2U}{\partial R^2}=\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0K_6(1+\alpha)^{2\alpha}4(1-R)^2}{(2R-R^2)(1+X)^{\alpha}}(1+X)^2(-2)=\frac{3}{2}\varphi^{\frac{2}{3}}\nu_0^2K_6(1+X)^{2\alpha}\tag{9}$$

将式(1)～式(9)各项对$(1+X)$取对数，得

$$ReU^3\frac{\partial U}{\partial X}=\frac{16}{81}Re(1+X)^{4\alpha-1}\alpha=\frac{16}{81}Re(4\alpha-1)\alpha\ln(1+X)\tag{1'}$$

$$-\frac{1}{2}ReU^3\frac{\partial\rho_0}{\rho_0\partial X}=\frac{4}{27}Re(1+X)^{3\alpha-1}\alpha=\frac{4}{27}Re(3\alpha-1)\alpha\ln(1+X)\tag{2'}$$

$$\frac{1}{2}\varphi\nu_0U\frac{\partial^2U}{\partial X\partial R}=\frac{\varphi}{6}\nu_0\alpha(1+X)^{2\alpha-1}=\frac{1}{6}\varphi\nu_0\alpha(2\alpha-1)\ln(1+X)\tag{3'}$$

$$\frac{1}{2}\nu_0\frac{\partial U}{\partial R}\left(\frac{U\partial\rho_0}{\rho_0\partial X}+\frac{\partial U}{\partial X}\right)=0\tag{4'}$$

$$\frac{K_3\lambda_0}{\rho_0}\left(\frac{\partial^2T_0}{\partial X^2}+\frac{\partial^2T_0}{\partial R^2}\right)=\frac{2}{3}K_3\lambda_0\beta(\beta-1)(1+X)^{\beta-2}(1+X)^{\alpha}$$

$$=\frac{2}{3}K_3\lambda_0\beta(\beta-1)(1+X)^{\alpha+\beta-2}$$

$$=\frac{2}{3}K_3\lambda_0\beta(\beta-1)(\alpha+\beta-2)\ln(1+X) \tag{5)'}$$

$$\frac{K_3\lambda_0}{\rho_0}\left(\frac{\partial^2 T_0}{\partial X^2}+\frac{\partial^2 T_0}{\partial R^2}\right)$$

$$=3K_4(1+X)^{\alpha+\beta-1}(\alpha-\beta)+\frac{3}{2}\nu_0\varphi^{\frac{2}{3}}K_5(\beta-1)(1+X)^{\beta-1}$$

$$=3K_4(\alpha+\beta-1)(\alpha-\beta)\ln(1+X)\frac{3}{2}\nu_0\varphi^{\frac{2}{3}}K_5(\beta-1)^2\ln(1+X) \tag{6)'}$$

$$\frac{4}{3}\nu_0 U\frac{\partial^2 U}{\partial X^2}=\frac{16}{27}\nu_0\alpha(\alpha-1)(1+X)^{\alpha-2}=\frac{16}{27}\nu_0\alpha(\alpha-1)(\alpha-2)\ln(1+X) \tag{7)'}$$

$$-\frac{2}{3}\frac{\nu_0^2\varphi^{\frac{2}{3}}}{URe}\frac{\partial U}{\partial R}\frac{\partial^2 U}{\partial X^2}=-\frac{2}{3}\varphi^{\frac{2}{3}}\nu_0^2\alpha(\alpha-1)(\alpha-2)\ln(1+X) \tag{8)'}$$

$$\frac{1}{2}\frac{\varphi^{\frac{2}{3}}\nu_0K_6}{U}\left(\frac{\partial U}{\partial R}\right)^2\frac{\partial^2 U}{\partial R^2}=\frac{3}{2}\varphi^{\frac{2}{3}}\nu_0^2K_6 2\alpha\ln(1+X)=3\varphi^{\frac{2}{3}}\nu_0^2K_6\alpha\ln(1+X) \tag{9)'}$$

将式(1)′～式(9)′各项除以$\alpha\ln(1+X)$代入式(8-2-14)，并取$\nu'_0\approx 1$，得

$$Re(0.792\alpha-0.05)+0.167(2\alpha-1)$$

$$=0.667K_3\lambda_0\beta\frac{(\beta-1)(\alpha+\beta-2)}{\alpha}+3K_4\frac{\alpha+\beta-1}{\alpha}(\alpha-\beta)$$

$$+1.5\varphi^{\frac{2}{3}}K_5(\beta-1)^2+(0.592+\varphi^{\frac{2}{3}}0.667)(\alpha-1)(\alpha-2)+3\varphi^{\frac{2}{3}}K_6 \tag{8-2-29}$$

式(8-2-28)与式(8-2-29)均各含有α，β两个未知数，两个方程理论上可解。计算过程复杂。现举例说明其解法。

例 8-1 由空气压缩罐引出直径为 0.04m 的管道，沿程温度 $t=30$℃，罐内温度 $t=40$℃，压力为 10 个标准大气压。管口处测得压力为 8 个标准大气压，温度为 $t=35$℃，进口处局部阻力系数 $\xi=0.5$。确定进口速度、密度、质量流量、速度分布参变常数 α、温度分布参变常数 β。

解 进口速度 $u_e=v_e$，应用公式

$$u_e=v_e=\sqrt{\frac{2R(T_b-T_e)}{\xi+1}}=\sqrt{\frac{2\times 287(313-308)}{0.5+1}}$$

$$=\sqrt{\frac{2\times 287\times 5}{1.5}}=\sqrt{1913.33}=43.742\text{m/s}$$

进口处密度 ρ_e，由气体状态方程

$$\frac{p_e}{\rho_e}=RT_e=287\times(273+35)=88396$$

$$\rho_e=\frac{8\times 1.013\times 10^5}{88396}=9.174\text{kg/m}^3$$

进口通过的质量流量

$$Q_m = \rho_e v_e \pi r_0^2 = 9.174 \times 43.742 \times \pi \times 0.02^2 = 0.504\text{kg/s}$$

要确定速度与温度分布参变常数 α，β，必须应用式(8-2-28)与式(8-2-29)联立求解。为此必选确定相应本题的 n 个无因次数。

$$Re = \frac{v_e r_0}{\nu_e} = \frac{43.742 \times 0.02}{16.5 \times 10^{-6}} = 0.53 \times 10^5$$

$$K_1 = \frac{p_e r_0}{\rho_e \nu_e u_e} = \frac{8 \times 1.013 \times 10^5 \times 0.02}{9.174 \times 16.5 \times 10^{-6} \times 43.742} = \frac{0.162 \times 10^5 \times 10^6}{0.662 \times 10^4} = 0.245 \times 10^7$$

$$K_2 = \frac{6tv_e}{r_0} = \frac{6 \times 0.01 \times 43.742}{0.02} = 131.226$$

$$K_3 = \frac{\lambda_e T_e}{\rho_e \nu_e u_e^2} = \frac{2.71 \times 308 \times 10^6}{10^2 \times 16.5 \times 43.742^2} = 28.8$$

$$K_4 = \frac{p_e r_0}{\rho_e u_e} = \frac{8 \times 1.013 \times 10^5 \times 0.02}{9.174 \times 43.742} = 40$$

$$K_5 = \frac{p_e r_0}{\rho_e u_e^2} = \frac{8 \times 1.013 \times 10^5 \times 0.02}{9.174 \times 43.742^2} = 0.92$$

$$K_6 = \frac{2v_e t}{r_0^2} = \frac{2 \times 16.5 \times 0.01}{10^6 \times 0.02^2} = 0.000825$$

$$\frac{K_1}{Re} = \frac{0.247 \times 10^7}{0.53 \times 10^5} = 46.17$$

$$\frac{K_2}{Re} = \frac{131.226}{0.53 \times 10^5} = 0.00247$$

将以上有关比值代入式(8-2-28)，则有

$$0.425\alpha^2 - (0.425 + 17.314\beta - 17.314)\alpha - 17.314\beta(\beta - 1) = 0 \qquad (8\text{-}2\text{-}28)'$$

将式(8-2-29)除以 Re，则为

$$\begin{aligned}&0.742\alpha - 0.05 + \frac{0.167}{Re}(2\alpha - 1) \\ &= 0.667\frac{K_3}{Re}\beta(\beta - 1)\left(\frac{\alpha + \beta - 2}{\alpha}\right) + \frac{3K_4}{Re}\left(\frac{\alpha + \beta - 1}{\alpha}\right)(\alpha - \beta) \\ &\quad + \frac{1.5\varphi^{\frac{2}{3}}K_5(\beta - 1)^2 + (0.592 + 0.667\varphi^{\frac{2}{3}})(\alpha - 1)(\alpha - 2)}{Re} + \frac{3\varphi^{\frac{2}{3}}K_6}{Re}\end{aligned} \qquad (8\text{-}2\text{-}29)'$$

进行量级比较，忽略微小量，则有

$$\begin{aligned}&0.742\alpha - 0.05 + \frac{0.167}{Re}(2\alpha - 1) \\ &= 0.667\frac{K_3}{Re}\beta(\beta - 1)\left(\frac{\alpha + \beta - 2}{\alpha}\right) + \frac{3K_4}{Re}\left(\frac{\alpha + \beta - 1}{\alpha}\right)(\alpha - \beta) + \frac{1.5\varphi^{\frac{2}{3}}K_5}{Re}(\beta - 1)^2\end{aligned} \qquad (8\text{-}2\text{-}29)''$$

将 K_3，K_4，K_5，K_6，K_e 值代入上式，则有

$$(0.792\alpha - 0.05)\alpha = 0.00036\beta(\beta - 1)(\alpha + \beta - 2) + 0.00226(\alpha + \beta - 1)(\alpha - \beta) -$$

$$0.000026\varphi^{\frac{2}{3}}(\beta-1)^2 \tag{8-2-29)''}$$

对式(8-2-29)‴再次进行量级分析，涡旋体积分数，约计 $\varphi=0.1$，而 $\varphi^{\frac{2}{3}}\approx 0.2$。而 α 与 β 最大范围 2，-2。设 $\alpha=2$，$\beta=-2$，则计算式(8-2-29)‴右边各项：

$$0.00036(-2)(-2-1)(2-2-2)=0.00036\times(-12)=-0.0043$$

$$0.00226(2-2-1)(2-(-2))=0.00226\times(-4)=-0.00904$$

$$0.000026\varphi^{\frac{2}{3}}(\beta-1)^2=0.000026\times 0.2(-2-1)^2=0.000046$$

而左边

$$(0.792\times 2-0.05)2=3.168-0.1=3.068$$

两边对比，说明右边量值很小，可作为零。

$$(0.792\alpha-0.05)\alpha=0 \tag{8-2-29)''''}$$

由它解出

$$\alpha=\frac{0.05}{0.792}=0.063$$

将它代入式(8-2-28)′，解出 β：

$$0.425\times 0.063^2-(0.425+17.314\beta-17.314)0.063-17.314\beta(\beta-1)=0 \tag{a}$$

$$0.00169-0.0268-1.09078\beta+1.09078-17.314\beta(\beta-1)=0 \tag{a)'}$$

$$17.314\beta^2-16.223\beta-1.066=0 \tag{a)''}$$

$$\beta^2-0.937\beta-0.062=0 \tag{a)'''}$$

$$\beta=\frac{0.937-\sqrt{0.937^2+4\times 0.062}}{2}=\frac{0.937-1.061}{2}=\frac{-0.124}{2}=-0.062$$

结果取 $\alpha=0.063$，$\beta=-0.062$。

速度分布为

$$U=(1+X)^{0.063}(2R-R^2) \tag{Ⅰ}$$

温度分布为

$$T_0=(1+X)^{-0.062} \tag{Ⅱ}$$

8.2.1.4　管长确定公式

动量微分方程，没有办法纳入壁面造成的能量损失，可压缩流体在管内运动过程，是压力下降，温度变化，密度变小，速度加大，而且伴随着能量损失。给它建立起能量守恒公式就是

$$\frac{p_e}{\rho_e}+\frac{v_e^2}{2}=\frac{p_a}{\rho_a}+\frac{v_a^2}{2}+E_f \tag{8-2-30}$$

式中：p_e——管道进口处压力；

ρ_e——管道进口处流体密度；

v_e——管道进口处速度，它是均布的；

v_a——管道出口处速度，是断面平均速度；

ρ_a——管道出口处流体密度；

E_f——管道全长损失能量。

设管道无因次长度为 L，它是待求的。现在计算管道出口断面平均速度 v_a。

$$v_a = \frac{1}{2}v_e(1+L)^{\alpha}\int_0^1(2R-R^2)\mathrm{d}R = \frac{1}{2}v_e(1+L)^{\alpha}\frac{2}{3} = \frac{1}{3}v_e(1+L)^{\alpha} \tag{8-2-31}$$

因为研究问题管道，壁面糙度对摩擦损失起主要作用，所以应用水力学中沿程阻力系数 λ 来计算阻力损失功率。

$$\frac{\gamma h_f}{\rho} = \frac{\chi}{\rho}\frac{\lambda l}{0}\frac{v_a^2}{2g} = \frac{\lambda}{2}\frac{l}{0}v_a^2 = \left(\frac{\lambda L}{2}\right)\frac{v_a^2}{2} \tag{8-2-32}$$

将式(8-2-31)与式(8-2-32)代入式(8-2-30)，则有

$$\frac{p_e}{\rho_e} - \frac{p_a}{\rho_a} + \frac{v_e^2}{2} = \left(1+\frac{\lambda L}{2}\right)\frac{1}{2}\left[\frac{v_e}{3}(1+L)^{\alpha}\right]^2 \tag{8-2-33}$$

式中：λ——沿程阻力系数。

举例说明其应用方法。

例 8-2　题意如例 8-1 所示，现确定其进口段(等质量流)长度 l 为多少？

解　$$\frac{p_e}{\rho_e} - \frac{p_a}{\rho_a} + \frac{v_e^2}{2} = 287\times(273+35) - 287(273+30) + \frac{1}{2}(43.742)^2$$

$$= 88396 - 86961 + 956.681 = 2391.681$$

代入式(8-2-33)

$$2391.681 = \left(1+\frac{0.01L}{2}\right)\frac{1}{2}\left[\frac{v_e}{3}(1+L)^{0.063}\right]^2$$

$$= \left(1+\frac{0.01L}{2}\right)\left(\frac{1+0.01L}{9}\right)(1+L)^{0.126}\frac{(43.742)^2}{2} \tag{a}$$

$$2.5 = (1+0.005L)\frac{1}{9}(1+L)^{0.126} \tag{b}$$

设不同 L 值，列表计算，结果如表 8-1。

表 8-1　　**计算 L 表**

L	(A)　$1+0.005L$	(B)　$(1+L)^{0.126}$	$\frac{1}{9}(A\times B)$
500	3.5	2.187	0.851
1000	6.0	2.388	1.592
1500	7.5	2.512	2.094
2000	10.0	2.606	2.895

由图 8-4 可知，$L=1700$，则 $l=1700\times0.02=34$m。

8.2.1.5　进口段(等质量)压力分布

将式(8-2-26)代入式(8-2-13)，整理后为

$$\frac{\mathrm{d}p_0}{\mathrm{d}X} = \frac{3}{2K_1}\left[-\left(1+\frac{17}{15}Re\right)\alpha(1+X)^{\alpha-1} + \frac{8}{3}\alpha(\alpha-1)(1+X)^{\alpha-1} - (1-\varphi^{\frac{2}{3}})(1+X)^{\alpha}\right.$$

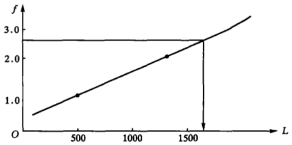


图 8-4 例 8-2 计算 L 值

$$-2K_2\varphi^{\frac{2}{3}}(1+X)^{2\alpha}\Big] \tag{a}$$

将上式积分为

$$p_0=\frac{1}{K_1}\frac{3}{2}\Big[-\left(1+\frac{17}{15}Re\right)(1+X)^{\alpha}+\frac{8}{3}(\alpha-1)(1+X)^{\alpha}-(1-\varphi^{\frac{2}{3}})\frac{(1+X)^{\alpha+1}}{\alpha+1}$$

$$-2K_2\varphi^{\frac{2}{3}}\frac{(1+X)^{2\alpha+1}}{2\alpha+1}+c\Big] \tag{b}$$

定积分常数，当 $X=0$，$p_0=1$，则有

$$c=1-\frac{3}{2}\frac{1}{K_1}\Big[-\left(1+\frac{17}{15}Re\right)+\frac{8}{3}(\alpha-1)-\frac{(1-\varphi^{\frac{2}{3}})}{\alpha+1}-\frac{2K_2\varphi^{\frac{2}{3}}}{2\alpha+1}\Big] \tag{c}$$

将式(c)代回式(b)，则有

$$p_0=1+\frac{3}{2}\frac{1}{K_1}\left(1+\frac{17}{15}Re\right)[1-(1+X)^{\alpha}]+\frac{3}{2}\frac{1}{K_1}$$

$$\left\{\frac{8}{3}(\alpha-1)[(1+X)^{\alpha}-1]+\frac{(1-\varphi^{\frac{2}{3}})}{\alpha+1}[1-(1+X)^{\alpha+1}]+\frac{2K_2\varphi^{\frac{2}{3}}}{2\alpha+1}[1-(1+X)^{2\alpha+1}]\right\} \tag{8-2-34}$$

式中，$p_0=\dfrac{p}{p_e}$，将其代入上式，则为有因次压力分布。p_e 是进口出处压力。

8.2.1.6 边层流界面位置(R_B)

将式(8-2-31)代入式(8-2-26)，则有

$$\frac{1}{3}(1+X)^{\alpha}=(1+X)^{\alpha}(2R_B-R_B^2)$$

$$\frac{1}{3}=2R_B-R_B^2 \tag{d}$$

$$R_B=\frac{2-\sqrt{2^2-4\times\frac{1}{3}}}{2}=\frac{2-\sqrt{4-1.333}}{2}=\frac{2-1.633}{2}=\frac{0.367}{2}=0.189 \tag{e}$$

8.2.1.7 边层流界面涡旋强度与大小

依式(8-2-26)与涡旋定义，可以导得

$$\omega\big|_{r=r_B}=\frac{v_e}{r_0}(1+X)^{\alpha}(1-R_B)=\frac{v_e}{r_0}(1+X)^{\alpha}(1-0.189)=0.811\,\frac{v_e}{r_0}(1+X)^{\alpha} \tag{8-2-35}$$

涡旋直径

$$d_s=2r_s=2\sqrt{\frac{10\nu}{\omega}}$$

$$d_s=2\sqrt{\frac{10\nu r_0}{0.811v_e(1+X)^{\alpha}}} \tag{8-2-36}$$

8.2.1.8　涡旋于界面上径向速度 u_r^*

依式(4-3-27)，结合本问题，则有

$$u_r^*=\frac{12\nu t\left(\omega\big|_{r=r_B}\right)^2}{v}=\frac{12\nu t\left[0.811\,\frac{v_e}{r_0}(1+X)^{\alpha}\right]^2}{\frac{1}{3}(1+X)^{\alpha}} \tag{f}$$

因为式中

$$t=\frac{r_0-r_B}{u_r^*} \tag{8-2-37}$$

代入整理，化简

$$u_r^*\sqrt{\frac{3\times12\times(0.811)^2\nu v_e(1+X)^{\alpha}(r_0-r_B)}{r_0^2}}$$

$$u_r^*=4.866\sqrt{\frac{\nu(r_0-r_B)v_e(1+X)^{\alpha}}{r_0^2}} \tag{8-2-38}$$

8.2.1.9　涡旋体积分数 φ 公式

根据第 4 章建立的涡旋体积分数公式，可得

$$\varphi=k\,\frac{[秒]\pi u_r^*}{48r_0} \tag{8-2-39}$$

式中，k 为经验或实验系数。

公式应用举例如下。

例 8-3　题意同例 8-1。此处求进口段末端，即 $X=1400$ 处边层流厚度 r_B，涡旋强度与大小，涡旋径向速度 u_r^*，该断面上涡旋体积分数 φ。已知 30℃时 $\nu=16.04\times10^{-6}\text{m}^2/\text{s}$。

解　由式(e)，已知 $R_B=0.189$，则有边层流厚度

$$\frac{r_B}{r_0}=R_B=0.189$$

$$r_B=0.189\times0.02=0.0038\text{m}$$

涡旋强度$\omega\big|_{r=r_B}$，由式(8-2-35)

$$\omega|_{r=r_B}=0.811\frac{v_e}{r_0}(1+X)^a=0.811\frac{43.742}{0.02}(1+1700)^{0.063}$$

$$=\frac{35.475(1701)^{0.063}}{0.02}=1773.738\times1.598=2834.157\text{rad/s}$$

涡旋直径 $d_s=2r_s$，由式(8-2-36)

$$d_s=2\sqrt{\frac{10\nu r_0}{0.811v_e(1+X)^a}}=\frac{2}{10^2}\sqrt{\frac{10\times0.16\times0.02}{0.811\times43.742(1+1700)^{0.063}}}$$

$$=\frac{2}{10^2}\sqrt{\frac{0.32}{0.811\times43.742\times1.598}}=\frac{2}{10^2}\sqrt{0.0056}=\frac{2\times0.075}{10^2}=0.0015\text{m}$$

涡旋于边层流界面上径向速度 u_r^*，依式(8-2-38)，则有

$$u_r^*=4.866\sqrt{\frac{\nu(r_0-r_B)(1+X)^a}{r_0^2}}=4.866\sqrt{\frac{0.16(0.02-0.0038)43.742(1+1700)^{0.063}}{10^4\times(0.02)^2}}$$

$$=4.866\sqrt{\frac{0.16\times0.0162\times43.742\times1.598}{2}}=4.866\sqrt{\frac{0.181}{2}}$$

$$=4.866\sqrt{0.09059}=4.866\times0.301=1.465\text{m/s}$$

涡旋体积分数 φ，由式(8-2-39)

$$\varphi=k\frac{[秒]u^*}{480}=k\frac{1.465}{48\times0.04}=k\times0.72=0.144$$

式中，k 是实验系数，现取 $k=0.2$。

8.2.2 可压缩湍流矩形管进口段

根据数学分析需要，坐标系与原点的选定如图 8-5 所示。

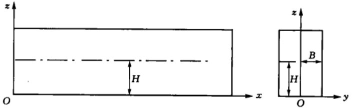


图 8-5　可压缩湍流矩形管道坐标系与原点选定

8.2.2.1 运动控制方程与边界条件

(1) 动量方程

依式(4-5-16)，结合本节条件为

$$\frac{\nu}{2}\frac{\partial\rho}{\rho\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+u\frac{\partial u}{\partial x}+\frac{\nu}{2}\left[\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\right]\nabla\cdot V$$

$$=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left[\frac{4}{3}\frac{\partial^2u}{\partial x^2}+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}\right)+6\varphi^{\frac{2}{3}}t\left(\frac{\partial u}{\partial y}\frac{\partial^2u}{\partial y^2}+\frac{\partial u}{\partial z}\frac{\partial^2u}{\partial z^2}\right)\right]\qquad(8\text{-}2\text{-}40)$$

(2) 能量方程

由式(4-5-41)，结合本问题，在定常流，忽略质量力与分子能，不考虑涡旋温度与连续相的差别，可得

$$u^2\frac{\partial u}{\partial x}-\frac{u^2}{2}\frac{u}{\rho}\frac{\partial p}{\partial x}+\frac{1}{2}\varphi\nu u\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial x\partial z}\right)+\frac{\nu}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\left(\frac{u}{\rho}\frac{\partial p}{\partial x}+\frac{\partial u}{\partial x}\right)$$
$$=\frac{\lambda}{\rho}\left(\frac{\partial^T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+u\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}\right)-\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$
$$\left[-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}+2\nu t\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right)\right]\tag{8-2-41}$$

气体状态方程与质量守恒方程均可用式(8-2-3)至式(8-2-6)。

边界条件：

$$u(x,\ y,\ z)\big|_{x=0}=v_e=u_e\tag{8-2-42}$$
$$u(x,\ y,\ z)\big|_{y=B}=0\tag{8-2-43}$$
$$u(x,\ y,\ z)\big|_{z=0}=0\tag{8-2-44}$$
$$\left.\frac{\partial u(x,\ y,\ z)}{\partial y}\right|_{y=0}=0\tag{8-2-45}$$
$$\left.\frac{\partial u(x,\ y,\ z)}{\partial z}\right|_{z=H}=0\tag{8-2-46}$$
$$\frac{\partial^2 u(x,\ y,\ z)}{\partial y^2}<0\tag{8-2-47}$$
$$\frac{\partial^2 u(x,\ y,\ z)}{\partial z^2}<0\tag{8-2-48}$$
$$p(x,\ y,\ z)\big|_{x=0}=p_e\tag{8-2-49}$$
$$T(x,\ y,\ z)\big|_{x=0}=T_e\tag{8-2-50}$$

8.2.2.2 无因次控制方程与边界条件

为了应用数学工具，必须将方程与边界条件变成无因次形式。为此取

$$\frac{x}{H}=X,\ \frac{z}{H}=Z,\ \frac{y}{B}=Y,\ \frac{u}{v_e}=U,\ \frac{p}{p_e}=p_0,\ \frac{T}{T_e}=T_0,\ \frac{\nu}{\nu_e}=\nu_0$$

将比值代回控制方程与边界条件。

(1) 动量方程

$$\frac{\nu_e v_e}{2H^2}\frac{\nu_0}{\rho_0}\frac{\partial p_0}{\partial X}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)+\frac{v_e^2}{H}U\frac{\partial U}{\partial X}+\frac{\nu_e v_e}{2H^2}\frac{\nu_0}{U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\frac{\partial U}{\partial X}$$
$$=-\frac{p_e H}{\rho_e\nu_e v_e}\frac{\partial p_0}{\rho_0\partial X}+\nu_0\left\{\frac{4}{3}\frac{\nu_e v_e}{H^2}\frac{\partial^2 U}{\partial X^2}+(1-\varphi^{\frac{2}{3}})\frac{\nu_e v_e}{H^2}\left[\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right]\right.$$
$$\left.+\frac{\nu_e v_e^2}{H^3}6t\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]\right\}\tag{8-2-40$'$}$$

将全式除以$\frac{\nu_e v_e}{H^2}$，则有

$$\frac{\nu_0}{2}\frac{\partial p_0}{\rho_0\partial X}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)+ReU\frac{\partial U}{\partial X}+\frac{\nu_0}{2U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\frac{\partial U}{\partial X}$$

$$=-K_1\frac{1}{\rho_0}\frac{\partial p_0}{\partial X}+\nu_0\left\{\frac{4}{3}\frac{\partial^2 U}{\partial X^2}+(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right]\right.$$

$$\left.+K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]\right\}\tag{8-2-51}$$

(2) 能量方程

$$\frac{v_e^3}{H}\frac{U^2\partial U}{\partial X}-\frac{1}{2}\frac{v_e^3}{H}\frac{U^3}{\rho_0}\frac{\partial\rho_0}{\partial X}+\frac{\varphi\nu_e}{2}\frac{v_e^2}{H^2}\nu_0U\left(\frac{H}{B}\frac{\partial^2 U}{\partial X\partial Y}+\frac{\partial^2 U}{\partial X\partial Z}\right)+\frac{\nu_e}{2}\frac{v_e^2}{H^2}\nu_0\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left(\frac{U}{\rho_0}\frac{\partial p_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$=\frac{\lambda_e T_e\lambda_0}{\rho_e H^2\rho_0}\left[\frac{\partial^2 T_0}{\partial X^2}+\left(\frac{H}{B}\right)^2\frac{\partial^2 T_0}{\partial Y^2}+\frac{\partial^2 T_0}{\partial Z^2}\right]+U\left(-\frac{p_e v_e}{\rho_e H}\frac{\partial p_0}{\rho_0\partial X}+\frac{4}{3}\frac{\nu_e v_e}{H^2}\nu_0\frac{\partial^2 U}{\partial X^2}\right)-\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0}{U}$$

$$\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left\{-\frac{\nu_e}{H^2}\frac{p_e}{\rho_e}\frac{\partial p_0}{\rho_0\partial X}+\frac{4}{3}\frac{\nu_e^2 v_e\nu_0}{H^3}\frac{\partial^2 U}{\partial X^2}+2\nu_e^2 t\,\frac{v_e^2}{H^4}\nu_0\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]\right\}$$

$$\tag*{(8-2-41)$'$}$$

除以$\frac{\nu_e v_e^2}{H^2}$，则有

$$ReU^3\frac{\partial U}{\partial X}-\frac{1}{2}Re\frac{U^3}{\rho_0}\frac{\partial\rho_0}{\partial X}+\frac{\varphi}{2}\nu_0U\left(\frac{H}{B}\frac{\partial^2 U}{\partial X\partial Y}+\frac{\partial^2 U}{\partial X\partial Z}\right)+\frac{\nu_0}{2}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left(\frac{U}{\rho_0}\frac{\partial\rho_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$=K_3\frac{\lambda_0}{\rho_0}\left[\frac{\partial^2 T_0}{\partial X^2}+\left(\frac{H}{B}\right)^2\frac{\partial^2 T_0}{\partial Y^2}+\frac{\partial^2 T_0}{\partial Z^2}\right]+U\left(-K_4\frac{\partial p_0}{\rho_0\partial X}+\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}\right)-\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0}{U}$$

$$\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left\{-K_5\frac{\partial p_0}{\rho_0\partial X}+\frac{4}{3}\left(\frac{\nu_e^2}{Re}\right)\frac{\partial^2 U}{\partial X^2}+K_6\nu_0\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]\right\}$$

$$\tag{8-2-52}$$

边界条件：

$$U(X,\ Y,\ Z)\big|_{X=0}=1\tag{8-2-53}$$

$$U(X,\ Y,\ Z)\big|_{Y=1}=0\tag{8-2-54}$$

$$U(X,\ Y,\ Z)\big|_{Z=0}=0\tag{8-2-55}$$

$$\left.\frac{\partial U(X,\ Y,\ Z)}{\partial Y}\right|_{Y=0}=0\tag{8-2-56}$$

$$\left.\frac{\partial U(X,\ Y,\ Z)}{\partial Z}\right|_{Z=1}=0\tag{8-2-57}$$

$$\frac{\partial^2 U(X,\ Y,\ Z)}{\partial Y^2}<0\tag{8-2-58}$$

$$\frac{\partial^2 U(X,\ Y,\ Z)}{\partial Z^2}<0\tag{8-2-59}$$

$$P_0(X,\ Y,\ Z)\big|_{X=0}=1 \tag{8-2-60}$$

$$T_0(X,\ Y,\ Z)\big|_{X=0}=1 \tag{8-1-61}$$

以上各式中：

$$Re=\frac{v_e H}{\nu_e} \tag{8-2-62}$$

$$K_1=\frac{p_e H}{\rho_e \nu_e v_e} \tag{8-2-63}$$

$$K_2=\frac{6tv_e}{H} \tag{8-2-64}$$

$$K_3=\frac{\lambda_e T_e}{\rho_e \nu_e v_e^2} \tag{8-2-65}$$

$$K_4=\frac{p_e H}{\rho_e v_e} \tag{8-2-66}$$

$$K_5=\frac{p_e H}{\rho_e v_e^2} \tag{8-2-67}$$

$$K_6=\frac{2\nu_e t}{H^2} \tag{8-2-68}$$

8.2.2.3　速度与温度分布

根据边界条件，选定速度与温度分布为

$$U=(1+X)^{\alpha}(2Z-Z^2)(1-Y^2) \tag{8-2-69}$$

$$T=(1+X)^{-\beta} \tag{8-2-70}$$

应用上两式之前，首先确定参变常数 α，β。为此，将两式代入式(8-2-51)与式(8-2-52)中，然后利用气态方程与质量守恒，将压力 p_0 与密度 ρ_0 均转化为速度与温度，结果，最后则得两个均含 α，β 的独立方程，联立求解则得 α，β。

首先将式(8-2-69)代入式(8-2-51)，逐项分别计算

$$\begin{aligned}\frac{\nu_0}{2}\frac{\partial\rho_0}{\rho_0\partial X}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)&=\frac{\nu_0}{2U}\frac{\partial U}{\partial X}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\\&=\nu_0\alpha(1+X)^{\alpha-1}\left[(1-Z)(1-Y^2)-\frac{H}{B}Y(2Z-Z^2)\right]\\&=\nu_0\alpha(1+X)^{\alpha-1}\left[\frac{1}{2}\times\frac{2}{3}-\frac{H}{B}\frac{1}{2}\times\frac{2}{3}\right]\\&=\frac{1}{3}\nu_0\alpha(1+X)^{\alpha-1}\left(1-\frac{H}{B}\right)=\frac{1}{3}\nu_0\alpha\left(1-\frac{H}{B}\right)(1+X)^{\alpha-1}\end{aligned} \tag{1}$$

$$\begin{aligned}ReU\frac{\partial U}{\partial X}&=Re(1+X)^{2\alpha-1}\alpha\int_0^1(2Z-Z^2)^2(1-Y^2)^2\mathrm{d}Z\mathrm{d}Y\\&=ReU\frac{\partial U}{X}=0.285Re\alpha(1+X)^{2\alpha-1}\end{aligned} \tag{2}$$

$$\frac{\nu_0}{2U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\frac{\partial U}{\partial X}$$

$$=\frac{1}{2}\nu_0\alpha(1+X)^{\alpha-1}\left[\frac{H}{B}(-2Y)(2Z-Z^2)+(1-Z)(1-Y^2)\right]$$

$$=\nu_0\alpha(1+X)^{\alpha-1}\left[\int_0^1\int_0^1(1-Z)(1-Y^2)\mathrm{d}Z\mathrm{d}Y-\frac{H}{B}\int_0^1\int_0^1 Y(2Z-Z^2)\mathrm{d}Y\mathrm{d}Z\right]$$

$$=\nu_0\alpha(1+X)^{\alpha-1}\frac{1}{3}\left(1-\frac{H}{B}\right)\tag{3}$$

$$-K_1\frac{\partial p_0}{\rho_0\partial X}=-K_1U\frac{\partial p_0}{\partial X}=-K_1U\left(T_0\frac{\partial\rho_0}{\partial X}+\rho_0\frac{\partial T_0}{\partial X}\right)=-K_1U\left[T_0\frac{\partial}{\partial X}\left(\frac{1}{U}\right)+\frac{1}{U}\frac{\partial T_0}{\partial X}\right]$$

$$=K_1\left(\frac{T_0}{U}\frac{\partial U}{\partial X}-\frac{\partial T_0}{\partial X}\right)=K_1\alpha(1+X)^{\beta-1}(\alpha-\beta)\tag{4}$$

$$\frac{4}{3}\nu_0\frac{\partial^2U}{\partial X^2}=\frac{4}{3}\nu_0\alpha(\alpha-1)(1+X)^{\alpha-2}(2Z-Z^2)(1-Y^2)$$

$$=\frac{4}{3}\nu_0\alpha(\alpha-1)(1+X)^{\alpha-2}\int_0^1\int_0^1(2Z-Z^2)(1-Y^2)\mathrm{d}Z\mathrm{d}Y$$

$$=\frac{16}{27}\nu_0\alpha(\alpha-1)(1+X)^{\alpha-2}\tag{5}$$

$$\nu_0(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2\frac{\partial^2U}{\partial Y^2}+\frac{\partial^2U}{\partial Z^2}\right]$$

$$=-2\nu_0(1-\varphi^{\frac{2}{3}})(1+X)^{\alpha}\left[\left(\frac{H}{B}\right)^2(2Z-Z^2)+(1-Y^2)\right]$$

$$=-2\nu_0(1-\varphi^{\frac{2}{3}})(1+X)^{\alpha}\int_0^1\int_0^1\left[\left(\frac{H}{B}\right)^2(2Z-Z^2)+(1-Y^2)\right]\mathrm{d}Y\mathrm{d}Z$$

$$=-\frac{4}{3}\nu_0(1-\varphi^{\frac{2}{3}})(1+X)^{\alpha}\left[\left(\frac{H}{B}\right)^2+1\right]\tag{6}$$

$$\nu_0K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2U}{\partial Z^2}\right]$$

$$=\nu_0K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3(1+X)^{2\alpha}(2Z-Z^2)^2(-2Y)(-2)+(1+X)^{2\alpha}(1-Y^2)^2 2(1-Z)(-2)\right]$$

$$=4\nu_0K_2\varphi^{\frac{2}{3}}(1+X)^{2\alpha}\left[\left(\frac{H}{B}\right)^3\int_0^1\int_0^1(2Z-Z^2)^2Y\mathrm{d}Z\mathrm{d}Y-\int_0^1\int_0^1(1-Y^2)^2(1-Z)\mathrm{d}Y\mathrm{d}Z\right]$$

$$=2\nu_0K_2\varphi^{\frac{2}{3}}(1+X)^{2\alpha}0.533\left[\left(\frac{H}{B}\right)^3-1\right]=1.066\nu_0K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right](1+X)^{2\alpha}\tag{7}$$

将式(1)～式(7)代回式(8-2-51)，有

$$\frac{1}{3}\nu_0\alpha\left(1-\frac{H}{B}\right)(1+X)^{\alpha-1}+0.355Re\alpha(1+X)^{2\alpha-1}-2.08\nu_0\alpha(1+X)^{\alpha-1}$$

$$=K_1(1+X)^{\beta-1}(\alpha-\beta)+0.593\nu_0\alpha(\alpha-1)(1+X)^{\alpha-2}-\frac{4}{3}\nu_0(1-\varphi^{\frac{2}{3}})(1+X)^{\alpha}$$

$$+1.066\nu_0K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right](1+X)^{2\alpha}\tag{8-2-51$'$}$$

全式取含$(1+X)$项的对数，然后除 $\alpha\ln(1+X)$，取 $\nu_0\approx1$，则有

$$\frac{1}{3}\left(1-\frac{H}{B}\right)(\alpha-1)+0.355Re(2\alpha-1)-2.08(\alpha-1)$$

$$=K_1\frac{(\alpha-\beta)(\beta-1)}{\alpha}+0.593(\alpha-1)(\alpha-2)-\frac{4}{3}(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right]+2.132K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right] \tag{8-2-71}$$

将式(8-2-69)与式(8-2-70)代入式(8-2-52)，由于过长，分别计算各项：

$$\begin{aligned}ReU^3\frac{\partial U}{\partial X}&=Re(1+X)^{3\alpha}(2Z-Z^2)^3(1-Y^2)^3\alpha(1+X)^{\alpha-1}(2Z-Z^2)(1-Y^2)\\&=Re\alpha(1+X)^{4\alpha-1}(2Z-Z^2)^4(1-Y^2)^4\\&=Re\alpha(1+X)^{4\alpha-1}\int_0^1(Z^8-8Z^7+24Z^6-32Z^5+16Z^4)\mathrm{d}Z\\&\quad\int_0^1(Y^8-4Y^6+6Y^4-4Y^2+1)\mathrm{d}Y\\&=Re\alpha(1+X)^{4\alpha-1}(6.745-6.333)(2.325-1.904)\\&=0.412\times0.421Re\alpha(1+X)^{4\alpha-1}=0.173Re\alpha(1+X)^{4\alpha-1}\end{aligned} \tag{1}$$

$$\begin{aligned}-\frac{1}{2}Re\frac{U^3\partial\rho_0}{\rho_0\partial X}&=\frac{1}{2}ReU^3\left(-\frac{1}{U}\frac{\partial U}{\partial X}\right)=\frac{1}{2}ReU^2\frac{\partial U}{\partial X}\\&=\frac{1}{2}Re\alpha(1+X)^{3\alpha-1}(2Z-Z^2)^3(1-Y^2)^3\\&=\frac{1}{2}Re\alpha(1+X)^{3\alpha-1}\int_0^1\int_0^1(8Z^3-12Z^4+6Z^5-Z^6)(1-3Y^2+3Y^4-Y^6)\mathrm{d}Z\mathrm{d}Y\\&=\frac{1}{2}Re\alpha(1+X)^{3\alpha-1}=\frac{0.2088}{2}Re\alpha(1+X)^{3\alpha-1}\\&=0.104Re\alpha(1+X)^{3\alpha-1}\end{aligned} \tag{2}$$

$$\begin{aligned}&\frac{\varphi}{2}\nu_0U\left(\frac{H}{B}\frac{\partial^2U}{\partial X\partial Y}+\frac{\partial^2U}{\partial X\partial Z}\right)\\&=\frac{\varphi}{2}\nu_0(1+X)^{\alpha}(2Z-Z^2)(1-Y^2)\\&\quad\left[\frac{H}{B}\alpha(1+X)^{\alpha-1}(2Z-Z^2)(-2Y)+\alpha(1+X)^{\alpha-1}(1-Y^2)2(1-Z)\right]\\&=\frac{\varphi}{2}\nu_0\alpha(1+X)^{2\alpha-1}\left\{\int_0^1\int_0^1\left[(1-Z)(1-Y^2)^2-\frac{H}{B}Y(2Z-Z^2)^2\right]\right\}\mathrm{d}Y\mathrm{d}Z\\&=-0.216\varphi\nu_0\alpha(1+X)^{2\alpha-1}\end{aligned} \tag{3}$$

$$\frac{\nu_0}{2}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left(\frac{U}{\rho_0}\frac{\partial\rho_0}{\partial X}+\frac{\partial U}{\partial X}\right)=\frac{\nu_0}{2}\left(\frac{H}{B}\frac{U}{Y}+\frac{\partial U}{\partial Z}-\frac{\partial U}{\partial X}+\frac{\partial U}{\partial X}\right)=0 \tag{4}$$

$$\begin{aligned}&K_3\frac{\lambda_0}{\rho_0}\left[\frac{\partial^2T_0}{\partial X^2}+\left(\frac{H}{B}\right)^2\frac{\partial^2T_0}{\partial Y^2}+\frac{\partial^2T_0}{\partial Z^2}\right]\\&=K_3\lambda_0(1+X)^{\alpha}(2Z-Z^2)(1-Y^2)\beta(\beta-1)(1+X)^{\beta-2}\\&=K_3\lambda_0\beta(\beta-1)(1+X)^{\alpha+\beta-1}\int_0^1\int_0^1(2Z-Z^2)(1-Y^2)\mathrm{d}Y\mathrm{d}Z\end{aligned}$$

$$=\frac{4}{9}K_3\lambda_0\beta(\beta-1)(1+X)^{\alpha+\beta-1} \tag{5}$$

$$U-K_4\frac{\partial p_0}{\rho_0\partial X}+\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0}{U}K_5\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\frac{\partial p_0}{\rho_0\partial X}$$

$$=\left[K_5\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0}{U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)-UK_4\right]\frac{\partial p_0}{\rho_0\partial X}$$

$$=\left[K_5\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0}{U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)-UK_4\right]\left(\frac{\partial T_0}{\partial X}-\frac{T_0}{U}\frac{\partial U}{\partial X}\right)$$

$$=K_5\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0}{U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left(\frac{\partial T_0}{\partial X}-\frac{T_0}{U}\frac{\partial U}{\partial X}\right)-UK_4\left(\frac{\partial T_0}{\partial X}-\frac{T_0}{U}\frac{\partial U}{\partial X}\right)$$

$$=K_5\varphi^{\frac{2}{3}}\nu_0\left(\int_0^1\frac{-Y}{1-Y^2}\mathrm{d}Y+\int_0^1\frac{1-Z}{1Z-Z^2}\mathrm{d}Z\right)[\beta(1+X)^{\beta-1}-\alpha(1+X)^{\beta-1}]$$

$$-K_4(\alpha-\beta)(1+X)^{\alpha+\beta-1}\int_0^1\int_0^1(2Z-Z^2)(1-Y^2)\mathrm{d}Y\mathrm{d}Z$$

$$=1.039K_5\varphi^{\frac{2}{3}}\nu_0(\alpha-\beta)(1+X)^{\beta-1}+\frac{4}{9}K_4(\alpha-\beta)(1+X)^{\alpha+\beta-1}$$

$$=(\alpha-\beta)\left[1.039K_4\varphi^{\frac{2}{3}}\nu_0(1+X)^{\beta-1}+\frac{4}{9}K_4(1+X)^{\alpha+\beta-1}\right] \tag{6}$$

$$\frac{4}{3}\nu_0U\frac{\partial^2U}{\partial X^2}=\frac{4}{3}\nu_0(1+X)^{2\alpha-2}\alpha(\alpha-1)\int_0^1\int_0^1(2Z-Z^2)^2(1-Y^2)^2\mathrm{d}Z\mathrm{d}Y$$

$$=0.379\nu_0\alpha(\alpha-1)(1+X)^{2\alpha-2} \tag{7}$$

$$-\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0}{U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\frac{4}{3}\frac{\nu_0}{Re}\frac{\partial^2U}{\partial X^2}$$

$$=1.039\nu_0\frac{4}{3}\frac{\nu_0\varphi^{\frac{2}{3}}}{Re}\alpha(\alpha-1)(1+X)^{\alpha-2}\int_0^1\int_0^1(2Z-Z^2)(1-Y^2)\mathrm{d}Z\mathrm{d}Y$$

$$=0.462\nu_0^2\varphi^{\frac{2}{3}}\frac{\alpha}{Re}(\alpha-1)(1+X)^{\alpha-2} \tag{8}$$

$$-\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0}{U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)K_6\nu_0\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2U}{\partial Z^2}\right]$$

$$=\frac{1.039}{2}\varphi^{\frac{2}{3}}K_6\nu_0^2\left[\left(\frac{H}{B}\right)^3(1+X)^{2\alpha}(2Z-Z^2)^2(-2Y)(-2)+(1+X)^{2\alpha}(1-Y^2)^2 2(1-Z)(-2)\right]$$

$$=0.519\varphi^{\frac{2}{3}}K_6\nu_0^2(1+X)^{2\alpha}\left[\left(\frac{H}{B}\right)^3\int_0^1\int_0^1Y(2Z-Z^2)^2\mathrm{d}Y\mathrm{d}Z-\int_0^1\int_0^1(1-Y^2)^2(1-Z)\mathrm{d}Y\mathrm{d}Z\right]$$

$$=0.519\varphi^{\frac{2}{3}}K_6\nu_0^2(1+X)^{2\alpha}\left[0.6\left(\frac{H}{B}\right)^3-0.333\right] \tag{9}$$

将式(1)～式(9)各项代回式(8-2-52)，有

$$0.173Re\alpha(1+X)^{4\alpha-1}+0.104Re\alpha(1+X)^{3\alpha-1}-0.216\nu_0\alpha(1+X)^{2\alpha-1}+0$$

$$=0.444K_3\lambda_0\beta(\beta-1)(1+X)^{\alpha+\beta-1}+(\alpha-\beta)[1.039K_5\varphi^{\frac{2}{3}}\nu_0(1+X)^{\beta-1}+0.444K_4(1+X)^{\alpha+\beta-1}]+0.379\nu_0\alpha(\alpha-1)(1+X)^{2(\alpha-1)}+0.462\nu_0^2\varphi^{\frac{2}{3}}\frac{\alpha}{Re}(\alpha-1)(1+X)^{\alpha-2}+$$

$$0.519\varphi^{\frac{2}{3}}K_6\nu_0^2\left[0.6\left(\frac{H}{B}\right)^3-0.333\right](1+X)^{2\alpha} \tag{8-2-52)'$$

对含$(1+X)$的项取对数，$\nu_0\approx1$，则有

$$\begin{aligned}&0.173Re\alpha(4\alpha-1)\ln(1+X)+0.104Re\alpha(3\alpha-1)\ln(1+X)-0.216\varphi\alpha(2\alpha-1)\ln(1+X)+0\\&=0.444K_3\lambda_0\beta(\beta-1)(\alpha+\beta-1)\ln(1+X)+(\alpha-\beta)[1.039K_5(\beta-1)\ln(1+X)+\\&\quad 0.444K_4(\alpha+\beta-1)\ln(1+X)]+0.379\alpha(\alpha-1)(2\alpha-1)\ln(1+X)+0.462\varphi^{\frac{2}{3}}\frac{\alpha}{Re}(\alpha-\\&\quad 1)(\alpha-2)\ln(1+X)+0.519\varphi^{\frac{2}{3}}K_6\left[0.6\left(\frac{H}{B}\right)^3-0.333\right]2\alpha\ln(1+X)\end{aligned} \tag{8-2-52)''$$

全式除以 $\alpha\ln(1+X)$，则有

$$\begin{aligned}&0.173Re(4\alpha-1)+0.104Re(3\alpha-1)-0.216\varphi(2\alpha-1)\\&=\frac{0.444K_3\lambda_0\beta(\beta-1)(\alpha+\beta-1)}{\alpha}+\frac{\alpha-\beta}{\alpha}[1.039K_5\varphi^{\frac{2}{3}}(\beta-1)+0.444K_4(\alpha+\beta-1)]+\\&\quad 0.379(\alpha-1)(2\alpha-1)+\frac{0.462}{Re}(\alpha-1)(\alpha-2)+1.038\varphi^{\frac{2}{3}}K_6\left[0.6\left(\frac{H}{B}\right)^3-0.333\right]\end{aligned}$$

(8-2-72)

式(8-2-71)与式(8-2-22)是独立方程，均含 α，β 两个未知数，联立求解，可得其二数，现举例说明其应用方法。

例 8-4　由空气压缩罐引出矩形断面宽为 0.04m，高为 0.03m 的管道，罐内温度 $t=40$℃，压力为 10 个标准大气压。测得管道进口处压力为 8.5 个标准大气压。温度为 36℃，进口处局部阻力系数 $\xi=0.2$。确定进口速度、密度、质量流量、管道温度与速度变化规律。

解　进口速度公式：

$$v_e=\sqrt{\frac{2R(T_b-T_e)}{\xi+1}}=\sqrt{\frac{2\times287(313-309)}{0.2+1}}=\sqrt{\frac{2\times287\times4}{1.2}}$$

$$=\sqrt{\frac{2296}{1.2}}=\sqrt{1913.3}=43.741\text{m/s}$$

进口处密度：

$$\rho_e=\frac{p_e}{RT_e}=\frac{8.5\times1.013\times10^5}{287\times309}=9.718\text{kg/m}^3$$

进口质量流量：

$$Q_m=\rho_e v_e A=9.718\times43.741\times0.04\times0.03=0.510\text{kg/s}$$

要确定速度与温度分布，必须先确定 α 与 β，应用式(8-2-71)与式(8-2-72)联立可解出两值。为此先确定几个有关的无因次数。

$$Re=\frac{v_e H}{\nu_e}=\frac{43.741\times0.015}{16.6\times10^{-6}}=0.395\times10^5$$

$$K_1=\frac{p_e H}{\rho_e\nu_e v_e}=\frac{8.5\times1.013\times10^5\times0.015}{9.718\times16.6\times10^{-6}\times43.741}=\frac{0.129\times10^5}{0.705\times10^{-2}}=0.180\times10^7$$

$$K_2=\frac{6tv_e}{H}=\frac{6\times0.01\times43.741}{0.015}=\frac{6\times0.01\times43.741}{0.015}=174.964$$

$$K_3=\frac{\lambda_e T_e}{\rho_e \nu_e v_e^2}=\frac{2.71\times309\times10^6}{10^2\times16.6\times(43.741)^2}=\frac{837.39\times10^6}{10^2\times16.6\times1913.275}=\frac{837.39\times10^4}{31760.366}$$
$$=0.026\times10^4$$
$$K_4=\frac{p_e H}{\rho_e v_e}=\frac{8.5\times1.013\times10^5\times0.015}{9.718\times43.741}=\frac{0.129\times10^5}{425.075}=0.0003\times10^5=30$$
$$K_5=\frac{p_e H}{\rho_e v_e^2}=\frac{8.5\times1.013\times10^5\times0.015}{9.718\times(43.741)^2}=\frac{0.129\times10^5}{9.718\times1913.275}=\frac{0.129\times10^5}{18593.207}$$
$$=0.0000069\times10^5=0.69$$
$$K_6=\frac{2\nu_e t}{H^2}=\frac{2\times16.6\times0.01}{10^6\times(0.015)^2}=\frac{0.332}{10^6\times0.00025}=\frac{1328}{10^6}=0.001328$$
$$\frac{K_1}{Re}=\frac{0.180\times10^7}{0.395\times10^5}=\frac{0.180}{0.395}\times10^2=0.456\times10^2=45.6$$
$$\frac{K_2}{Re}=\frac{174.964}{0.395\times10^5}=0.0044$$

将以上有关值代入式(8-2-71)，则有

$$\frac{1}{3}\left(1-\frac{0.015}{0.02}\right)(\alpha-1)+0.355\times0.395\times10^5(2\alpha-1)-2.08(\alpha-1)$$
$$=0.18\times10^7\frac{(\alpha+\beta)(\beta+1)}{\alpha}+0.593(\alpha-1)(\alpha-2)-\frac{4}{3}(1-\varphi^{\frac{2}{3}})\left[\left(\frac{0.015}{0.02}\right)^2+1\right]+2.132$$
$$\times0.004\varphi^{\frac{2}{3}}\left[\left(\frac{0.015}{0.02}\right)^3+1\right] \tag{8-2-71$'$}$$
$$0.083(\alpha-1)+0.140\times10^5(2\alpha-1)-2.08(\alpha-1)$$
$$=0.18\times10^7\frac{(\alpha-\beta)(\beta-1)}{\alpha}+0.593(\alpha-1)(\alpha-2)-2.08(1-\varphi^{\frac{2}{3}})-233.853\varphi^{\frac{2}{3}} \tag{8-2-71$''$}$$

全式除以 0.14×10^5，忽略小量

$$2\alpha-1=-1.286\times10^2\frac{(\alpha-\beta)(\beta-1)}{\alpha} \tag{8-2-71$'''$}$$

将式(8-2-72)除以 Re，则为

$$0.713(4\alpha-1)+0.104(3\alpha-1)-\frac{0.216\varphi}{Re}(2\alpha-1)$$
$$=\frac{0.444K_3\lambda_0\beta(\beta-1)(\alpha+\beta-1)}{\alpha Re}+\frac{\alpha-\beta}{\alpha}\left[\frac{1.039K_5\varphi^{\frac{2}{3}}(\beta-1)}{Re}+\frac{0.444K_4}{Re}(\alpha+\beta-1)\right]$$
$$+\frac{0.379(\alpha-1)(2\alpha-1)}{Re}+\frac{0.462}{Re}(\alpha-1)(\alpha-2)+\frac{1.038\varphi^{\frac{2}{3}}K_6}{Re}\left[0.6\left(\frac{H}{B}\right)^3-0.333\right] \tag{8-2-72$'$}$$

进行量级比较，略去小量，则有

$$[0.173(4\alpha-1)+0.104(3\alpha-1)]\alpha=0 \tag{8-2-72$''$}$$

$$\alpha(1.004\alpha-0.27)=0 \tag{8-2-72)''''}$$

$$\alpha=\frac{0.27}{1.004}=0.268 \tag{8-2-73}$$

代入式(8-2-71)''''，得

$$(2\alpha-1)\alpha=-128.6(\alpha+\beta)(\beta+1) \tag{8-2-71)'''''}$$

$$(2\times0.208-1)0.268=-128.6(0.268+\beta)(\beta+1)$$

$$\beta^2+1.269\beta+0.267=0$$

$$\beta=\frac{-1.269+\sqrt{(1.269)^2-4\times0.267}}{2}=\frac{-1.269+\sqrt{1.610-1.068}}{2}=\frac{-1.269+0.735}{2}$$

$$=-0.266 \tag{8-2-74}$$

结果速度分布：

$$U=(1+X)^{0.268}(2Z-Z^2)(1-Y^2) \tag{8-2-75}$$

温度分布：

$$T_0=(1+X)^{-0.266} \tag{8-2-76}$$

8.2.2.4 等质量流管长确定

利用式(8-2-30)，将式中 v_a 结合矩形断面管道计算出来，将能量损失率 E_f 中管道直径变成当量直径，即4倍水力半径 $D=4R'$。

$$R'=\frac{ab}{2(a+b)} \tag{8-2-77}$$

式中：a——矩形断面高度；

b——矩形断面宽度；

R'——水力半径。

(1) 1kg 流体损失功率

依据式(8-2-32)可得将1kg流体损失功率为

$$\frac{\gamma h_f}{\rho}=gh_f=g\lambda\frac{l}{D}\frac{v_a^2}{2g}=\frac{\lambda l v_a}{D\ 2}\doteq\lambda\frac{l}{4R'}\frac{v_a^2}{2}=\frac{1}{8}\lambda\frac{H}{R'}\frac{l}{H}v_a^2=\frac{1}{8}\lambda\frac{H}{R'}v_a^2 \tag{8-2-78}$$

(2) 矩形断面下平均速度

$$v_a=\frac{1}{2}v_e(1+L)^a\int_0^1\int_0^1(2Z-Z^2)(1-Y^2)\mathrm{d}Y\mathrm{d}Z=\frac{2}{9}v_e(1+L)^a \tag{8-2-79}$$

(3) 确定等质量流管长公式

$$\frac{p_e}{\rho_e}+\frac{v_e^2}{2}=\frac{p_a}{\rho_a}+\frac{v_a^2}{2}+\frac{\gamma h_f}{\rho_a} \tag{8-2-80}$$

将式(8-2-78)与式(8-2-79)代入式(8-2-80)

$$\frac{p_e}{\rho_e}-\frac{p_a}{\rho_a}+\frac{v_e^2}{2}=\frac{4}{81}\left(1+\frac{\lambda}{4}\frac{H}{R'}L\right)\frac{1}{2}v_e^2(1+X)^{2\alpha}=\frac{2}{81}\left(1+\frac{\lambda}{4}\frac{H}{R'}L\right)v_e^2(1+X)^{2\alpha} \tag{8-2-81}$$

例8-5 续作例8-4题，已知沿程阻力系数 $\lambda=0.001$，管道出口温度为30℃，试确定等质量流管段长度。

解 首先确定水力半径 R'，由式(8-2-77)

$$R' = \frac{ab}{2(a+b)} = \frac{0.03 \times 0.04}{2(0.03+0.04)} = 0.0086$$

将有关数据代入式(8-2-81)，$\frac{p_e}{\rho_e} = RT_e$，$\frac{p_a}{\rho_a} = RT_a$，则有

$$R(T_e - T_a) + \frac{(43.741)^2}{2} = \frac{2}{81}\left[1 + \frac{0.001}{4}\left(\frac{0.015}{0.0086}\right)L\right](43.741)^2(1+L)^{2\times 0.268} \tag{8-2-81}'$$

$$287(309-303) + 956.638 = 0.0247(1 + 0.00025 \times 1.744L)1913.275(1+L)^{0.536} \tag{8-2-81}''$$

$$2678.638 = 47.258(1 + 0.0004L)(1+L)^{0.536} \tag{8-2-81}'''$$

绘图如图 8-6 所示，L 值计算如表 8-2 所列。

表 8-2　　例 8-5 题计算 L 值

L	$1+0.0004L$	$(1+L)^{0.536}$	$(1+0.0004L)(1+L)^{0.536}$	$47.258(1+0.0004L)(1+L)^{0.536}$
100	1.04	11.866	12.344	582.899
300	1.12	21.306	23.863	1127.704
500	1.20	27.997	33.597	1587.727
1000	1.40	47.573	56.802	2684.330
1500	1.60	50.413	80.660	3811.846
2000	1.80	58.812	105.802	5002.836

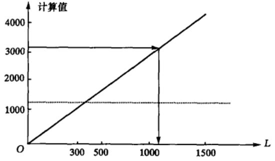


图 8-6　例 8-5 计算 L 值

由图 8-6 可知进口段 $L = 1000$，$l = 1000 \times 0.015 = 15$m。也就是说等质量流管段长 $l = 15$m。若管长超过 15m，则变为变质量流管段。

8.2.2.5　进口段(等质量流)压力分布

利用质量守恒，将式(8-2-51)改写为

$$-\frac{\nu_0}{2}\frac{1}{U}\frac{\partial U}{\partial X}\left(\frac{H}{B}\frac{\partial U}{\partial Y} + \frac{\partial U}{\partial Z}\right) + ReU\frac{\partial U}{\partial X} + \frac{\nu_0}{U}\left(\frac{H}{B}\frac{\partial U}{\partial Y} + \frac{\partial U}{\partial Z}\right)\frac{\partial U}{\partial X}$$
$$= -K_1 U\frac{\partial p_0}{\partial X} + \nu_0\left\{\frac{4}{3}\frac{\partial^2 U}{\partial X^2} + (1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2} + \frac{\partial^2 U}{\partial Z^2}\right]\right.$$

$$+K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]\Bigg\} \tag{8-2-51$''$}$$

将式(8-2-69)代入式(8-2-51)″，则有

$$-\frac{\nu_0}{2}\frac{1}{U}\frac{\partial U}{\partial X}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)=\frac{1}{3}\nu_0\left(\frac{H}{B}-1\right)\alpha(1+X)^{\alpha-1} \tag{1}$$

$$ReU\frac{\partial U}{\partial X}=0.285Re\alpha(1+X)^{2\alpha-1} \tag{2}$$

$$\frac{\nu_0}{U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)=0.222\left(1-\frac{H}{B}\right)\alpha(1+X)^{\alpha-1} \tag{3}$$

$$-K_1U\frac{\partial p_0}{\partial X}=-\frac{4}{9}K_1(1+X)^{\alpha}\frac{\partial p_0}{\partial X} \tag{4}$$

$$\nu_0\frac{4}{3}\frac{\partial^2 U}{\partial X^2}=0.593\nu_0\alpha(\alpha-1)(1+X)^{\alpha-2} \tag{5}$$

$$\nu_0(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right]=\frac{4}{3}\nu_0\left[\left(\frac{H}{B}\right)^2+1\right](1+X)^{\alpha} \tag{6}$$

$$\nu_0K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]=2.132\nu_0K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right] \tag{7}$$

将(1)～式(7)代入式(8-2-51)″，取 $\nu_0\approx1$，则有

$$\begin{aligned}&0.111\left(\frac{H}{B}-1\right)\alpha(1+X)^{\alpha-1}+0.285Re\alpha(1+X)^{2\alpha-1}\\&=-0.444K_1(1+X)^{\alpha}\frac{\partial p_0}{\partial X}+0.593\alpha(\alpha-1)(1+X)^{\alpha-2}-1.333(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right]\\&\quad(1+X)^{\alpha}+2.132K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right](1+X)^{2\alpha}\end{aligned} \tag{8-2-51$'''$}$$

除以 $(1+X)^{\alpha}$，整理为

$$\begin{aligned}\frac{\partial p_0}{\partial X}=\frac{1}{0.444K_1}\Bigg\{&0.593\alpha(\alpha-1)(1+X)^{-2}+2.132K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right](1+X)^{\alpha}-1.333(1-\varphi^{\frac{2}{3}})\\&\left[\left(\frac{H}{B}\right)^2+1\right]-0.111\alpha\left(\frac{H}{B}-1\right)(1+X)^{-1}-0.285Re\alpha(1+X)^{\alpha-1}\Bigg\}\end{aligned} \tag{8-2-51$''''$}$$

积分后，得

$$\begin{aligned}p_0=\frac{1}{0.444K_1}\Bigg\{&-\frac{0.593\alpha(\alpha-1)}{1+X}+\frac{2.132K_2\varphi^{\frac{2}{3}}}{\alpha+1}\left[\left(\frac{H}{B}\right)^3-1\right](1+X)^{\alpha+1}-1.333(1-\varphi^{\frac{2}{3}})\\&\left[\left(\frac{H}{B}\right)^2+1\right](1+X)-0.111\left(\frac{H}{B}-1\right)\alpha\ln(1+X)-0.285Re(1+X)^{\alpha}\Bigg\}+c\end{aligned} \tag{A}$$

当 $X=0$，$p_0=1$，则 c 为

$$\begin{aligned}c=1+\frac{1}{0.444K_1}\Bigg\{&-0.285Re+1.333(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right]+0.593\alpha(\alpha-1)\\&-\frac{2.132K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right]}{\alpha+1}\Bigg\}\end{aligned} \tag{B}$$

将式(B)代回式(A)，则有

$$p_0=1+\frac{1}{0.444K_1}\left\{0.285Re[1-(1+X)^{\alpha}]+1.333(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right][1-(1+X)]\right.$$
$$+0.593\alpha(\alpha-1)\left[1-\frac{1}{1+X}\right]+2.132\,\frac{K_2\varphi^{\frac{2}{3}}}{\alpha+1}\left[\left(\frac{H}{B}\right)^3-1\right][(1+X)^{\alpha+1}-1]$$
$$\left.-0.111\left(\frac{H}{B}-1\right)\alpha\ln(1+X)\right\} \tag{8-2-82}$$

式中 $p_0=\frac{p}{p_e}$，将它代入，则有因次压力。

8.2.2.6 边层流界面位置(R_B)

将式(8-2-79)代入式(8-2-69)

$$\frac{2}{9}v_e(1+X)^{\alpha}=v_e(1+X)^{\alpha}(2Z-Z^2)(1-Y^2)$$

$$\frac{2}{9}=(2Z-Z^2)(1-Y^2) \tag{8-2-83}$$

当确定垂直 z 方向两个壁面上的边层流界面时，则上式取 $Z=Z_B$，当确定垂直 y 方向两个壁面上的边层流界面时，则上式 $Y=Y_B$。

$$\frac{2}{9}=(2Z_B-Z_B^2)(1-Y^2) \tag{8-2-84}$$

$$\frac{2}{9}=(2Z-Z^2)(1-Y^2) \tag{8-2-85}$$

例如，当 $Y=0$，由式(8-2-84)

$$Z_b=1-\sqrt{1-\frac{2}{9}}=0.118$$
$$z_b=0.015\times0.118=0.0018\text{m}$$

当 $Y=0.5$ 时

$$Z_b=1-\sqrt{1-\frac{2}{9\times0.75}}=1-\sqrt{1-0.296}=0.161$$
$$z_b=0.015\times0.161=0.0024\text{m}$$

当 $Z=1$，由式(8-2-85)

$$Y^2=1-\frac{2}{9}=1-0.222=0.778$$
$$Y=0.882$$

由坐标选定关系

$$Y_B=B-Y=1-0.882=0.118$$
$$y_B=0.04\times0.118=0.0047\text{m}$$

当 $Z=0.5$，由式(8-2-85)

$$\frac{2}{9}=(1-(0.5)^2)(1-Y^2)=0.75(1-Y^2)$$

$$Y^2=1-0.296=0.704$$
$$Y=0.839$$
$$Y_B=B-Y=1-0.839=0.161$$
$$y_B=0.04\times0.161=0.0064\text{m}$$

8.2.2.7　边层流界面上涡旋强度与大小

依式(8-2-69)及涡旋定义式，可得分别垂直于 z 轴与 y 轴边层流界面上的涡旋与大小。首先垂直于 z 轴。

$$\omega\big|_{z=z_b}=\frac{v_e}{H}(1+X)^a(1-Y^2)(1-Z_B) \tag{8-2-86}$$

涡旋直径

$$d_{sz}=2\sqrt{\frac{10\nu}{\omega\big|_{z=z_b}}} \tag{8-2-87}$$

垂直于 y 轴

$$\omega\big|_{y=y_b}=\frac{1}{2}\frac{v_e}{B}(1+X)^a(2Z-Z^2)(-2Y)=\frac{v_e}{B}(1+X)^a(2Z-Z)Y_B \tag{8-2-88}$$

涡旋直径

$$d_{sy}=2\sqrt{\frac{10\nu}{\omega\big|_{y=y_b}}} \tag{8-2-89}$$

例 8-6　继例 8-4 题，确定 $X=500$，垂直于 z 轴的边层流界面上 $Y=\frac{1}{2}$点的涡旋强度与大小，确定垂直于 y 轴的边层流界面上 $Z=\frac{1}{2}$点的涡旋强度与大小。

解　由式(8-2-86)计算 z 方向的涡旋与大小。

$$\omega\big|_{z=z_b}=\frac{v_e}{H}(1+X)^a(1-Y^2)(1-Z_B)=\frac{43.741}{0.015}(1+500)^{0.268}(1-0.5^2)(1-0.161)$$
$$=2916.067\times5.291\times0.75\times0.839=9708.642^\circ/\text{s}$$

首先计算在 $X=500$ 处，温度为多少，由式(8-2-76)

$$t=t_e(1+X)^{-0.266}=36(1+50)^{0.266}=36\times\frac{1}{5.226}=36\times0.191=6.877$$

$t=6.877$ 是下降温度，实际上 $t=36-6.877=29.113$℃，按 29℃ 取 $\nu=15.94\times10^{-6}$。

$$d_{sz}=2\sqrt{\frac{10\nu}{\omega\big|_{z=z_b}}}=\frac{2}{10^2}\sqrt{\frac{1.594}{9708}}=\frac{2\times0.0128}{10^2}=0.000256\text{m}$$

$$\omega\big|_{y=y_B}=\frac{v_e}{B}(1+X)^a(2Z-Z^2)Y_B=\frac{43.741}{0.04}\times5.291\times0.75\times0.161=1863^\circ/\text{s}$$

$$d_{sy}=2\sqrt{\frac{10\nu}{\omega\big|_{y=y_b}}}=\frac{2}{10^2}\sqrt{\frac{1.594}{1863}}=0.00058\text{m}$$

依式(4-3-27)，结合本节问题，在垂直于 z 轴的边层流界面上的涡旋速度 u_z^* 为

$$u_z^* = \frac{12\nu t\left(\omega_{z=z_B}\right)}{v}$$

$$u_z^{*2} = \frac{12(H-z_B)\nu v_e^2(1+X)^{2\alpha}(1-Y^2)^2(1-Z_B)^2}{\frac{2}{9}(1+X)^{\alpha}v_e H^2}$$

$$u_z^* = \sqrt{\frac{54(H-z_B)\nu v_e(1+X)^{\alpha}(1-Y^2)^2(1-Z_B)^2}{H^2}} \tag{8-2-90}$$

在垂直于 y 轴的边层流界面上的涡旋速度

$$u_y^* = \sqrt{\frac{54(B-y_B)\nu v_e(1+X)^{\alpha}(2Z-Z^2)^2 Y_B^2}{B^2}} \tag{8-2-91}$$

例 8-7 继例 8-6 题，$X=500$ 处，分别确定垂直于 z 轴的边层流界面上 $Y=0.5$ 处的涡旋速度 u_z^*，垂直于 y 轴的边层流界面上 $Z=0.5$ 处的涡旋速度 u_y^* 各为多少？

解 由式(8-2-89)计算 u_z^*：

$$u_z^* = \sqrt{\frac{54(H-z_B)\nu v_e(1+X)^{\alpha}(1-Y^2)^2(1-Z_B)^2}{H^2}}$$

$$= \sqrt{\frac{54(0.015-0.0024)15.94\times43.741(1+500)^{0.268}(1-(0.5)^2)(1-0.161)^2}{10^6(0.015)^2}}$$

$$= \sqrt{\frac{54\times0.0126\times697.232\times5.291\times0.75\times0.704}{22.5\times10}} = \sqrt{\frac{795.178}{225}}$$

$$= \sqrt{3.534} = 1.88\text{m/s}$$

由式(8-2-90)计算 u_y^*：

$$u_y^* = \sqrt{\frac{54(B-y_B)\nu v_e(1+X)^{\alpha}(2Z-Z^2)^2 Y_B^2}{B^2}}$$

$$= \sqrt{\frac{54(0.04-0.0064)15.94\times43.741(1+500)^{0.268}(2\times0.5-(0.5)^2)(0.161)^2}{10^6(0.02)^2}}$$

$$= \sqrt{\frac{54\times0.0336\times697.232\times5.291\times0.75\times0.026}{10^6\times0.0004}} = \sqrt{\frac{1.814\times3689.055\times0.0195}{400}}$$

$$= \sqrt{\frac{130.493}{400}} = \sqrt{0.326} = 0.571\text{m/s}$$

8.2.2.8 涡旋体积分数 φ 公式

根据第 4 章，有涡旋体积分数定义

$$\varphi = \frac{t(Hu_y^* + Bu_z^*)}{48HB} \tag{8-2-92}$$

例 8-8 继例 8-7 题，试确定 $X=500$ 断面上涡旋体积分数 φ 为多少？

解

$$\varphi = \frac{t(Hu_y^* + Bu_z^*)}{48HB} = \frac{t(0.015\times0.571+0.02\times1.88)}{48\times0.015\times0.02} = \frac{t\times0.0462}{0.0144} = t\times3.208$$

当准定常时间 t 测定为 $t=0.01$ 时，$\varphi=0.032$；$t=0.001$ 时，$\varphi=0.0032$；当 $t=0.1$ 时，$\varphi=0.32$。一般 t 可以实测而定。

8.3　可压缩湍流变质量流管道

8.2 节讨论了可压缩湍流等质量流管道流动，它的长度一般比较短。如果在出口压力不变的条件下，管道长度超过等质量流管长，质量流沿程则逐渐减少，成为变质量管道。于是提出流速在断面上分布沿程如何变化？管道通过的质量流与管长有何关系？

根据等质量流的讨论，一切推出均以出口压力 p_a 为依据。因此，等质量流的长度处的压力是 p_a。所以，变质量的管道中压力均为出口处压力 p_a，即其管道压力不会沿程变化。

等质量管道其长度比较大，一般分为等温与绝热两种情况。等温管道内压力、密度均不变，绝热的管道内温度与密度变化。

本节将讨论等温圆管与绝热圆管的变质量流，以及等温与绝热矩形管道的变质量流运动。

8.3.1　等温圆管道

流动情况、坐标系与原点的选定如图 8-7 所示。

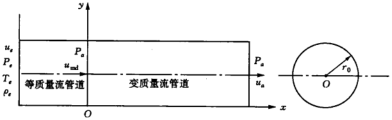


图 8-7　变质量流管道段流动示意图

(1) 运动物理控制方程与边界条件

由式(4-5-16)，并结合所讨论的问题，为

$$u\frac{\partial u}{\partial x}+\frac{\nu}{2}\left(\frac{1}{u}\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial x}=\nu\left[\frac{4}{3}\frac{\partial^2 u}{\partial x^2}+(1-\varphi^{\frac{2}{3}})\frac{\partial^2 u}{\partial y^2}+6\varphi^{\frac{2}{3}}t\,\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right] \tag{8-3-1}$$

$$u(x,\ y)\big|_{x=0}=v_{md} \tag{8-3-2}$$

$$\left.\frac{\partial u(x,\ y)}{\partial y}\right|_{y=r_0}=0 \tag{8-3-3}$$

$$\frac{\partial^2 u(x,\ y)}{\partial y^2}<0 \tag{8-3-4}$$

(2) 无因次运动控制方程与边界条件

取

$$\frac{u}{v_{\mathrm{md}}}=U,\ \frac{y}{r_0}=Y,\ \frac{x}{r_0}=X,\ \frac{\nu}{\nu_d}=\frac{\nu}{\nu_a}=\nu_0$$

式中：v_{md}——等质量流管段末端断面平均速度。

将比值代入式(8-3-1)，各项分别如下：

$$u\frac{\partial u}{\partial x}=\frac{v_{\mathrm{md}}^2}{r_0}U\frac{\partial U}{\partial X} \tag{1}$$

$$\frac{\nu}{2}\left(\frac{1}{u}\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial x}=\frac{\nu_e\nu_0}{2}\left(\frac{1}{v_{\mathrm{md}}U}\frac{v_{\mathrm{md}}}{r_0}\frac{\partial U}{\partial Y}\right)\frac{v_{\mathrm{md}}}{r_0}\frac{\partial U}{\partial X}=\frac{1}{2}\frac{\nu_e v_{\mathrm{md}}}{r_0^2}\frac{\nu_0}{U}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X} \tag{2}$$

$$\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}=\frac{4}{3}\frac{\nu_e v_{\mathrm{md}}}{r_0^2}\frac{\partial^2 U}{\partial X^2} \tag{3}$$

$$\nu(1-\varphi^{\frac{2}{3}})\frac{\partial u^2}{\partial y^2}=\nu_e(1-\varphi^{\frac{2}{3}})\nu_0\frac{v_{\mathrm{md}}}{r_0^2}\frac{\partial^2 U}{\partial Y^2} \tag{4}$$

$$6\varphi^{\frac{2}{3}}t\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}=\frac{6t\nu_e\varphi^{\frac{2}{3}}\nu_0}{r_0^3}v_{\mathrm{md}}^2\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} \tag{5}$$

将式(1)～式(5)代回式(8-3-1)，得

$$\begin{aligned}&\frac{v_{\mathrm{md}}^2}{r_0}U\frac{\partial U}{\partial X}+\frac{1}{2}\frac{\nu_e v_{\mathrm{md}}}{r_0^2}\frac{\nu_0}{U}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}\\&=\frac{4}{3}\frac{\nu_e v_{\mathrm{md}}}{r_0^2}\nu_0\frac{\partial^2 U}{\partial X^2}+\nu_e\frac{v_{\mathrm{md}}}{r_0^2}\nu_0\frac{\partial^2 U}{\partial Y^2}+\frac{6t\nu_e v_{\mathrm{md}}^2}{r_0^3}\varphi^{\frac{2}{3}}\nu_0\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\end{aligned} \tag{8-3-1$'$}$$

除以$\frac{\nu_e v_{\mathrm{md}}}{r_0^2}$，则有

$$ReU\frac{\partial U}{\partial X}+\frac{1}{2}\frac{\nu_0}{U}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}=\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}+\nu_0(1-\varphi^{\frac{2}{3}})\frac{\partial^2 U}{\partial Y^2}+K_2\varphi^{\frac{2}{3}}\nu_0\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} \tag{8-3-5}$$

式中：

$$Re=\frac{r_0 v_{\mathrm{md}}}{\nu_e} \tag{8-3-6}$$

$$K_2=\frac{6tv_{\mathrm{md}}}{r_0} \tag{8-3-7}$$

无因次边界条件：

$$U(X,\ Y)\big|_{X=0}=1 \tag{8-3-2$'$}$$

$$\left.\frac{\partial U(X,\ Y)}{\partial Y}\right|_{Y=1}=0 \tag{8-3-3$'$}$$

$$\frac{\partial^2 U(X,\ Y)}{\partial Y^2}<0 \tag{8-3-4$'$}$$

(3) 速度分布

根据边界条件，选无因次速度为

$$U=(1+X)^{-\alpha}(2Y-Y^2) \tag{8-3-8}$$

将式(8-3-8)代入式(8-3-5)，以便确定 α，即

$$-Re(2Y-Y^2)^2\alpha(1+X)^{-(2\alpha+1)}+\nu_0(1-Y)\alpha(1+X)^{-(\alpha+1)}$$
$$=\frac{4}{3}\nu_0(2Y-Y^2)\alpha(\alpha+1)(1+X)^{-(\alpha+2)}-2\nu_0(1-\varphi^{\frac{2}{3}})(1+X)^{-\alpha}-4K\varphi^{\frac{2}{3}}\nu_0(1-Y)(1+X)^{-2\alpha} \tag{8-3-5$'$}$$

将含 Y 项积分为

$$\int_0^1(2Y-Y^2)^2\mathrm{d}Y=\int_0^1(4Y^2-4Y^3+Y^4)\mathrm{d}Y=0.533 \tag{a}$$

$$\int_0^1(1-Y)\mathrm{d}Y=\left(Y-\frac{Y^2}{2}\right)\Big|_0^1=0.5 \tag{b}$$

$$\int_0^1(2Y-Y^2)\mathrm{d}Y=\left(Y^2-\frac{Y^3}{3}\right)\Big|_0^1=0.667 \tag{c}$$

将式(a)，式(b)，式(c)代入式(8-3-5)′，得

$$-0.533Re\alpha(1+X)^{-(2\alpha+1)}+0.5\nu_0\alpha(1+X)^{-(\alpha+1)}$$
$$=0.889\alpha(\alpha+1)(1+X)^{-(\alpha+2)}-2(1-\varphi^{\frac{2}{3}})\nu_0(1+X)^{-\alpha}-2K\varphi^{\frac{2}{3}}\nu_0(1+X)^{-2\alpha} \tag{8-3-9}$$

将式(8-3-9)对含 $(1+X)$ 的项取对数，而 $\nu_0\approx1$，得

$$-0.533Re\alpha(2\alpha+1)\ln(1+X)+0.5\alpha(\alpha+1)\ln(1+X)$$
$$=0.889\alpha(\alpha+1)(\alpha+2)\ln(1+X)+2(1-\varphi^{\frac{2}{3}})\alpha\ln(1+X)+2K\varphi^{\frac{2}{3}}2\alpha\ln(1+X) \tag{d}$$

将式(d)除以 $\alpha\ln(1+X)$，整理为

$$0.533Re(2\alpha+1)+0.5(\alpha-1)=0.889(\alpha+2)+2(1-\varphi^{\frac{2}{3}})+4K\varphi^{\frac{2}{3}} \tag{e}$$

式(e)除以 Re，则为

$$1.066\alpha+0.533+\frac{0.5(\alpha+1)}{Re}=-\frac{0.889(\alpha+2)}{Re}+\frac{2(1-\varphi^{\frac{2}{3}})}{Re}+\frac{4K\varphi^{\frac{2}{3}}}{Re} \tag{f}$$

式中：

$$\frac{K}{Re}=\frac{6t v_{\mathrm{d}}}{r_0}\times\frac{v_{\mathrm{e}}}{v_{\mathrm{d}}r_0}=\frac{6t\nu_{\mathrm{e}}}{r_0^2} \tag{8-3-10}$$

代入式(f)，则

$$\alpha=-0.34+\frac{22.514t\nu_{\mathrm{e}}}{r_0^2} \tag{8-3-11}$$

式中：t——湍流管道准定常流时间，一般为 0.001～0.01s，具体应实测而定。

当空气温度为 $t=30℃$ 时，$\nu_{\mathrm{e}}=16.56\times10^{-6}\ \mathrm{m^2/s}$，$r_0=0.02\mathrm{m}$，$t=0.01$，则式(8-3-11)中

$$\frac{22.514t\nu_{\mathrm{e}}}{r_0^2}=\frac{22.514\times0.01\times16.56}{0.4\times10^{-3}\times10^6}=\frac{0.225}{0.4\times10^3}=\frac{0.563}{10^3}=0.000563\ll0.1 \tag{g}$$

一般情况下，α 可以取近似值 $\alpha=-0.34$，则

$$U=(1+X)^{-0.34}(2Y-Y^2) \tag{8-3-12}$$

(4) 流量衰减与距离关系

$$Q(x)=\pi r_0^2 v^2=\pi r_0^2 v_d(1+X)^{-0.34}\frac{1}{2}\int_0^1(2Y-Y^2)\mathrm{d}Y$$

$$=\frac{1}{2}\pi r_0^2 v_d(1+X)^{-0.34}\frac{2}{3}=\frac{v_d}{3}\pi r_0^2(1+X)^{-0.34} \tag{8-3-13}$$

式(8-3-13)可以计算在已知等质量流条件下，确定距离超过等质量流管道任何位置处的流量。v_d 是等质量流管段末端断面平均速度。

(5) 边层流界面位置(Y_B)

将断面平均速度代入式(8-3-12)，得

$$\frac{1}{3}v_d(1+X)^{-0.34}=v_d(1+X)^{-0.34}(2Y_B-Y_B^2)$$

$$\frac{1}{3}=2Y_B-Y_B^2 \tag{8-3-14}$$

由上式解出

$$Y_B=0.189 \tag{8-3-15}$$

(6) 边层流界面上涡旋强度与大小

依式(8-3-12)与涡旋定义可得

$$\omega\big|_{y=y_B}=\frac{v_d}{r_0}(1+X)^{-0.34}(1-Y_B)=\frac{v_d}{r_0}(1+X)^{-0.34}(1-0.189)=0.811\frac{v_d}{r_0}(1+X)^{-0.34} \tag{8-3-16}$$

涡旋直径

$$d_s=2r_s=2\sqrt{\frac{10\nu}{\omega\big|_{y=y_B}}} \tag{8-3-17}$$

(7) 涡旋于边层流界面上径向速度 u_y^*

依第 4 章有关公式，结合本节讨论问题，涡旋径向分速 u_y^* 为

$$u_y^*=\frac{12\nu t}{v}\left(\omega\big|_{y=y_B}\right)^2=\frac{12\nu t\left[0.811\frac{v_d}{r_0}(1+X)^{-0.34}\right]^2}{\frac{1}{3}v_d(1+X)^{-0.34}}$$

$$=\frac{3\times 12\nu(r_0-y_B)(0.811)^2 v_d}{u_y^*}\frac{v_d}{r_0}(1+X)^{-0.34}$$

$$u_y^*=4.866\sqrt{\frac{\nu(r_0-y_B)v_d(1+X)^{-0.34}}{r_0^2}} \tag{8-3-18}$$

(8) 涡旋体积分数 φ 公式

依第 4 章建立的涡旋体积分数公式，可得

$$\varphi = k\frac{\pi t u_y^*}{48 r_0} \tag{8-3-19}$$

式中，k 为实验系数。

现举例说明公式应用。

例 8-9　由空压罐引出直径为 0.04m 的管道，如图 8-8 所示。罐内温度 $t=40℃$，压力为 10 个标准大气压；管道进口处测得压力为 8 个标准大气压，温度为 $t=35℃$；出口为大气压，温度为 30℃，进口处局部阻力系数 $\xi=0.5$。确定 l_2 为 150m 时，出口处的流量。

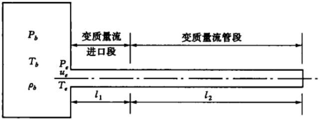


图 8-8　例 8-9 用图

解　等质量流管长，通过的流量均在例 8-1 与例 8-2 中计算出，$Q_m=0.504\text{kg/s}$，$l_1=34\text{m}$。本题首先计算 l_1 处断面平均速度。由例 8-1 已得速度分布公式

$$U=(1+X)^{0.063}(2R-R^2) \tag{a}$$

其平均速度

$$v_d=\frac{1}{3}v_e(1+X)^{0.063} \tag{b}$$

$$v_d=\frac{43.742}{3}\left(1+\frac{34}{0.02}\right)^{0.063}=14.581\times1701^{0.063}=14.581\times1.598=22.819\text{m/s}$$

由式(8-3-13)计算 $l_2=150\text{m}$ 处出口流量 Q

$$\begin{aligned}Q&=\frac{v_d}{3}\pi r_0^2(1+X)^{-0.34}=\frac{22.189}{3}3.1416(0.02)^2\left(1+\frac{150}{0.02}\right)^{-0.34}\\&=7.396\times3.1416\times0.0004(1+7500)^{-0.34}=23.235\times0.0004(7501)^{-0.34}\\&=0.0093\times\frac{1}{20.775}=0.00045\text{m}^3/\text{s}\end{aligned}$$

$$Q_m=\rho_e Q=9.174\times0.00045=0.004\text{kg/s}$$

原来流量 $Q_m=0.504\text{kg/s}$。

例 8-10　继例 8-9 题，求 l_2 长之半处边层流界面上涡旋强度与大小。

解　由式(8-3-16)

$$\begin{aligned}\omega\big|_{y=y_B}&=0.811\frac{v_d}{r_0}(1+X)^{-0.34}=0.811\frac{22.819}{0.02}\left(1+\frac{75}{0.02}\right)^{-0.34}\\&=0.811\times1140.95(1+3750)^{-0.34}=\frac{925.316}{16.413}=56.376°/\text{s}\end{aligned}$$

涡旋直径 d_s，式(8-3-17)

$$d_s=2\sqrt{\frac{10\nu}{\omega|_{y=y_B}}}=\frac{2}{100}\sqrt{\frac{1.656}{56.376}}=\frac{2}{100}\times0.171=0.00342\text{m}$$

例 8-11 继例 8-9 题，求 l_2 之半处边层流界面上涡旋径向速度 $u_y^*=u_r^*$ 为多少？它的涡旋体积分数 φ 为多少？

解 首先求

$$y_B=Y_Br_0=0.189\times0.02=0.00378\text{m}$$

由式(8-3-18)

$$u_y^*=u_r^*=4.866\sqrt{\frac{\nu(r_0-y_B)v_d(1+X)^{-0.34}}{r_0^2}}$$

$$=4.866\sqrt{\frac{0.165(0.02-0.00378)22.819\left(1+\frac{75}{0.02}\right)^{-0.34}}{10^4(0.02)^2}}$$

$$=4.866\sqrt{\frac{0.165\times0.0162\times22.819(1+3750)^{-0.34}}{4}}=4.866\sqrt{\frac{0.061}{4\times16.414}}$$

$$=4.866\sqrt{\frac{0.061}{65.652}}=4.866\sqrt{0.0009}=4.866\times0.03=0.148\text{m/s}$$

涡旋体积分数 φ 可由式(8-3-19)计算。

$$\varphi=\frac{t\pi u_y^*}{48r_0^2}=\frac{0.01\times3.1416\times0.148}{48\times0.02}=\frac{0.0046}{0.96}=0.0047$$

8.3.2 绝热圆管道

绝热变质量流圆管道与等温变质量流圆管道不同之处，是密度与温度变化相关，而密度变化又与速度变化相联。而等温情况则是密度不变，速度衰减是直接摩擦产生的。它们共同之点是在变质量管道中压力均为出口处压，沿管道不变。

流动情况，坐标系选定，原点位置与图 8-8 相同。

8.3.2.1 运动物理控制方程与边界条件

(1) 动量方程

依式(4-5-16)，结合本问题，得

$$\frac{\nu}{2}\frac{1}{\rho}\frac{\partial\rho}{\partial x}\frac{\partial u}{\partial y}+u\frac{\partial u}{\partial x}+\frac{\nu}{2}\left(\frac{1}{u}\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial x}=\nu\left[\frac{4}{3}\frac{\partial^2u}{\partial x^2}+(1-\varphi^{\frac{2}{3}})\frac{\partial^2u}{\partial y^2}+6\varphi^{\frac{2}{3}}t\frac{\partial^2u}{\partial y^2}\frac{\partial u}{\partial y}\right]\quad(8\text{-}3\text{-}20)$$

(2) 能量微分方程

绝热变质量流管道特点，就是沿管道有温度变化，要找到它的温度变化规律，必须引入能量方程。依式(4-5-41)，结合本问题，则有

$$u^2\frac{\partial u}{\partial x}-\frac{u^3}{2}\frac{\partial\rho}{\partial x}+\frac{\varphi}{2}\nu u\frac{\partial^2u}{\partial x\partial y}+\frac{1}{2}\nu\frac{\partial u}{\partial y}\left(\frac{u}{\rho}\frac{\partial p}{\partial x}+\frac{\partial u}{\partial x}\right)$$
$$=\frac{\lambda}{\rho}\left(\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}\right)+u\left(\frac{4}{3}\nu\frac{\partial^2u}{\partial x^2}-\frac{1}{2}\varphi^{\frac{2}{3}}\nu\frac{\partial u}{u\partial y}\right)\left(\frac{4}{3}\nu\frac{\partial^2u}{\partial x^2}+2\nu t\frac{\partial u}{\partial y}\frac{\partial^2u}{\partial y^2}\right)\quad(8\text{-}3\text{-}21)$$

(3) 气体绝热条件

绝热条件为

$$\frac{p}{\rho^{k}}=c \tag{8-3-22}$$

现在讨论的问题是绝热变质量流管道，其中压力 p 沿程不变。则它应写成

$$p=c\rho^{k}=\text{常数} \tag{8-3-23}$$

将它代入气态方程

$$p=R\rho T \tag{8-3-24}$$

$$c\rho^{k}=\rho RT \tag{8-3-25}$$

$$\rho^{k-1}=\frac{R}{c}T \tag{8-3-26}$$

对 x 微分，则有

$$(k-1)\rho^{k-2}\frac{\partial\rho}{\partial x}=\frac{R}{c}\frac{\partial T}{\partial x} \tag{8-3-27}$$

(4) 连续性方程

$$\rho u=1 \tag{8-3-28}$$

$$\rho\frac{\partial u}{\partial x}=-u\frac{\partial\rho}{\partial x} \tag{8-3-29}$$

$$u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial\rho}{\partial x} \tag{8-3-30}$$

边界条件

$$u(x,\ y)\big|_{x=0}=v_{\mathrm{md}} \tag{8-3-31}$$

$$\left.\frac{\partial u(x,\ y)}{\partial y}\right|_{y=r_0}=0 \tag{8-3-32}$$

$$\frac{\partial^2 u(x,\ y)}{\partial y^2}<0 \tag{8-3-33}$$

$$T(x,\ y)\big|_{x=0}=T_{\mathrm{md}} \tag{8-3-34}$$

式中：v_{md}——等质量流管道末端断面平均速度；

T_{md}——等质量流管道末端断面上平均温度；

R——气体常数。

8.3.2.2　无因次运动控制方程与边界条件

取

$$\frac{u}{v_{\mathrm{md}}}=U,\ \frac{x}{r_0}=X,\ \frac{y}{r_0}=Y,\ \frac{T}{T_{\mathrm{md}}}=T_0,\ \frac{\nu}{\nu_{\mathrm{d}}}=\nu_0,\ \frac{\rho}{\rho_{\mathrm{d}}}=\rho_0。$$

将比值代入以上方程与边界条件。

动量方程：

$$\frac{\nu_{\mathrm{d}}}{2}\frac{v_{\mathrm{md}}}{r_0^2}\frac{\nu_0}{\rho_0}\frac{\partial\rho_0}{\partial X}\frac{\partial U}{\partial Y}+\frac{v_{\mathrm{md}}^2}{r_0}U\frac{\partial U}{\partial X}+\frac{\nu_{\mathrm{d}}}{2}\frac{v_{\mathrm{md}}}{r_0^2}\frac{\nu_0}{U}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}$$

$$= \frac{4}{3}\frac{\nu_d v_{md}}{r_0^2}\nu_0 \frac{\partial^2 U}{\partial X^2} + (1-\varphi^{\frac{2}{3}})\frac{\nu_d}{r_0^2}\nu_0 v_{md}\frac{\partial^2 U}{\partial Y^2} + \frac{6t\varphi^{\frac{2}{3}}\nu_d v_{md}^2 \nu_0}{r_0^3}\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} \tag{8-3-20$'$}$$

除以$\frac{\nu_e v_{md}}{r_0^2}$，则有

$$\frac{\nu_0}{2}\frac{\partial \rho_0}{\rho_0 \partial X}\frac{\partial U}{\partial Y} + ReU\frac{\partial U}{\partial X} + \frac{\nu_0}{2}\frac{1}{U}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X} = \frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2} + (1-\varphi^{\frac{2}{3}})\nu_0\frac{\partial^2 U}{\partial Y^2} + K_2\varphi^{\frac{2}{3}}\nu_0\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} \tag{8-3-20$''$}$$

能量方程：

$$\frac{v_{md}^3}{r_0}U^3\frac{\partial U}{\partial X} - \frac{v_{md}^3}{2r_0}\frac{U^3}{\rho_0}\frac{\partial \rho_0}{\partial X} + \frac{\varphi}{2}\frac{\nu_d}{r_0^2}v_{md}^2\nu_0 U\frac{\partial^2 U}{\partial X \partial Y} + \frac{1}{2}\frac{\nu_e v_{md}^2}{r_0^2}\nu_0\frac{\partial U}{\partial Y}\left(\frac{U}{\rho_0}\frac{\partial \rho_0}{\partial X} + \frac{\partial U}{\partial X}\right)$$

$$= \frac{\lambda_d T_e \lambda_0}{\rho_d r_0^2 \rho_0}\left(\frac{\partial^2 T_0}{\partial X^2} + \frac{\partial^2 T_0}{\partial Y^2}\right) + U\left(\frac{\nu_e v_{md}^2}{r_0^2}\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}\right)$$

$$- \frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu_0}{U}\frac{\partial U}{\partial Y}\left(\frac{4}{3}\frac{\nu_e v_{md}}{r_0^3}\nu_0\frac{\partial^2 U}{\partial X^2} + \frac{2\nu_d^2 t}{r_0^4}v_{md}^2\nu_0\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right) \tag{8-3-21$'$}$$

全式除以$\frac{\nu_d v_d^2}{r_0^2}$，则有

$$ReU^3\frac{\partial U}{\partial X} - \frac{1}{2}ReU^3\frac{\partial \rho_0}{\rho_0 \partial X} + \frac{\varphi}{2}\nu_0 U\frac{\partial^2 U}{\partial X \partial Y} + \frac{\nu_0}{2}\frac{\partial U}{\partial Y}\left(\frac{U}{\rho_0}\frac{\partial \rho_0}{\partial X} + \frac{\partial U}{\partial X}\right)$$

$$= \frac{K_3\lambda_0}{\rho_0}\left(\frac{\partial^2 T_0}{\partial X^2} + \frac{\partial^2 T_0}{\partial Y^2}\right) + U\left(\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}\right) - \frac{\varphi^{\frac{2}{3}}}{2}\nu_0\frac{1}{U}\frac{\partial U}{\partial Y}\left(\frac{4}{3}\frac{\nu_0}{Re}\frac{\partial^2 U}{\partial X^2} + K_6\nu_0\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right) \tag{8-3-21$''$}$$

$$K_3 = \frac{\lambda_d T_d}{\rho_d \nu_d v_{md}^2} \tag{8-3-35}$$

$$K_6 = \frac{6t\nu_d}{r_0^2} \tag{8-3-36}$$

气体绝热条件无因次化，将有关比值代入式(8-3-27)，则有

$$\frac{\rho_0^{(K-1)}\partial \rho_0}{r_0 \rho_0 \partial X} = \frac{R_d T_d}{(K-1)c_d \rho_d^{(K-1)}}\frac{R_0}{c_0 r_0}\frac{\partial T_0}{\partial X} \tag{8-3-27$'$}$$

实际 $R_0=1$，$c_0=1$，$R_d=R$，$c_d=c$，则式(8-3-27)′可能写成

$$\rho_0^{(K-1)}\frac{\partial \rho_0}{\rho_0 \partial X}K_7\frac{1}{K-1}\frac{\partial T_0}{\partial X} \tag{8-3-37}$$

式中：

$$K_7 = \frac{RT_d}{c\rho_d^{(K-1)}} \tag{8-3-38}$$

连续性方程无因次化：

$$\frac{1}{U}\frac{\partial U}{\partial X}=\frac{\partial \rho_0}{\rho_0 \partial X} \tag{8-3-39}$$

8.3.2.3　综合微分方程

将式(8-3-37)代入式(8-3-20)″与(8-3-21)″中。考虑到式(8-3-28)，则得动量方程：

$$\frac{\nu_0}{2}\frac{K_7}{k-1}U^{k-1}\frac{\partial T_0}{\partial X}\frac{\partial U}{\partial Y}+ReU\frac{\partial U}{\partial X}+\frac{\nu_0}{2}\frac{1}{U}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}$$

$$=\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}+(1-\varphi^{\frac{2}{3}})\nu_0\frac{\partial^2 U}{\partial Y^2}+K_2\varphi^{\frac{2}{3}}\nu_0\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} \tag{8-3-40}$$

能量方程：

$$ReU^3\frac{\partial U}{\partial X}-\frac{1}{2}Re\frac{K_7}{k-1}U^{2+k}\frac{\partial T_0}{\partial X}+\frac{\varphi}{2}\nu_0 U\frac{\partial^2 U}{\partial X\partial Y}+\frac{\nu_0}{2}\left(U^k\frac{K_7}{k-1}\frac{\partial T_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$=K_3\lambda_0 U\left(\frac{\partial^2 T_0}{\partial X^2}+\frac{\partial^2 T_0}{\partial Y^2}\right)+U\left(\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}\right)-\frac{1}{2}\varphi^{\frac{2}{3}}\nu_0\frac{\partial U}{\partial Y}\left(\frac{4}{3}\frac{\nu_0}{Re}\frac{\partial^2 U}{\partial X^2}+K_6\nu_0\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right) \tag{8-3-41}$$

无因次边界条件：

$$U(X,\ Y)\big|_{X=0}=1 \tag{8-3-42}$$

$$\left.\frac{\partial U(X,\ Y)}{\partial Y}\right|_{Y=1}=0 \tag{8-3-43}$$

$$\frac{\partial^2 U(X,\ Y)}{\partial Y^2}<0 \tag{8-3-44}$$

$$T_0(X,\ Y)\big|_{X=0}=1 \tag{8-3-45}$$

管道中，中心速度最大，温度最低，边壁速度小温度高。因此，其边界条件为

$$\left.\frac{\partial T_0(X,\ Y)}{\partial Y}\right|_{Y=0}=0 \tag{8-3-46}$$

$$\frac{\partial^2 T_0(X,\ Y)}{\partial Y^2}>0 \tag{8-3-47}$$

8.3.2.4　温度与速度分布

根据边界条件，选无因次速度为

$$U=(1+X)^{-\alpha}(2Y-Y^2) \tag{8-3-48}$$

根据边界条件，选无因次温度为

$$T_0=(1+X)^{\beta}(1+Y^2) \tag{8-3-49}$$

以上两式要利用动量方程(8-3-40)与能量方程(8-3-41)联合起来确定 α 与 β。为此将两式代入动量与能量方程。首先代入式(8-3-40)

$$\frac{\nu_0 K_7\beta}{k-1}(1+X)^{\beta-(\alpha k-1)}[(2)^{k-1}(Y^{k-1}-Y^k+Y^{k+1}-Y^{k+2})(k-1)(2)^{k-2}(Y^{k+2}-Y^{k+1}+$$

$$Y^{k+2}-Y^{k+3})]-Re\alpha(2Y-Y^2)^2(1+X)^{-(2\alpha+1)}-\nu_0\alpha(1-Y)(1+X)^{-(\alpha+1)}$$

$$=\frac{4}{3}\nu_0(2Y-Y^2)\alpha(\alpha+1)(1+X)^{-(\alpha+2)}-2(1-\varphi^{\frac{2}{3}})\nu_0(1+X)^{-\alpha}-4K_2\varphi^{\frac{2}{3}}\nu_0(1-Y)(1+X)^{-2\alpha}$$

(8-3-40)′

将式中含 Y 项积分

$$\frac{\nu_0K_7\beta}{k-1}(1+X)^{\beta-(2k+1)}\Big[(2)^{k-1}\Big(\frac{1}{k}-\frac{1}{k+1}+\frac{1}{k+2}-\frac{1}{k+3}\Big)-(k-1)(2)^{k-2}$$

$$\Big(\frac{1}{k+3}-\frac{1}{k+2}+\frac{1}{k+3}-\frac{1}{k+4}\Big)\Big]-0.533Re\alpha(1+X)^{-(2\alpha+1)}-0.5\nu_0\alpha(1+X)^{-(\alpha+1)}$$

$$=0.889\nu_0\alpha(\alpha+1)(1+X)^{-(\alpha+2)}-2(1-\varphi^{\frac{2}{3}})\nu_0(1+X)^{-\alpha}-2K_2\varphi^{\frac{2}{3}}\nu_0(1+X)^{-2\alpha}\quad(8\text{-}3\text{-}40)''$$

对式中$(1+X)$取对数

$$\frac{\nu_0K_7\beta}{k-1}\Big[(2)^{k-1}\Big(\frac{1}{k}-\frac{1}{k+1}+\frac{1}{k+2}-\frac{1}{k+3}\Big)-(k-1)(2)^{k-2}\Big(\frac{1}{k+3}-\frac{1}{k+2}+\frac{1}{k+3}-\frac{1}{k+4}\Big)\Big]$$

$$[\beta-(2k+1)]\ln(1+X)+0.533Re\alpha(2\alpha+1)\ln(1+X)+0.5\nu_0\alpha(\alpha+1)\ln(1+X)$$

$$=-0.889\nu_0\alpha(\alpha+1)(\alpha+2)+2(1-\varphi^{\frac{2}{3}})\nu_0\ln(1+X)+4K_2\varphi^{\frac{2}{3}}\nu_0\alpha\ln(1+X)\qquad(8\text{-}3\text{-}40)'''$$

全式除以 $\alpha\ln(1+X)$，则有

$$\frac{\nu_0K_7\beta}{(k-1)\alpha}\Big[(2)^{k-1}\Big(\frac{1}{k}-\frac{1}{k+1}+\frac{1}{k+2}-\frac{1}{k+3}\Big)-(k-1)(2)^{k-2}\Big(\frac{1}{k+3}-\frac{1}{k+2}+\frac{1}{k+3}-\frac{1}{k+4}\Big)\Big]$$

$$+0.533Re(2\alpha+1)+0.5\nu_0(\alpha+1)=-0.889\nu_0(\alpha+1)(\alpha+2)+2(1-\varphi^{\frac{2}{3}})\nu_0+4K_2\varphi^{\frac{2}{3}}\nu_0$$

(8-3-50)

式中，对于具体问题，只有 α，β 是未知数，也是待求之值。

将式(8-3-48)与式(8-3-49)代入式(8-3-41)，则有

$$-Re\alpha(2Y-Y^2)^4(1+X)^{-(4\alpha+1)}-\frac{1}{2}Re\frac{K_7}{k-1}\beta(1+X)^{-[(1+\alpha)k+\alpha-\beta]}[2^{k+2}(Y^{k+2}+Y^{k+4})$$

$$+(k+2)(2)^{k+1}(Y^{k+3}+Y^{k+5})]-\varphi\nu_0\alpha(1+X)^{-(2\alpha+1)}(1-Y)(2Y-Y^2)+\nu_0$$

$$\Big[\frac{K_7\beta}{k-1}(1-Y)(1+Y^2)(2Y-Y^2)^k(1+X)^{\beta-[\alpha(1+k)+1]}-\alpha(1-Y)(2Y-Y^2)(1+X)^{-(2\alpha+1)}\Big]$$

$$=K_3\lambda_0[(1+Y^2)(2Y-Y^2)\beta(\beta-1)(1+X)^{\beta-\alpha-2}+2(2Y-Y^2)(1+X)^{\beta-\alpha}]+\frac{4}{3}\nu_0\alpha(\alpha+1)(2Y-$$

$$Y^2)^2(1+X)^{-2(\alpha+1)}-\varphi^{\frac{2}{3}}\nu_0\Big[\frac{4}{3}\frac{\nu_0}{Re}(1-Y)(2Y-Y^2)(1+X)^{-2(\alpha+1)}-4K_6\nu_0(1-Y)^2(1+X)^{-3\alpha}\Big]$$

(8-3-41)′

进一步改写为

$$-Re\alpha(16Y^4-32Y^5+24Y^6-8Y^7+Y^8)(1+X)^{-(4\alpha+1)}-\frac{1}{2}Re\frac{K_7\beta}{k-1}(1+X)^{-[\alpha(k+1)-\beta+1]}$$

$$[(2)^{k+2}(Y^{k+2}+Y^{k+4})-(k+2)(2)^{k+1}(Y^{k+3}+Y^{k+5})]-\varphi\nu_0\alpha(1+X)^{-(2\alpha+1)}(2Y-3Y^2$$

$$+Y^3)+\nu_0\Big\{\frac{K_7\beta}{k-1}[(2)^k(Y^k-Y^{k+1}+Y^{k+2}-Y^{k+3})+k(2)^{k-1}(Y^{k+4}-Y^{k+3}+Y^{k+2}-Y^{k+1})]$$

$$(1+X)^{\beta-[\alpha(1+k)+1]}-\alpha(2Y-3Y^2+Y^3)(1+X)^{-(2\alpha+1)}\Big\}$$

$$=K_3\lambda_0[(2Y-Y^2+2Y^3-Y^4)\beta(\beta-1)(1+X)^{\beta-\alpha-2}+2(2Y-Y^2)(1+X)^{\beta-\alpha}]$$

$$+\frac{4}{3}\nu_0\alpha(\alpha+1)(4Y^2-4Y^3+Y^4)(1+X)^{-2(\alpha+1)}-\varphi^{\frac{2}{3}}\nu_0\Big[\frac{4}{3}\frac{\nu_0}{Re}(2Y-3Y^2+Y^3)$$

$$(1+X)^{-2(\alpha+1)}-4K_6\nu_0(1-2Y+Y^2)(1+X)^{-3\alpha}\Big] \tag{8-3-41$''$}$$

将含 Y 项积分

$$-0.404Re\alpha(1+X)^{-(4\alpha+1)}-\frac{1}{2}Re\frac{K_7\beta}{k-1}\Big[(2)^{k+2}\Big(\frac{1}{k+3}+\frac{1}{k+5}\Big)-(k+2)(2)^{k+1}\Big(\frac{1}{k+4}+\frac{1}{k+6}\Big)\Big]$$

$$(1+X)^{-[\alpha(k+1)-\beta+1]}-0.25\varphi\nu_0\alpha(1+X)^{-(2\alpha+1)}+\nu_0\Big\{\frac{K_7\beta}{k-1}\Big[(2)^k\Big(\frac{1}{k+1}-\frac{1}{k+2}+\frac{1}{k+3}-\frac{1}{k+4}\Big)$$

$$-k(2)^{k-1}\Big(\frac{1}{k+5}-\frac{1}{k+4}+\frac{1}{k+3}-\frac{1}{k+2}\Big)\Big](1+X)^{\beta-[\alpha(1+k)+1]}-0.25\alpha(1+X)^{-(2\alpha+1)}\Big\}$$

$$=K_3\lambda_0\Big[1.05\beta(\beta-1)(1+X)^{\beta-\alpha-2}+\frac{4}{3}(1+X)^{\beta-\alpha}\Big]+0.533\nu_0\alpha(\alpha+1)(1+X)^{-(2\alpha+1)}\varphi^{\frac{2}{3}}\nu_0$$

$$\Big[\frac{1}{3}\frac{\nu_0}{Re}(1+X)^{-2(\alpha+1)}-\frac{1}{3}K_6\nu_0(1+X)^{-3\alpha}\Big] \tag{8-3-41$'''$}$$

对式中$(1+X)$取对数

$$-0.404Re\alpha(4\alpha+1)\ln(1+X)+\frac{1}{2}Re\frac{K_7\beta}{k-1}[\alpha(k+1)-\beta+1]\Big[(2)^{k+2}\Big(\frac{1}{k+3}+\frac{1}{k+5}\Big)$$

$$-(k+2)(2)^{k+1}\Big(\frac{1}{k+4}+\frac{1}{k+6}\Big)\Big]\ln(1+X)+0.25\varphi\nu_0\alpha(2\alpha+1)\ln(1+X)$$

$$+\nu_0\frac{K_7\beta}{k-1}\Big[(2)^k\Big(\frac{1}{k+1}-\frac{1}{k+2}+\frac{1}{k+3}-\frac{1}{k+4}\Big)-k(2)^{k-1}\Big(\frac{1}{k+5}-\frac{1}{k+4}+\frac{1}{k+3}-\frac{1}{k+2}\Big)\Big]$$

$$\{\beta-[\alpha(1+k)+1]\}\ln(1+X)+0.25\nu_0\alpha(2\alpha+1)\ln(1+X)$$

$$=K_3\lambda_0\Big[1.05\beta(\beta-1)(\beta-\alpha-2)\ln(1+X)+\frac{4}{3}(\beta-\alpha)\ln(1+X)\Big]-0.533\nu_0\alpha(\alpha+1)$$

$$(2\alpha+1)\ln(1+X)+\varphi^{\frac{2}{3}}\nu_0\Big[\frac{2}{3}\frac{\nu_0}{Re}(\alpha+1)\ln(1+X)+K_6\nu_0\alpha\ln(1+X)\Big] \tag{8-3-41$''''$}$$

除以 $\alpha\ln(1+X)$，则有

$$-0.404Re(4\alpha+1)+\frac{1}{2}Re\frac{K_7\beta}{k-1}\alpha[\alpha(k+1)-\beta+1]\Big[(2)^{k+2}\Big(\frac{1}{k+3}+\frac{1}{k+5}\Big)-(k+2)(2)^{k+1}$$

$$\Big(\frac{1}{k+4}+\frac{1}{k+6}\Big)\Big]+0.25\varphi\nu_0(2\alpha+1)+\nu_0\frac{K_7\beta}{k-1}\alpha\Big[(2)^k\Big(\frac{1}{k+1}-\frac{1}{k+2}+\frac{1}{k+3}-\frac{1}{k+4}\Big)-k(2)^{k-1}$$

$$\Big(\frac{1}{k+5}-\frac{1}{k+4}+\frac{1}{k+3}-\frac{1}{k+2}\Big)\Big](\beta-\alpha+\alpha k+1)+0.25\nu_0(2\alpha+1)$$

$$=K_3\lambda_0\Big[1.05\beta\frac{(\beta-1)}{\alpha}(\beta-\alpha-2)+\frac{4}{3}\frac{(\beta-\alpha)}{\alpha}\Big]+0.533\nu_0(\alpha+1)(1+X)^{-(2\alpha+1)}$$

$$+\varphi^{\frac{2}{3}}\nu_0\Big[\frac{2}{3}\frac{\nu_0}{Re}\frac{(\alpha+1)}{\alpha}+K_6\nu_0\Big] \tag{8-3-51}$$

式(8-3-51)只有 α，β 是未知数，它与式(8-3-50)联立可以解出 α，β，而 $\varphi^{\frac{2}{3}}$，φ 值在近似

过程中自然被去掉。

8.3.3 等温矩形管道

坐标系与原点的选定，如图 8-5 所示。当定质量流管长超过其进口段后，必然有等温或绝热变质量流管道与其衔接。此处只讨论等温管道。

等温变质量管道特点是沿程压力、密度、温度均不变，只是速度在壁面摩擦力作用下逐渐减少。实际生产中，要提出多长管道，质量流尚存多少？要回答这个问题，就需要找到速度分布，其他问题则均能迎刃而解。

(1) 运动控制方程与边界条件

由于等温的特点，等温矩形管道变质量流运动的控制方程，只有一个动量微分方程，依式(4-5-16)，结合本问题为

$$u\frac{\partial u}{\partial x}+\frac{\nu}{2}\left[\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\right]\nabla\cdot\mathbf{V}$$
$$=\nu\left[\frac{4}{3}\frac{\partial^2 u}{\partial x^2}+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+6\varphi^{\frac{2}{3}}t\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right)\right]\tag{8-3-52}$$

边界条件为

$$u(x,\ y,\ z)\big|_{x=0}=v_d\tag{8-3-53}$$

式中，v_d——进口段末端断面平均速度，它可以用进口段速度分布公式计算出来，在本题中是已知的边界条件：

$$u(x,\ y,\ z)\big|_{y=B}=0\tag{8-3-54}$$

$$u(x,\ y,\ z)\big|_{z=0}=0\tag{8-3-55}$$

$$\left.\frac{\partial u(x,\ y,\ z)}{\partial y}\right|_{y=0}=0\tag{8-3-56}$$

$$\left.\frac{\partial u(x,\ y,\ z)}{\partial z}\right|_{z=H}=0\tag{8-3-57}$$

$$\frac{\partial^2 u(x,\ y,\ z)}{\partial y^2}<0\tag{8-3-58}$$

$$\frac{\partial^2 u(x,\ y,\ z)}{\partial z^2}<0\tag{8-3-59}$$

(2) 无因次控制运动方程与边界条件

为进行数学分析，必须将方程与边界条件无因次化。为此，取

$$\frac{x}{H}=X,\ \frac{z}{H}=Z,\ \frac{y}{B}=Y,\ \frac{u}{v_d}=U$$

将以上比值代入运动方程与边界条件

$$\frac{v_d^2}{H}U\frac{\partial U}{\partial X}-\frac{\nu_d v_d\nu_0}{2H^2U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)$$
$$=\nu_0\left\{\frac{4}{3}\frac{\nu_d v_d H}{H^2}\frac{\partial^2 U}{\partial X^2}+(1-\varphi^{\frac{2}{3}})\frac{\nu_d v_d}{H^2}\left[\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right]+\frac{\nu_d v_d^2}{H^3}6t\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]\right\}\tag{8-3-53$'$}$$

除以$\dfrac{\nu_d v_d}{H^2}$，则有

$$ReU\frac{\partial U}{\partial X}+\frac{\nu_0}{2U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)$$
$$=\nu_0\left\{\frac{4}{3}\frac{\partial^2 U}{\partial X^2}+(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right]+K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]\right\}$$
$$\tag{8-3-60}$$

无因次边界条件：

$$U(X,\ Y,\ Z)|_{Z=0}=1 \tag{8-3-54$'$}$$

$$U(X,\ Y,\ Z)|_{Y=1}=0 \tag{8-3-55$'$}$$

$$U(X,\ Y,\ Z)|_{Z=0}=0 \tag{8-3-56$'$}$$

$$\left.\frac{\partial U(X,\ Y,\ Z)}{\partial Y}\right|_{Y=0}=0 \tag{8-3-57$'$}$$

$$\left.\frac{\partial U(X,\ Y,\ Z)}{\partial Z}\right|_{Z=1}=0 \tag{8-3-58$'$}$$

$$\frac{\partial^2 U(X,\ Y,\ Z)}{\partial Y^2}<0 \tag{8-3-59$'$}$$

$$\frac{\partial^2 U(X,\ Y,\ Z)}{\partial Z^2}<0 \tag{8-3-60$'$}$$

式中：

$$Re=\frac{v_d H}{\nu_d} \tag{8-3-61}$$

$$K_2=\frac{6tv_d}{H} \tag{8-3-62}$$

(3) 速度分布

根据边界条件，选定速度分布为

$$U=(1+X)^{-\alpha}(2Z-Z^2)(1-Y^2) \tag{8-3-63}$$

为确定 α，必须将式(8-3-63)代入式(8-3-60)

$$ReU\frac{\partial U}{\partial X}=-Re\alpha(1+X)^{-(2\alpha+1)}\int_0^1\int_0^1(2Z-Z^2)(1-Y^2)^2\mathrm{d}Z\mathrm{d}Y$$
$$=-0.284Re\alpha(1+X)^{-(2\alpha+1)} \tag{1}$$

$$\frac{\nu_0}{2U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\frac{\partial U}{\partial X}=\frac{\nu_0\alpha}{(1+X)^{(\alpha+1)}}\left[\frac{H}{B}Y(2Z-Z^2)-(1-Y^2)(1-Z)\right]$$
$$=\frac{\nu_0\alpha}{(1+X)^{(\alpha+1)}}\left[\int_0^1\int_0^1\frac{H}{B}Y(2Z-Z^2)\mathrm{d}Y\mathrm{d}Z\right.$$
$$\left.-\int_0^1\int_0^1(1-Y^2)(1-Z)\mathrm{d}Y\mathrm{d}Z\right]$$

$$=\frac{1}{3}\frac{\nu_0\alpha}{(1+X)^{(\alpha+1)}}\left(\frac{H}{B}-1\right)=\frac{1}{3}\nu_0\alpha(1+X)^{-(\alpha+1)}\left(\frac{H}{B}-1\right) \tag{2}$$

$$\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}=\frac{4}{3}\nu_0\alpha(\alpha+1)(1+X)^{-(\alpha+2)}(2Z-Z^2)(1-Y^2)$$

$$=\frac{4}{3}\nu_0\alpha(\alpha+1)(1+X)^{-(\alpha+2)}\int_0^1\int_0^1(2Z-Z^2)(1-Y^2)\mathrm{d}Z\mathrm{d}Y$$

$$=\frac{16}{27}\nu_0\alpha(\alpha+1)(1+X)^{-(\alpha+2)} \tag{3}$$

$$\nu_0(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right]=\nu_0(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2(-2)(2Z-Z^2)-2(1-Y^2)\right](1+X)^{-\alpha}$$

$$=-2(1-\varphi^{\frac{2}{3}})(1+X)^{-\alpha}\left[\left(\frac{H}{B}\right)^2(2Z-Z^2)+(1-Y^2)\right]\nu_0$$

$$=-2(1-\varphi^{\frac{2}{3}})(1+X)^{-\alpha}$$

$$\left[\left(\frac{H}{B}\right)^2\int_0^1(2Z-Z^2)\mathrm{d}Z+\int_0^1(1-Y^2)\mathrm{d}Y\right]\nu_0$$

$$=-\frac{4}{3}\nu_0(1-\varphi^{\frac{2}{3}})(1+X)^{-\alpha}\left[\left(\frac{H}{B}\right)^2+1\right] \tag{4}$$

$$\nu_0K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]$$

$$=\nu_0K_2\varphi^{\frac{2}{3}}(1+X)^{-2\alpha}\left[\left(\frac{H}{B}\right)^3(-2Y)(-2)(2Z-Z^2)^2+2(1-Z)(-2)(1-Y^2)^2\right]$$

$$=4\nu_0K_2\varphi^{\frac{2}{3}}(1+X)^{-2\alpha}\left[\left(\frac{H}{B}\right)^3\int_0^1\int_0^1 Y(2Z-Z^2)^2\mathrm{d}Z\mathrm{d}Y-\int_0^1\int_0^1(1-Z)(1-Y^2)^2\mathrm{d}Z\mathrm{d}Y\right]$$

$$=4\times0.5\times0.533\nu_0K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right](1+X)^{-2\alpha}$$

$$=1.066\nu_0K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right](1+X)^{-2\alpha} \tag{5}$$

将式(1)～式(5)代回式(8-3-60)，则有

$$-0.284Re\alpha(1+X)^{-(2\alpha+1)}+\frac{1}{3}\nu_0\alpha(1+X)^{-(\alpha+1)}\left(\frac{H}{B}-1\right)$$

$$=\frac{16}{27}\nu_0(\alpha+1)\alpha(1+X)^{-(\alpha+2)}-\frac{4}{3}\nu_0(1-\varphi^{\frac{2}{3}})(1+X)^{-\alpha}\left[\left(\frac{H}{B}\right)^2+1\right]+1.066\nu_0K_2\varphi^{\frac{2}{3}}$$

$$\left[\left(\frac{H}{B}\right)^3-1\right](1+X)^{-2\alpha} \tag{8-3-61$'$}$$

对$(1+X)$取对数，则有

$$0.284Re\alpha(2\alpha+1)\ln(1+X)-\frac{1}{3}\nu_0\alpha(\alpha+1)\ln(1+X)\left(\frac{H}{B}-1\right)$$

$$=-\frac{16}{27}\nu_0(\alpha+1)\alpha(\alpha+2)\ln(1+X)+\frac{4}{3}\nu_0(1-\varphi^{\frac{2}{3}})\alpha\left[\left(\frac{H}{B}\right)^2+1\right]\ln(1+X)-$$

$$1.066\nu_0K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right]2\alpha\ln(1+X) \tag{8-3-61$''$}$$

全式除以 $\alpha\ln(1+X)$，则有

$$0.284Re\alpha(2\alpha+1)-\frac{1}{3}\nu_0(\alpha+1)\left(\frac{H}{B}-1\right)$$

$$=-\frac{16}{27}\nu_0(\alpha+1)(\alpha+2)+\frac{4}{3}\nu_0(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right]-2.132\nu_0K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right] \tag{8-3-61)'''$$

全式除以 Re，去掉小量，则

$$0.284(2\alpha+1)-\frac{0.333}{Re}\nu_0(\alpha+1)\left(\frac{H}{B}-1\right)$$

$$=-\frac{0.593}{Re}\nu_0(\alpha+1)(\alpha+2)+\frac{1.333}{Re}\nu_0(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right]-\frac{2.132}{Re}\nu_0K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right] \tag{8-3-64}$$

式中 $\nu_0\approx1$，去掉小量后，$\frac{K_2}{Re}=\frac{6tv_d^2}{\nu_d}$，则

$$0.284(2\alpha+1)=-\frac{2.132\times6tv_d^2\varphi^{\frac{2}{3}}}{\nu_d}\left[\left(\frac{H}{B}\right)^3-1\right] \tag{8-3-65}$$

$$\alpha=-\frac{1}{0.584}\left\{0.284+\frac{12.74\times tv_d^2\varphi^{\frac{2}{3}}}{\nu_d}\left[\left(\frac{H}{B}\right)^3-1\right]\right\} \tag{8-3-66}$$

式中：t——准定常时间，可实测而定，一般在0.01～0.001s范围；

φ——涡旋体积分数，一般在0.01～0.1范围，可以通过公式计算出来。

近似处理

$$\alpha=-\frac{0.284}{0.584}=-0.485 \tag{8-3-67}$$

$$U=(1+X)^{-0.485}(2Z-Z^2)(1-Y^2) \tag{8-3-68}$$

(4) 流量衰减与距离关系

$$Q(x)=4HBv=4HB\ \frac{1}{2}\int_0^1\int_0^1 v_d(1+X)^{-0.485}(2Z-Z^2)(1-Y^2)\mathrm{d}Z$$

$$=2HBv_d(1+X)^{-0.485}\int_0^1\int_0^1(2Z-Z^2)\mathrm{d}Z(1-Y^2)\mathrm{d}Y$$

$$=\frac{8}{9}HBv_d(1+X)^{-0.485} \tag{8-3-69}$$

8.3.4 绝热矩形管道

坐标系与原点的选定，如图8-5所示。绝热变质量管道的特点，温度会沿流程增加，由此会引起密度的减少，由于压力不变，从能量守恒的观点，总的质量流一定会沿程逐渐减少。

8.3.4.1 运动控制方程与边界条件

(1) 动量方程

由式(4-5-16)，结合本节条件，为

$$\frac{\nu}{2}\frac{\partial\rho}{\rho\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+u\frac{\partial u}{\partial x}+\frac{\nu}{2}\left[\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\right]\frac{\partial u}{\partial x}$$

$$=\nu\left[\frac{4}{3}\frac{\partial^2 u}{\partial x^2}+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+6\varphi^{\frac{2}{3}}t\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right)\right] \tag{8-3-70}$$

(2) 能量方程

由式(3-19-27)，结合本题，可得

$$u^2\frac{\partial u}{\partial x}-\frac{u^2}{2}\frac{\partial\rho}{\partial x}+\frac{1}{2}\varphi\nu u\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial x\partial z}\right)+\frac{\nu}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\left(\frac{u}{\rho}\frac{\partial\rho}{\partial x}+\frac{\partial u}{\partial x}\right)$$

$$=\frac{\lambda}{\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+u\left(\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}\right)-\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$\left[\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}+2\nu t\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right)\right] \tag{8-3-71}$$

(3) 绝热条件

可用式(8-3-27)

$$(k-1)\rho^{k-2}\frac{\partial\rho}{\partial x}=\frac{R}{c}\frac{\partial T}{\partial x}$$

(4) 连续性方程

可用式(8-3-30)

$$u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial\rho}{\partial x}$$

边界条件：

$$u(x,\ y,\ z)\big|_{x=0}=v_d \tag{8-3-72}$$

$$u(x,\ y,\ z)\big|_{y=B}=0 \tag{8-3-73}$$

$$u(x,\ y,\ z)\big|_{z=0}=0 \tag{8-3-74}$$

$$\left.\frac{\partial u(x,\ y,\ z)}{\partial y}\right|_{y}=0 \tag{8-3-75}$$

$$\left.\frac{\partial u(x,\ y,\ z)}{\partial z}\right|_{z=H}=0 \tag{8-3-76}$$

$$\frac{\partial^2 u(x,\ y,\ z)}{\partial y^2}<0 \tag{8-3-77}$$

$$\frac{\partial^2 u(x,\ y,\ z)}{\partial z^2}<0 \tag{8-3-78}$$

$$T(x,\ y,\ z)\big|_{x=0}=T_d \tag{8-3-79}$$

$$\rho(x,\ y,\ z)\big|_{x=0}=\rho_d \tag{8-3-80}$$

T_d，ρ_d 是进口段末端断面上平均温度与断面上平均密度。

8.3.4.2　无因次运动控制方程与边界条件

为方程与边界条件无因次化，取

$$\frac{x}{H}=X,\ \frac{z}{H}=Z,\ \frac{y}{B}=Y,\ \frac{u}{v_d}=U,\ \frac{\rho}{\rho_d}=\rho_0,\ \frac{T}{T_d}=T_0,\ \frac{\nu}{\nu_d}=\nu_0$$

将以上比值代入运动方程与边界条件。

动量方程：

$$\frac{\nu}{2}\frac{\partial\rho}{\rho\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)=\frac{1}{2}\frac{\nu_d v_d}{H^2}\frac{\nu_0\partial\rho_0}{\rho_0\partial X}\left(\frac{H\partial U}{B\partial Y}+\frac{\partial U}{\partial Z}\right) \tag{1}$$

$$u\frac{\partial u}{\partial x}=\frac{v_d^2}{H}U\frac{\partial U}{\partial X} \tag{2}$$

$$\frac{\nu}{2}\frac{1}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{\partial u}{\partial x}=\frac{\nu_d v_d\nu_0}{2H^2U}\left(\frac{H\partial U}{B\partial Y}+\frac{\partial U}{\partial Z}\right)\frac{\partial U}{\partial X} \tag{3}$$

$$\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}=\frac{4}{3}\frac{\nu_d v_d}{H^2}\nu_0\frac{\partial^2 U}{\partial X^2} \tag{4}$$

$$\nu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\nu_d v_d}{H^2}(1-\varphi^{\frac{2}{3}})\nu_0\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2} \tag{5}$$

$$\nu 6\varphi^{\frac{2}{3}}t\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right)=\frac{6\varphi^{\frac{2}{3}}t\nu_d v_d^2}{H^3}\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right] \tag{6}$$

将式(1)～式(6)代回式(8-3-70)后，得

$$\begin{aligned}&\frac{\nu_d v_d}{2H^2}\frac{\nu_0\partial\rho_0}{\rho_0\partial X}\left(\frac{H\partial U}{B\partial Y}+\frac{\partial U}{\partial Z}\right)+\frac{v_d^2}{H}U\frac{\partial U}{\partial X}+\frac{\nu_d v_d}{2H^2}\nu_0\frac{1}{U}\left(\frac{H\partial U}{B\partial Y}+\frac{\partial U}{\partial Z}\right)\frac{\partial U}{\partial X}\\&=\nu_0\left\{\frac{4}{3}\frac{\nu_d v_d}{H^2}\frac{\partial^2 U}{\partial X^2}+(1-\varphi^{\frac{2}{3}})\frac{\nu_d v_d}{H^2}\left[\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right]+\frac{6\varphi^{\frac{2}{3}}t\nu_d v_d^2}{H^3}\right.\\&\left.\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]\right\}\end{aligned} \tag{8-3-71$'$}$$

除以$\frac{\nu_d v_d}{H^2}$，则有

$$\begin{aligned}&\frac{1}{2}\frac{\nu_0\partial\rho_0}{\rho_0\partial X}\left(\frac{H\partial U}{B\partial Y}+\frac{\partial U}{\partial Z}\right)+ReU\frac{\partial U}{\partial X}+\frac{\nu_0}{2U}\left(\frac{H\partial U}{B\partial Y}+\frac{\partial U}{\partial Z}\right)\\&=\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}+\nu_0(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right]+K_2\varphi^{\frac{2}{3}}\nu_0\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]\end{aligned} \tag{8-3-81}$$

以下推导无因次能量方程，分各单项计算：

$$u^2\frac{\partial u}{\partial x}=\frac{v_d^3}{H}U^2\frac{\partial U}{\partial X} \tag{1}$$

$$-\frac{u^2}{2}\frac{u}{\rho}\frac{\partial\rho}{\partial x}=-\frac{v_d^3U^3\partial\rho_0}{2H\rho_0\partial X} \tag{2}$$

$$\frac{1}{2}\varphi\nu u\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial x\partial z}\right)=\frac{1}{2}\frac{\nu_d v_d^2\nu_0 U}{H^2}\varphi\left(\frac{H}{B}\frac{\partial^2 U}{\partial X\partial Y}+\frac{\partial^2 U}{\partial X\partial Z}\right) \tag{3}$$

$$\frac{\nu}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\left(\frac{u}{\rho}\frac{\partial \rho}{\partial x}+\frac{\partial u}{\partial x}\right)=\frac{\nu_d}{2}\frac{v_d^2}{H^2}\nu_0\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left(\frac{U}{\rho_0}\frac{\partial \rho_0}{\partial X}+\frac{\partial U}{\partial X}\right) \tag{4}$$

$$\frac{\lambda}{\rho}\left(\frac{\partial^2 T_0}{\partial x^2}+\frac{\partial^2 T_0}{\partial y^2}+\frac{\partial^2 T_0}{\partial z^2}\right)=\frac{\lambda_d T_e}{\rho_d H^2}\frac{\lambda_0}{\rho_0}\left[\frac{\partial^2 T_0}{\partial X^2}+\left(\frac{H}{B}\right)^2\frac{\partial^2 T_0}{\partial Y^2}+\frac{\partial^2 T_0}{\partial Z^2}\right] \tag{5}$$

$$\frac{4}{3}\nu u\frac{\partial^2 u}{\partial x^2}=\frac{4}{3}\frac{\nu_d v_d^2}{H^2}\nu_0\frac{\partial^2 U}{\partial X^2} \tag{6}$$

$$-\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}=-\frac{2}{3}\frac{\nu_d v_d^2}{H^3}\frac{\varphi^{\frac{2}{3}}\nu_0^2}{U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\frac{\partial^2 U}{\partial X^2} \tag{7}$$

$$-\frac{\varphi^{\frac{2}{3}}}{2}\frac{\nu}{u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)2\nu t\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right)$$

$$=-\frac{\nu_d^2 v_d^2 t}{H^4}\varphi^{\frac{2}{3}}\frac{\nu_0^2}{U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right] \tag{8}$$

将式(1)～式(8)代回

$$\frac{v_d^3}{H}U^2\frac{\partial U}{\partial X}-\frac{v_d^3}{2H}\frac{U^3}{\rho_0}\frac{\partial \rho_0}{\partial X}+\frac{1}{2}\frac{\nu_d v_d^2}{H^2}\nu_0\varphi U\left(\frac{H}{B}\frac{\partial^2 U}{\partial X\partial Y}+\frac{\partial^2 U}{\partial X\partial Z}\right)+\frac{\nu_d v_d^2}{2H^2}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left(\frac{U}{\rho_0}\frac{\partial \rho_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$=\frac{\lambda_d T_e}{\rho_d H^2}\frac{\lambda_0}{\rho_0}\left[\frac{\partial^2 T_0}{\partial X^2}+\left(\frac{H}{B}\right)^2\frac{\partial^2 T_0}{\partial Y^2}+\frac{\partial^2 T_0}{\partial Z^2}\right]+\frac{4}{3}\frac{\nu_d v_d^2}{H^2}\nu_0\frac{\partial^2 U}{\partial X^2}-\frac{2}{3}\frac{\nu_d v_d^2}{H^3}\frac{\varphi^{\frac{2}{3}}\nu_0^2}{U}$$

$$\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\frac{\partial^2 U}{\partial X^2}-\frac{\nu_d^2 v_d^2 t}{H^4}\varphi^{\frac{2}{3}}\frac{\nu_0^2}{U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right] \tag{8-3-72$'$}$$

除以$\frac{\nu_d v_d^2}{H^2}$，则有

$$ReU^3\frac{\partial U}{\partial X}-\frac{1}{2}Re\frac{U^3}{\rho_0}\frac{\partial \rho_0}{\partial X}+\frac{\varphi}{2}\nu_0 U\left(\frac{H}{B}\frac{\partial^2 U}{\partial X\partial Y}+\frac{\partial^2 U}{\partial X\partial Z}\right)+\frac{\nu_0}{2}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left(\frac{U}{\rho_0}\frac{\partial \rho_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$=\frac{K_3\lambda_0}{\rho_0}\left[\frac{\partial^2 T_0}{\partial X^2}+\left(\frac{H}{B}\right)^2\frac{\partial^2 T_0}{\partial Y^2}+\frac{\partial^2 T_0}{\partial Z^2}\right]+\frac{4}{3}\nu_0 U\frac{\partial^2 U}{\partial X^2}-\frac{2}{3}\frac{\nu_0^2}{Re}\varphi^{\frac{2}{3}}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)$$

$$-K_4\varphi^{\frac{2}{3}}\frac{\nu_0^2}{U}\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right) \tag{8-3-82}$$

边界条件：

$$U(X,\ Y,\ Z)|_{X=0}=1 \tag{8-3-83}$$

$$U(X,\ Y,\ Z)|_{Y=1}=0 \tag{8-3-84}$$

$$U(X,\ Y,\ Z)|_{Z=0}=0 \tag{8-3-85}$$

$$\left.\frac{\partial U(X,\ Y,\ Z)}{\partial Y}\right|_{Y=0}=0 \tag{8-3-86}$$

$$\left.\frac{\partial U(X,\ Y,\ Z)}{\partial Z}\right|_{Z=1}=0 \tag{8-3-87}$$

$$\frac{\partial^2 U(X,\ Y,\ Z)}{\partial Y^2}<0 \tag{8-3-88}$$

$$\frac{\partial^2 U(X,\ Y,\ Z)}{\partial Z^2}<0 \tag{8-3-89}$$

$$T_0(X,\ Y,\ Z)\big|_{X=0}=1 \tag{8-3-90}$$

$$\rho_0(X,\ Y,\ Z)\big|_{X=0}=1 \tag{8-3-91}$$

绝热条件无因次化，由式(8-3-27)

$$(K-1)\rho_d^{(K-2)}\frac{\rho_d\partial\rho_0}{H\partial X}=\frac{R_d R_0}{c_d\ c_0}\frac{T_d\partial T_0}{H\ \partial X} \tag{8-3-27$''$}$$

化简为

$$(K-1)\frac{\rho_0^{K-1}}{\rho_0}\frac{\partial\rho_0}{\partial X}=K_5\ \frac{R_0}{c_0}\frac{\partial T_0}{\partial X} \tag{8-3-92}$$

$$K_5=\frac{R_d T_d}{c_d\rho_d^{K-1}} \tag{8-3-93}$$

连续性方程无因次化，由式(8-3-29)

$$\rho_d\rho_0\ \frac{v_d\partial U}{H\partial X}=-\frac{v_d\rho_d}{H}U\ \frac{\partial\rho_0}{\partial X} \tag{8-3-29$'$}$$

$$-\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial X}=\frac{1}{U}\frac{\partial U}{\partial X} \tag{8-3-94}$$

以上各式中：

$$Re=\frac{v_d H}{\nu_d} \tag{8-3-95}$$

$$K_2=\frac{6tv_d}{H} \tag{8-3-96}$$

$$K_3=\frac{\lambda_d T_d}{\rho_d\nu_d v_d^2} \tag{8-3-97}$$

$$K_4=\frac{\nu_d t}{H^2} \tag{8-3-98}$$

8.3.4.3　速度与温度分布

根据边界条件，选无因次速度为

$$U=(1+X)^{-\alpha}(2Z-Z^2)(1-Y^2) \tag{8-3-99}$$

根据边界条件，选无因次温度分布为

$$T_0=(1+X)^{\beta}(1+Y^2)(1+Z^2) \tag{8-3-100}$$

将以上两式代入无因次动量方程与无因次能量方程，确定 α，β。首先分项代入式(8-3-81)。

对式(8-3-81)中$\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial X}$项，要利用式(8-3-91)与连续方程进行处理，然后代入温度与速

度分布，注意 $R_0=1$，$c_0=1$。

$$\frac{1}{2}\frac{\nu_0}{\rho_0}\frac{\partial\rho_0}{\partial X}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)=\frac{\nu_0}{2}\frac{K_5}{\rho_0^{k-1}}\frac{1}{k-1}\frac{\partial T_0}{\partial X}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)$$

$$=\frac{\nu_0}{2}\frac{K_5}{k-1}[(1+X)^{-\alpha}(2Z-Z^2)(1-Y^2)]^{k-1}2(1+X)^{-\alpha}[(1-Z)(1-Y^2)-Y(2Z-Z^2)]$$
$$\beta(1+X)^{\beta-1}(1+Y^2)(1+Z^2)$$

$$=\frac{\nu_0K_5\beta}{k-1}(2Z-Z^2)^{k-1}(1-Y^2)^{k-1}[(1-Z)(1-Y^2)-Y(2Z-Z^2)](1+X)^{[\beta-\alpha(k+2)-1]}$$

$$=\frac{\nu_0K_5\beta}{k-1}[2^{k-1}Z^{k-1}-(k-1)Z^{k-2}Z^k][1-(k-1)Y^2][(1-Z)(1-Y^2)-Y(2Z-Z^2)]$$
$$(1+X)^{[\beta-\alpha(k+2)-1]}$$

$$=\frac{\nu_0K_5\beta}{k-1}[2^{k-1}Z^{k-1}-(k-1)Z^{k-2}Z^k-(k-1)2^{k-1}Z^{k-1}Y^2+(k-1)^2 2^{k-2}Z^kY^2]$$
$$[(1-Z)(1-Y^2)-Y(2Z-Z^2)](1+X)^{[\beta-\alpha(k+2)-1]}$$

$$=\frac{\nu_0K_5\beta}{k-1}(1+X)^{[\beta-\alpha(k+2)-1]}\int_0^1\int_0^1[2^{k-1}Z^{k-1}-(k-1)Z^{k-2}Z^k-(k-1)2^{k-1}Z^{k-1}Y^2$$
$$+(k-1)^2 2^{k-2}Z^kY^2][(1-Z)(1-Y^2)-Y(2Z-Z^2)]\mathrm{d}Z\mathrm{d}Y$$

$$=-0.064\left[\frac{2^{k-1}}{k}-\frac{2^{k-2}}{k+1}-\frac{1}{3}\frac{(k-1)2^{k-1}}{k}+(k-1)^2 2^{k-2}\frac{1}{3(k+1)}\right]\frac{\nu_0K_5\beta}{k-1}(1+X)^{[\beta-\alpha(k+2)-1]} \tag{1}$$

$$ReU\frac{\partial U}{\partial X}=-Re\alpha(2Z-Z^2)^2(1-Y^2)^2(1+X)^{-(2\alpha+1)}$$
$$=-\alpha Re\int_0^1\int_0^1(4Z^2-4Z^3+Z^4)(1-2Y^2+Y^4)\mathrm{d}Z\mathrm{d}Y(1+X)^{-(2\alpha+1)}$$
$$=-0.285\alpha Re(1+X)^{-(2\alpha+1)} \tag{2}$$

$$\frac{\nu_0}{2U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)$$
$$=\frac{\nu_0(1+X)^{-\alpha}}{2(1+X)^{-\alpha}(2Z-Z^2)(1-Y^2)}\left[\frac{H}{B}(-2Y)(2Z-Z^2)+2(1-Z)(1-Y^2)\right]$$
$$=\nu_0\left[\frac{1-Z}{2Z-Z^2}-\frac{H}{B}\frac{Y}{1-Y^2}\right]=\nu_0\left[\int_0^1\int_0^1\left(\frac{1-Z}{2Z-Z^2}-\frac{H}{B}\frac{Y}{1-Y^2}\right)\mathrm{d}Y\mathrm{d}Z\right]$$
$$=-\nu_0\frac{H}{B}\ln 2=-0.693\nu_0\frac{H}{B} \tag{3}$$

$$\frac{4}{3}\nu_0\frac{\partial^2U}{\partial X^2}=\frac{4}{3}\nu_0(2Z-Z^2)(1-Y^2)(\alpha+1)\alpha(1+X)^{-(\alpha+2)}$$
$$=\frac{4}{3}\nu_0\alpha(\alpha+1)(1+X)^{-(\alpha+1)}\int_0^1\int_0^1(2Z-Z^2)(1-Y^2)\mathrm{d}Z\mathrm{d}Y$$
$$=\frac{16}{27}\nu_0(\alpha+1)\alpha(1+X)^{-(\alpha+2)} \tag{4}$$

$$\nu_0(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{\partial^2 U}{\partial Z^2}\right]$$

$$=\nu_0(1-\varphi^{\frac{2}{3}})(1+X)^{-\alpha}\left[\left(\frac{H}{B}\right)^2(-2)(2Z-Z^2)+(-2)(1-Y^2)\right]$$

$$=-2\nu_0(1-\varphi^{\frac{2}{3}})(1+X)^{-\alpha}\int_0^1\int_0^1\left[\left(\frac{H}{B}\right)^2(2Z-Z^2)-(1-Y^2)\right]\mathrm{d}Z\mathrm{d}Y$$

$$=-\frac{4}{3}\nu_0(1-\varphi^{\frac{2}{3}})(1+X)^{-\alpha}\left[\left(\frac{H}{B}\right)^2+1\right](1+X)^{-\alpha} \tag{5}$$

$$K_2\varphi^{\frac{2}{3}}\nu_0\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]$$

$$=4K_2\varphi^{\frac{2}{3}}\nu_0(1+X)^{-2\alpha}\left[\left(\frac{H}{B}\right)^3Y(2Z-Z^2)^2-(1-Z)(1-Y^2)\right]$$

$$=4K_2\varphi^{\frac{2}{3}}\nu_0(1+X)^{-2\alpha}\int_0^1\int_0^1\left[\left(\frac{H}{B}\right)^3Y(2Z-Z^2)^2-(1-Z)(1-Y^2)\right]\mathrm{d}Z\mathrm{d}Y$$

$$=1.066K_2\varphi^{\frac{2}{3}}\nu_0(1+X)^{-2\alpha}\left[\left(\frac{H}{B}\right)^3-1\right] \tag{6}$$

将式(1)～式(6)代回式(8-3-28)

$$-0.064\left[\frac{2^{k-1}}{k}-\frac{2^{k-2}}{k+1}-\frac{(k-1)2^{k-1}}{3k}+(k-1)^2 2^{k-2}\frac{1}{3(k+1)}\right]\frac{\nu_0K_5\beta}{k-1}(1+X)^{[\beta-\alpha(k+2)-1]}$$

$$=0.285\alpha Re(1+X)^{-(2\alpha+1)}-0.693\nu_0\frac{H}{B}$$

$$=0.593\nu_0\alpha(\alpha+1)(1+X)^{-(\alpha+2)}-\frac{4}{3}\nu_0(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right](1+X)^{-\alpha}$$

$$+1.066K_2\varphi^{\frac{2}{3}}\nu_0\left[\left(\frac{H}{B}\right)^3-1\right](1+X)^{-2\alpha} \tag{8-3-82$'$}$$

除以 Re，则有

$$-0.064\left[\frac{2^{k-1}}{k}-\frac{2^{k-2}}{k+1}-\frac{(k-1)2^{k-1}}{3k}+(k-1)^2\frac{2^{k-2}}{3(k+1)}\right]\frac{\nu_0K_5\beta}{(k-1)Re}(1+X)^{[\beta-\alpha(k+2)-1]}$$

$$=0.285\alpha(1+X)^{-(2\alpha+1)}-\frac{0.693\nu_0}{Re}\frac{H}{B}$$

$$=\frac{0.593}{Re}\nu_0\alpha(\alpha+1)(1+X)^{-(\alpha+2)}-\frac{1.333}{Re}\nu_0(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right](1+X)^{-\alpha}$$

$$+\frac{1.066}{Re}K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right](1+X)^{-2\alpha} \tag{8-3-82$''$}$$

进行量级比较，$\dfrac{0.693\nu_0}{Re}\dfrac{H}{B}<1$，$\nu_0\approx1$，然后对$(1+X)$取对数

$$-0.064\left[\frac{2^{k-1}}{k}-\frac{2^{k-2}}{k+1}-\frac{(k-1)2^{k-1}}{3k}+(k-1)^2\frac{2^{k-2}}{3(k+1)}\right]\frac{\nu_0K_5\beta}{(k-1)Re}[\beta-\alpha(k+2)-1]$$

$$\ln(1+X)+0.285\alpha(2\alpha+1)\ln(1+X)$$

$$=-\frac{0.593}{Re}\nu_0\alpha(\alpha+1)(\alpha+2)\ln(1+X)+\frac{1.333}{Re}\nu_0(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right]\alpha\ln(1+X)$$

$$-\frac{2.132}{Re}K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right]\alpha\ln(1+X) \tag{8-3-82$'''$}$$

将式上除以 $\alpha\ln(1+X)$，则有

$$-0.064\left[\frac{2^{k-1}}{k}-\frac{2^{k-2}}{k+1}-\frac{(k-1)2^{k-1}}{3k}+(k-1)^2\frac{2^{k-2}}{3(k+1)}\right]\frac{\nu_0\beta}{(k-1)}\frac{K_5}{Re\alpha}[\beta-\alpha(k+1)-1]+0.285(2\alpha+1)$$
$$=-\frac{0.593}{Re}\nu_0(\alpha+1)(\alpha+2)+\frac{1.333}{Re}\nu_0(1-\varphi^{\frac{2}{3}})\left[\left(\frac{H}{B}\right)^2+1\right]-\frac{2.132}{Re}K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right] \tag{8-3-101}$$

进行量级比较，由 $\nu_0\approx1$，$\varphi<1$，而 Re 很大，则可以忽略小项

$$-0.064\left[\frac{2^{k-1}}{k}-\frac{2^{k-2}}{k+1}-\frac{(k-1)2^{k-1}}{3k}+(k-1)^2\frac{2^{k-2}}{3(k+1)}\right]\frac{\beta}{\alpha(k-1)}\frac{K_5}{Re}[\beta-\alpha(k+1)-1]+0.285(2\alpha+1)$$
$$=-\frac{2.132}{Re}K_2\varphi^{\frac{2}{3}}\left[\left(\frac{H}{B}\right)^3-1\right] \tag{8-3-102}$$

上式只有 α，β，φ 是未知数，而 φ 以后又可以计算出来。

将式(8-3-98)与式(8-3-99)代入能量方程(8-3-82)，为计算方程，以分项进行。

$$ReU^3\frac{\partial U}{\partial X}=Re\alpha(1+X)^{-(4\alpha+1)}\int_0^1\int_0^1(2Z-Z^2)^4(1-Y^2)^4\mathrm{d}Y\mathrm{d}Z$$
$$=Re\alpha(1+X)^{-(4\alpha+1)}\int_0^1\int_0^1(Z^8-6Z^7+17Z^6-24Z^5+16Z^4)(Y^8-4Y^6+4Y^2+1)\mathrm{d}Y\mathrm{d}Z$$
$$=-0.162Re\alpha(1+X)^{-(4\alpha+1)} \tag{1}$$

$$-\frac{1}{2}Re\frac{U^3}{\rho_0}\frac{\partial\rho_0}{\partial X}=-\frac{1}{2}ReU^3\frac{K_5}{\rho_0^{k-1}}\frac{\partial T_0}{\partial X}$$
$$=-\frac{1}{2}Re\frac{K_5}{k-1}[(1+X)^{-\alpha}(2Z-Z^2)(1-Y^2)]^{2+k}\beta(1+X)^{\beta-1}(1+Y^2)(1+Z^2)$$
$$=-\frac{1}{2}Re\frac{\beta K_5}{k-1}(1+X)^{\beta-\alpha(2+k)}\int_0^1\int_0^1(2Z-Z^2)^{2+k}$$
$$(1-Y^2)^{2+k}(1+Y^2)(1+Z^2)\mathrm{d}Y\mathrm{d}Z$$
$$=-\frac{1}{2}\frac{Re\beta K_5}{k-1}(1+X)^{\beta-\alpha(2+k)}\int_0^1\int_0^1\{2^{2+k}Z^{2+k}$$
$$-2^{2+k}Z^2-kZ^{3+k}[1+Y^2+Z^2+Y^2+Z^2$$
$$-(2+k)Y^2-(2+k)Y^4-(2+k)(1+Y^2)Y^2Z^2]\}\mathrm{d}Y\mathrm{d}Z$$
$$=-\frac{1}{2}Re\frac{\beta K_5}{k+1}(1+X)^{\beta-\alpha(2+k)}\int_0^1\int_0^1\{2^{2+k}[Z^{2+k}$$
$$-Z^2+Z^{2+k}Y^2-Z^2Y^2+Z^{4+k}-Z^4+Z^{4+k}Y^2-Y^2Z^4$$
$$+(2+k)(Y^2Z^2-Z^{2+k}Y^2-Z^{2+k}Y^4+Y^4Z^2)]$$
$$-k[Z^{3+k}+Z^{3+k}Y^2+Z^{5+k}+(2+k)(Y^2Z^{3+k}$$
$$+Y^4Z^{3+k}+Z^{5+k}(1+Y^2)Y^2]\}\mathrm{d}Y\mathrm{d}Z$$
$$=-\frac{1}{2}Re\frac{\beta K_5}{k-1}(1+X)^{[\beta-\alpha(2+k)]}\left\{2^{(2+k)}\left[1.333\left(\frac{1}{3+k}+\frac{1}{5+k}\right)\right.\right.$$
$$\left.\left.-0.616(2+k)\left(0.178-\frac{0.533}{3+k}+\frac{1.067}{5+k}\right)\right]-k\left[\frac{1.333}{4+k}\right.\right.$$

$$+\frac{1}{6+k}+(2+k)\left(\frac{0.533}{4+k}\right)+\frac{0.533}{6+k}\Big)\Big]\Big\} \tag{2}$$

$$\frac{\varphi}{2}\nu_0 U\left(\frac{H}{B}\frac{\partial^2 U}{\partial X\partial Y}+\frac{\partial^2 U}{\partial X\partial Z}\right)$$

$$=\varphi\nu_0\alpha(1+X)^{-(2\alpha+1)}\left[\frac{H}{B}(2Z-Z^2)^2 Y(1-Y^2)+(1-Z)(2Z-Z^2)(1-Y^2)^2\right]$$

$$=\varphi\nu_0\alpha(1+X)^{-(2\alpha+1)}\Big[\frac{H}{B}(4Z^2Y-4Z^3Y+Z^4Y-4Z^2Y^3+4Z^3Y^3-Z^4Y^3)$$

$$+2Z-3Z^2-4Y^2Z+Z^3+6\varphi^2Z^2-2Y^2Z^3+2ZY^4-3Y^4Z^2+Y^4Z^3\Big]$$

$$=\varphi\nu_0\alpha(1+X)^{-(2\alpha+1)}\int_0^1\int_0^1\Big[\frac{H}{B}(4Z^2Y-4Z^3Y+Z^4Y-4Z^2Y^3+4Z^3Y^3-Z^4Y^3)$$

$$-2Z+3Z^2+4Y^2Z-Z^3-6Y^2Z^2+2Y^2Z^3-2ZY^4+3Y^4Z^2-Y^4Z^3\Big]\mathrm{d}Y\mathrm{d}Z$$

$$=\varphi\nu_0\alpha(1+X)^{-(2\alpha+1)}\left(0.134\frac{H}{B}-0.05\right) \tag{3}$$

$$\frac{\nu_0}{2}\left(\frac{H}{B}\frac{\partial U}{\partial Z}+\frac{\partial U}{\partial Z}\right)\left(\frac{U}{\rho_0}\frac{\partial\rho_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$=\frac{\nu_0}{2}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left(\frac{1}{\rho_0^{k-1}}\frac{K_5}{k-1}\frac{U\partial T_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$=\frac{\nu_0}{2}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left(\frac{K_5}{k-1}U\frac{k\partial T_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$=\frac{\nu_0}{2}\left(\frac{H}{B}\frac{K_5}{k-1}U^k\frac{\partial U}{\partial Y}\frac{\partial T_0}{\partial X}+\frac{H}{B}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}+\frac{K_5}{k-1}U\frac{k\partial T_0}{\partial X}\frac{\partial U}{\partial X}+\frac{\partial U}{\partial Z}\frac{\partial U}{\partial X}\right)$$

$$=\nu_0\Big[-\frac{H}{B}\frac{K_5}{k-1}\beta(1+X)^{[\beta-\alpha(k+1)-1]}\Big](2Z-Z^2)^{1+k}Y(1+Y^2)^k$$

$$+\frac{H}{B}\alpha(2Z-Z^2)^2Y(1-Y^2)(1+X)^{-(2\alpha+1)}+\frac{\beta K_5}{k-1}(1+X)^{[\beta-\alpha(k+1)]-1}$$

$$(2Z-Z^2)^k(1+Z^2)(1-Z)(1-Y^2)^{(k+1)}(1+Y^2)$$

$$-\alpha(1+X)^{-(2\alpha+1)}(1-Y^2)^2(1-Z)(2Z-Z^2)]$$

$$=\nu_0\Big\{{}^{*}_{*}-\frac{H}{B}\frac{K_5\beta}{(k-1)}(1+X)^{[\beta-\alpha(k+1)-1]}\Big[\frac{2}{k+2}-\frac{k(2)^k}{k+4}+\frac{1}{k+3}+\frac{k}{k+2}$$

$$-\frac{2}{5}\frac{k^2(2)^k}{(k+4)}+\frac{2}{5}\frac{k}{(k+3)}\Big]+0.133\alpha\frac{H}{B}(1+Z)^{-(2\alpha+1)}$$

$$+\frac{\beta k_5}{(k-1)}(1+X)^{[\beta-\alpha(k+1)-1]}\Big\{\frac{1}{k+1}(0.943-0.333k$$

$$+(0.19k-0.8)\Big[\Big(\frac{1}{k+2}-\frac{1}{k+3}+\frac{1}{k+4}\Big)$$

$$+k(2)^{k-1}\Big(\frac{1}{k+2}-\frac{1}{k+3}+\frac{1}{k+4}-\frac{1}{k+5}\Big)\Big]\Big\}-\alpha(1+X)^{(2\alpha+1)}\Big[\frac{1}{k+2}-\frac{k(2)^k}{k+4}$$

$$-\frac{1}{k+1}+\frac{1}{k+2}-\frac{2}{5}\frac{k(2)^k}{(k+4)}+\frac{2}{5}\frac{k}{(k+3)}\Big]^{*}_{*}\Big\} \tag{4}$$

$$\frac{K_3\lambda_0}{\rho_0}\left(\frac{\partial^2 T_0}{\partial X^2}+\left(\frac{H}{B}\right)^2\frac{\partial^2 T_0}{\partial Y^2}+\frac{\partial^2 T_0}{\partial Z^2}\right)=K_3\lambda_0 U\left(\frac{\partial^2 T_0}{\partial X^2}+\left(\frac{H}{B}\right)^2\frac{\partial^2 T_0}{\partial Y^2}+\frac{\partial^2 T_0}{\partial Z^2}\right)$$

$$=K_3\lambda_0\Big\{\Big[(1+X)^{\beta-\alpha-2}\beta(\beta-1)\int_0^1\int_0^1(1+Z^2)(1+Y^2)(2Z-Z^2)(1-Y^2)\mathrm{d}Y\mathrm{d}Z\Big]$$

$$+2(1+X)^{\beta-\alpha}\Big[\left(\frac{H}{B}\right)^2\int_0^1\int_0^1(1+Z^2)(2Z-Z^2)(1-Y^2)\mathrm{d}Y\mathrm{d}Z$$

$$+\int_0^1\int_0^1(1+Y^2)(2Z-Z^2)(1-Y^2)\mathrm{d}Y\mathrm{d}Z\Big]\Big\}$$

$$=K_3\lambda_0\left\{(1+X)^{\beta-\alpha-2}\beta(\beta-1)0.273+2(1+X)^{\beta-\alpha}\left[0.322\left(\frac{H}{B}\right)^2+0.543\right]\right\}$$

$$=K_3\lambda_0\left\{0.273\beta(\beta-1)(1+X)^{\beta-\alpha-2}+2(1+X)^{\beta-\alpha}\left[0.322\left(\frac{H}{B}\right)^2+0.543\right]\right\} \tag{5}$$

$$\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}=\frac{4}{3}\nu\alpha(1+X)^{-(\alpha+1)}(2Z-Z^2)(1-Y^2)$$

$$=\frac{4}{3}\nu_0\alpha(k+X)^{-(\alpha+1)}\int_0^1\int_0^1(2Z-Z^2)(1-Y^2)\mathrm{d}Y\mathrm{d}Z$$

$$=\frac{16}{27}\nu_0\alpha(1+X)^{-(\alpha+1)}=0.593\nu_0\alpha(1+X)^{-(\alpha+1)} \tag{6}$$

$$-\frac{2}{3}\frac{\nu_0^2}{Re}\varphi^{\frac{2}{3}}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)=-\frac{4}{3}\nu_0\varphi^{\frac{2}{3}}(1+X)^{-\alpha}[(1-Z)(1-Y^2)-Y(2Z-Z^2)]$$

$$=-\frac{4}{3}\nu_0\varphi^{\frac{2}{3}}(1+X)^{-\alpha}\int_0^1\int_0^1[(1-Z)(1-Y^2)-Y(2Z-Z^2)]\mathrm{d}Y\mathrm{d}Z$$

$$=-\frac{4}{3}\nu_0\varphi^{\frac{2}{3}}(1+X)^{-\alpha}\left(\frac{1}{3}-\frac{1}{3}\right)=0 \tag{7}$$

$$-K_4\varphi^{\frac{2}{3}}\frac{\nu_0^2}{U}\left(\frac{H}{B}\frac{\partial U}{\partial Y}+\frac{\partial U}{\partial Z}\right)\left[\left(\frac{H}{B}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}+\frac{\partial U}{\partial Z}\frac{\partial^2 U}{\partial Z^2}\right]$$

$$=-K_4\varphi^{\frac{2}{3}}\nu_0^2(1+X)^{-2\alpha}\frac{1}{(2Z-Z^2)(1-Y^2)}\left[-\frac{H}{B}2Y(2Z-Z^2)+(1-Z)(1-Y^2)\right]$$

$$\left[\left(\frac{H}{B}\right)(-2Y)(-2)(2Z-Z^2)^2+2(1-Z)(-2)(1-Y^2)^2\right]$$

$$=-16K_4\varphi^{\frac{2}{3}}\nu_0^2\left(-\frac{H}{B}\frac{Y}{1-Y^2}+\frac{1-Z}{2Z-Z^2}\right)\left[\left(\frac{H}{B}\right)^3Y(2Z-Z^2)^2+(1-Z)(1-Y^2)^2\right](1+X)^{-2\alpha}$$

$$=-16K_4\varphi^{\frac{2}{3}}\nu_0^2(1+X)^{-2\alpha}\Big[-\left(\frac{H}{B}\right)^4\frac{Y^2}{1-Y^2}(2Z-Z^2)^2-\frac{H}{B}Y(1-Y^2)(1-Z)+\left(\frac{H}{B}\right)^3$$

$$Y(1-Z)(2Z-Z^2)+\frac{(1-Z)^2}{2Z-Z^2}(1-Y^2)^2\Big]$$

$$=-16K_4\varphi^{\frac{2}{3}}\nu_0^2(1+X)^{-2\alpha}\Big[-\left(\frac{H}{B}\right)^4\int_0^1\frac{Y^2}{1-Y^2}\mathrm{d}Y\int_0^1(2Z-Z^2)\mathrm{d}Z-\frac{H}{B}\int_0^1Y(1-Y^2)\mathrm{d}Y$$

$$\int_0^1(1-Z)\mathrm{d}Z+\left(\frac{H}{B}\right)^3\int_0^1Y\mathrm{d}Y\int_0^1(1-Z)(2Z-Z^2)\mathrm{d}Z+\int_0^1\frac{(1-Z)^2}{2Z-Z^2}\mathrm{d}Z\int_0^1(1-Y^2)^2\mathrm{d}Y\Big]$$

$$= -16K_4\varphi^{\frac{2}{3}}\nu_0^2(1+X)^{-2\alpha}\left\{-\left(\frac{H}{B}\right)^4\left[-Y+\frac{1}{2}\ln\left(\frac{1+Y}{1-Y}\right)\right]\Big|_0^1\left(\frac{Z^2}{2}-\frac{Z^3}{3}\right)\Big|_0^1\right.$$

$$-\frac{H}{B}\left(\frac{Y^2}{2}-\frac{Y^4}{4}\right)\Big|_0^1\left(Z-\frac{Z^2}{2}\right)\Big|_0^1+\left(\frac{H}{B}\right)^3\frac{Y^2}{2}\Big|_0^1\left(Z-\frac{Z^2}{2}\right)\left(Z^2-\frac{Z^3}{3}\right)\Big|_0^1-\frac{1}{2}\ln\left(\frac{-Z+2}{Z}\right)\Big|_0^1$$

$$\left.-2[Z-\ln(-Z+1)]\big|_0^1+\left[\frac{Z^3}{3}\Big|_0^1+2\frac{Z^2}{2}\Big|_0^1-4\ln(2-Z)\big|_0^1\right]\left(Z-2\frac{Z^3}{3}+\frac{Z^4}{5}\right)\Big|_0^1\right\}$$

$$= -16K_4\varphi^{\frac{2}{3}}\nu_0^2(1+X)^{-2\alpha}\left\{\frac{3}{2}\left(\frac{H}{B}\right)^4-\frac{1}{8}\frac{H}{B}+\frac{1}{6}\left(\frac{H}{B}\right)^3+\left(1+\frac{1}{3}-0\right)(0.2)\right\}$$

$$= -16K_4\varphi^{\frac{2}{3}}\nu_0^2\left[0.266+\frac{3}{2}\left(\frac{H}{B}\right)^4+\frac{1}{6}\left(\frac{H}{B}\right)^3-\frac{H}{B}\right](1+X)^{-2\alpha} \tag{8}$$

将式(1)～式(8)代回式(8-3-93)，则

$$-0.162Re\alpha(1+X)^{-(4\alpha+1)}-\frac{1}{2}Re\frac{\beta K_5}{k-1}(1+X)^{\beta-\alpha(2+k)}$$

$$\left\{{}^{2+k}_{2}\quad\left[1.333\left(\frac{1}{3+k}+\frac{1}{5+k}\right)-0.616(2+k)\left(0.178-\frac{0.533}{3+k}+\frac{1.067}{5+k}\right)\right]\right.$$

$$\left.-k\left[\frac{1.333}{4+k}+\frac{1}{6+k}+(2+k)\left(\frac{0.533}{4+k}+\frac{0.533}{6+k}\right)\right]\right\}+\varphi\nu_0\alpha(1+X)^{-(2\alpha+1)}$$

$$\left(0.134\frac{H}{B}-0.05\right)+\nu_0\left\{-\frac{H}{B}\frac{K_5\beta}{(k-1)}(1+X)^{\beta-\alpha(k+1)-1}\left[\frac{2}{k+2}-\frac{k\cdot 2^k}{k+4}\right.\right.$$

$$\left.+\frac{1}{k+3}+\frac{k}{k+2}-\frac{2}{5}\left(\frac{k^2\cdot 2^k}{k+4}-\frac{k}{k+3}\right)\right]+0.133\alpha\frac{H}{B}(1+X)^{-(2\alpha+1)}$$

$$+\frac{\beta K_5}{k+1}(1+X)^{\beta-\alpha(k+1)-1}\left\{\frac{1}{k+1}(0.943-0.333k+(0.19k-0.8)\right.$$

$$\left.\left[\left(\frac{1}{k+2}-\frac{1}{k+3}+\frac{1}{k+4}\right)+k\cdot 2^{k-1}\left(\frac{1}{k+2}-\frac{1}{k+3}+\frac{1}{k+4}-\frac{1}{k+5}\right)\right]\right\}$$

$$\left.-\alpha(1+X)^{-(2\alpha+1)}\left[\frac{1}{k+2}-\frac{k\cdot 2^k}{k+2}+\frac{1}{k+1}+\frac{k}{k+2}-\frac{2}{5}\left(\frac{k\cdot 2^k}{k+4}-\frac{k}{5+3}\right)\right]\right\}$$

$$= K_3\lambda_0\left\{0.273\beta(\beta-1)(1+X)^{\beta-\alpha-2}+2(1+X)^{\beta-\alpha}\left[0.322\left(\frac{H}{B}\right)^2+0.534\right]\right\}+0.593\nu_0\alpha$$

$$(1+X)^{-(\alpha+1)}-16K_4\varphi^{\frac{2}{3}}\nu_0^2\left[0.266+\frac{2}{3}\left(\frac{H}{B}\right)^4+0.167\left(\frac{H}{B}\right)^3-\frac{H}{B}\right](1+X)^{-2\alpha} \tag{8-3-104}$$

对式(8-3-104)中的$(1+X)$取自然对数，则

$$0.162Re\alpha(4\alpha+1)\ln(1+X)-\frac{1}{2}Re\frac{\beta K_5}{k-1}[\beta-\alpha(2+k)]$$

$$\left\{{}^{2+k}_{2}\quad\left[1.333\left(\frac{1}{3+k}+\frac{1}{5+k}\right)-0.616(2+k)\left(0.178-\frac{0.533}{3+k}+\frac{1.067}{5+k}\right)\right]\right.$$

$$\left.-k\left[\frac{1.333}{4+k}+\frac{1}{6+k}+(2+k)\left(\frac{0.533}{4+k}+\frac{0.533}{6+k}\right)\right]\right\}\ln(1+X)$$

$$-\varphi\nu_0\alpha(2\alpha+1)\left(0.134\frac{H}{B}-0.05\right)\ln(1+X)-\nu_0\frac{H}{B}\frac{K_5\beta}{k-1}[\beta-\alpha(k+1)-1]$$

$$\left[\frac{2}{k+2}-\frac{k\cdot 2^k}{k+4}+\frac{1}{k+3}+\frac{k}{k+2}-\frac{2}{5}\left(\frac{k^2\cdot 2^k}{k+4}-\frac{k}{k+3}\right)\right]\ln(1+X)$$

$$
\begin{aligned}
&-0.133\nu_0(2\alpha+1)\alpha\frac{H}{B}\ln(1+X)+\frac{\nu_0\beta K_5}{k+1}[\beta-\alpha(k+1)-1]\\
&\left[\frac{1}{k+1}(0.943-0.333k+(0.19k-0.8)\left(\frac{1}{k+2}-\frac{1}{k+3}+\frac{1}{k+4}\right)\right.\\
&\left.+k\cdot 2^{k-1}\left(\frac{1}{k+2}-\frac{1}{k+3}+\frac{1}{k+4}-\frac{1}{k+5}\right)\right]\ln(1+X)\\
&+\nu_0\alpha(2\alpha+1)\left[\frac{1}{k+2}-\frac{k\cdot 2^k}{k+2}+\frac{1}{k+1}+\frac{k}{k+2}-\frac{2}{5}\left(\frac{k\cdot 2^k}{k+4}-\frac{k}{5+3}\right)\right]\\
=\;&K_3\lambda_0\left\{0.273\beta(\beta-1)(\beta-\alpha-2)\ln(1+X)+2(\beta-\alpha)\left[0.322\left(\frac{H}{B}\right)^2+0.534\right]\ln(1+X)\right\}\\
&-0.533\nu_0\alpha(\alpha+1)\ln(1+X)+32K_4\varphi^{\frac{2}{3}}\nu_0^2\left[0.266+\frac{2}{3}\left(\frac{H}{B}\right)^4+\frac{1}{6}\left(\frac{H}{B}\right)^3-\frac{H}{B}\right]\ln(1+X)
\end{aligned}
\tag{8-3-105}
$$

将式(8-3-105)除以 $\ln(1+X)$，然后全式除 Re，并进行数量级比较，忽略相对小量，则

$$
\begin{aligned}
&0.162\alpha(4\alpha+1)-\frac{1}{2}\frac{\beta K_5}{k-1}[\beta-\alpha(k+2)]\\
&\left\{^{2+k}\left[1.333\left(\frac{1}{3+k}+\frac{1}{5+k}\right)-0.616(2+k)\left(0.178-\frac{0.533}{3+k}+\frac{1.067}{5+k}\right)\right]\right.\\
&\left.-k\left[\frac{1.333}{4+k}+\frac{1}{6+k}+(2+k)\left(\frac{0.533}{4+k}+\frac{0.533}{6+k}\right)\right]\right\}-\frac{\nu_0}{Re}\frac{H}{B}\frac{K_5\beta}{k-1}[\beta-\alpha(k+1)-1]\\
&\left[\frac{2}{k+2}-\frac{k\cdot 2^k}{k+4}+\frac{1}{k+3}+\frac{k}{k+2}-\frac{2}{5}\left(\frac{k^2\cdot 2^k}{k+4}-\frac{k}{k+3}\right)\right]\\
&+\frac{\nu_0\beta K_5}{(k+1)Re}[\beta-\alpha(k+1)-1]\left[\frac{1}{k+1}\frac{1}{k+1}(0.943-0.333k+(0.19k-0.8)\right.\\
&\left.\left(\frac{1}{k+2}-\frac{1}{k+3}+\frac{1}{k+4}\right)+k\cdot 2^{k-1}\left(\frac{1}{k+2}-\frac{1}{k+3}+\frac{1}{k+4}-\frac{1}{k+5}\right)\right]\\
=\;&\frac{K_3\lambda_0}{Re}\left\{0.273\beta(\beta-1)(\beta-\alpha-2)+2(\beta-\alpha)\left[0.322\left(\frac{H}{B}\right)^2+0.534\right]\right\}\\
&+\frac{32}{Re}K_4\varphi^{\frac{2}{3}}\nu_0^2\alpha\left[0.266+\frac{2}{3}\left(\frac{H}{B}\right)^4+\frac{1}{6}\left(\frac{H}{B}\right)^3-\frac{H}{B}\right]
\end{aligned}
\tag{8-3-106}
$$

式(8-3-102)与式(8-3-106)是分别由动量方程与能量方程导出来的。式中 Re，K_2，K_3，K_4，K_5 对于具体问题，均可以计算，而 φ 涡旋体积分数，可以通过涡旋体积分数公式计算，则三个未知数 α，β，φ 可以由三个方程联立求解。只要速度分布公式得到，则其他深一层问题也就迎刃而解了。

8.3.4.4 流量沿程衰减计算公式

当流量沿程下降时，流量与距离关系公式是必须解决的，如下：

$$
Q(x)=4u_{\mathrm{md}}HB=4HB(1+X)^{-\alpha}\int_0^1\int_0^1(2Z-Z^2)(1-Y^2)\mathrm{d}Y\mathrm{d}Z=\frac{8}{9}HB(1+X)^{-\alpha}
\tag{8-3-107}
$$

8.3.4.5　湍流极限距离

现在研究的湍流管道，当速度沿程下降时，总是在一定距离后，管道内变成层流运动。那么这个长度是多少呢?

对于圆形管道，应用公式为

$$2320=\frac{v_d}{\nu}=\frac{2r_0v_d}{\nu}\frac{2}{9}(1+X)^{-\alpha} \tag{8-3-108}$$

对于非圆形管道，采用水力半径计算公式为

$$580=\frac{Rv}{\nu}=\frac{2}{9}\frac{HB(1+X)^{-\alpha}}{(H+B)\nu} \tag{8-3-109}$$

第 9 章　定常湍流边界层

本章研究定常流不可压缩与可压缩湍流边界层，按无压与有压力变化平板边界层分别讨论。

分析认为，平板边界层外理想流体运动为一维运动，一维变化；层内为二维变化，一维流动。如图 9-1 所示。

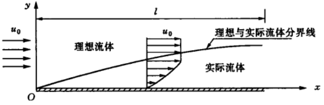


图 9-1　平板边界层示意图

本章分为 5 节：9.1 推导定常层流平板边界层厚度计算公式，目的是旁证本章研究方法的正确性；9.2 节确定边界层厚度，层内速度分布，边层流界面位置以及其上产生的涡旋强度、大小、速度，最后得出涡旋体积分数 φ；9.3 节研究定常流不可压缩有压力变化湍流平板边界层；9.4 节研究定常可压缩无压力变化湍流平板边界层；9.5 节研究定常可压缩有压力变化湍流平板边界层。

9.1　定常层流平板边界层

本节研究无压与有压力变化平板边界层，确定层内速度分布及边界层厚度。

9.1.1　无压力变化层流平板边界层

坐标系与原点选定如图 9-1 所示。

(1) 物理运动控制方程与边界条件

动量方程：依式(4-1-10)为

$$u\frac{\partial u}{\partial x}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{9-1-1}$$

边界条件：

$$u(x,\ y)|_{x=0}=0 \tag{9-1-2}$$

$$u(x,\ y)|_{y=\delta(x)}=u_c \tag{9-1-3}$$

$$\left.\frac{\partial u(x,\ y)}{\partial y}\right|_{y=\delta(x)}=0 \tag{9-1-4}$$

$$\frac{\partial^2 u(x,\ y)}{\partial y^2}<0 \tag{9-1-5}$$

式中：u_c——边界层外理想流体速度，即来流速度。

(2) 运动控制方程与边界条件无因次化

方法 1　取

$$\frac{u}{u_c}=U,\quad \frac{x}{l}=X,\quad \frac{y}{\delta(x)}=Y,\quad \frac{\nu}{\nu_c}=\nu_0$$

式中：l——平板长度；

$\delta(x)$——理想流体与实际流体运动分界线，即边界层厚度；

ν_c——来流处的流体运动黏性系数。

将以上比值代入运动物理方程与边界条件，则有

$$ReU\frac{\partial U}{\partial X}=\nu_0\left[\frac{\partial^2 U}{\partial X^2}+\left(\frac{l}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial Y^2}\right] \tag{9-1-6}$$

式中：

$$Re=\frac{u_c l}{\nu_c} \tag{9-1-7}$$

$$U(X,\ Y)|_{X=0}=0 \tag{9-1-8}$$

$$U(X,\ Y)|_{Y=1}=1 \tag{9-1-9}$$

$$\left.\frac{\partial U(X,\ Y)}{\partial Y}\right|_{Y=1}=0 \tag{9-1-10}$$

$$\frac{\partial^2 U(X,\ Y)}{\partial Y^2}<0 \tag{9-1-11}$$

方法 2　与方法 1 取法不同之处是

$$\frac{y}{l}=Y,\quad \frac{\delta(x)}{l}=\delta_Y$$

将它们代入运动物理控制方程与边界条件。将不同之处写出：

$$ReU\frac{\partial U}{\partial X}=\nu_0\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right) \tag{9-1-12}$$

$$U(X,\ Y)|_{Y=\delta_Y}=1 \tag{9-1-13}$$

$$\left.\frac{\partial U(X,\ Y)}{\partial Y}\right|_{Y=\delta_Y}=0 \tag{9-1-14}$$

(3) 确定边界层厚度 $\delta(x)$与层内速度分布

根据边界条件，选层内速度分布为

$$U=\eta X^{\frac{1}{2}}(2Y-Y^2) \tag{9-1-15}$$

式中：η——参变量。

为定出 η，应将式(9-1-5)代入式(9-1-6)，则有

$$\frac{1}{2}\eta Re(2Y-Y^2)^2=-\nu_0\left[\frac{1}{4}(2Y-Y^2)X^{-\frac{3}{2}}+2X^{\frac{1}{2}}\left(\frac{l}{\delta(x)}\right)^2\right] \tag{1}$$

根据问题实质，η 不能为负值，故应取正值；当 $X=0$ 时，出现∞项，应去掉；$\nu_0\approx1$。则上式应为

$$\frac{1}{2}\eta Re(2Y-Y^2)^2=2X^{\frac{1}{2}}\left(\frac{l}{\delta(x)}\right)^2 \tag{2}$$

$$\eta=\frac{4X^{\frac{1}{2}}\left(\frac{l}{\delta(x)}\right)^2}{Re(2Y-Y^2)^2} \tag{3}$$

将 η 代入式(9-1-15)，有

$$U=\frac{4X\left(\frac{l}{\delta(x)}\right)^2(2Y-Y^2)}{Re(2Y-Y^2)^2}=\frac{4X\left(\frac{l}{\delta(x)}\right)^2}{Re(2Y-Y^2)} \tag{4}$$

当 $Y=1$ 时，$U=1$，将此条件代入式(4)，则有

$$1=\frac{4X\left(\frac{l}{\delta(x)}\right)^2}{Re} \tag{5}$$

由式(5)，得边界层厚度为

$$\delta(x)=\frac{2X^{\frac{1}{2}}l}{\sqrt{Re}} \tag{9-1-16}$$

而式(4)作为速度分布，则无法满足$\left.\frac{\partial U}{\partial Y}\right|_{Y=1}=0$ 的条件。所以必须将式(9-1-15)代入式(9-1-12)，得

$$\eta=\frac{4X^{\frac{1}{2}}}{Re(2Y-Y^2)^2} \tag{9-1-17}$$

因为 η 是参变量，反映流体受力后的运动强度，将分母中的$(2Y-Y^2)^2$进行处理，对任一断面上均为平均值，即

$$(2Y-Y^2)^2=\frac{1}{2}\int_0^{\delta_Y}(2Y-Y^2)^2\mathrm{d}Y=\frac{1}{2}\delta_Y^3\left(\frac{4}{3}-\delta_Y+\frac{1}{5}\delta_Y^2\right) \tag{6}$$

$\delta_Y=\frac{\delta(x)}{l}\ll1$，故可近似为$\frac{2}{3}\delta_Y^3$，代入式(9-1-17)，得

$$\eta=\frac{6X^{\frac{1}{2}}}{Re\delta_Y^3} \tag{7}$$

将式(7)代入式(9-1-15)，得

$$U=\frac{6X^{\frac{1}{2}}}{Re\delta_Y^3}X^{\frac{1}{2}}(2Y-Y^2)=\frac{6X}{Re\delta_Y^3}(2Y-Y^2) \tag{9-1-15$'$}$$

要使式(9-1-15)′成为层内速度分布，而且满足 $Y=\delta_Y$ 时 $U=1$，式中必须乘以一个修正系数 ξ，即

$$U=\xi\frac{6X}{Re\delta_{Y}^{3}}(2Y-Y^{2}) \tag{9-1-18}$$

$$\xi=\frac{1}{3\left(1-\dfrac{X^{\frac{1}{2}}}{\sqrt{Re}}\right)} \tag{9-1-19}$$

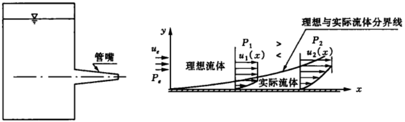
$$\delta_{Y}=\frac{\delta(x)}{l}=\frac{2X^{\frac{1}{2}}}{\sqrt{Re}} \tag{9-1-20}$$

式(9-1-18)满足边界条件

$$U|_{X=0}=0,\qquad \left.\frac{\partial U}{\partial Y}\right|_{Y=\delta_{Y}}=0,\qquad U|_{Y=\delta_{Y}}=1$$

9.1.2　有压力变化层流平板边界层

有压力变化层流平板边界层，一般出现在具体设备上。如图 9-2 所示，定水头水箱引出的收缩或扩散型的管嘴中，当雷诺数在层流范围内。



(a)　(b)

图 9-2　有压力变化定常流层流平板边界层

9.1.2.1　运动控制物理方程与边界条件

(1) 实际流区动量方程

依式(4-1-10)，结合本问题，为

$$u\frac{\partial u}{\partial x}=-\frac{\partial p}{\rho\partial x}+\nu\left(\frac{\partial^{2}u}{\partial x^{2}}+\frac{\partial^{2}u}{\partial y^{2}}\right) \tag{9-1-21}$$

实际流区流体运动边界条件：

$$u(x,\ y)|_{x=0}=0 \tag{9-1-22}$$

$$u(x,\ y)|_{y=\delta(x)}=u(x) \tag{9-1-23}$$

$$\left.\frac{\partial u(x,\ y)}{\partial y}\right|_{y=\delta(x)}=0 \tag{9-1-24}$$

$$\frac{\partial^{2}u(x,\ y)}{\partial y^{2}}<0 \tag{9-1-25}$$

(2) 理想流体运动区流体运动

动量方程：

$$u(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}=-\frac{\mathrm{d}p}{\rho\mathrm{d}x} \tag{9-1-26}$$

边界条件：

$$u(x)|_{x=0}=u_c \tag{9-1-27}$$

压力只是 x 的函数，与实际或理想流体运动无关，即

$$p(x)|_{x=0}=p_c \tag{9-1-28}$$

9.1.2.2 运动控制方程与边界条件无因次化

取

$$\frac{u(x,\ y)}{u_c}=U(X,\ Y),\ \frac{x}{l_c}=X,\ \frac{y}{\delta(x)}=Y,\ \frac{\rho}{\rho_c}=\rho_0,\ \frac{\nu}{\nu_c}=\nu_0,\ \frac{p}{p_c}=p_0$$

$u(x)$是理想流体运动速度，它只是 x 的函数；$u(x,\ y)$是实际流体运动速度；l_c 是进口处管嘴半径；p_c 是进口处流体压力；由于是不可压缩流体，故 $\nu_0=1$，$\rho_0=1$。将比值代入方程与边界条件。

实际流体运动区：

$$ReU\frac{\partial U}{\partial X}=-K_1\frac{\partial p_0}{\rho_0\partial X}+\nu_0\left[\frac{\partial^2U}{\partial X^2}+\left(\frac{l}{\delta(x)}\right)^2\frac{\partial^2U}{\partial Y^2}\right] \tag{9-1-29}$$

式中：

$$Re=\frac{u_c l}{\nu_c} \tag{9-1-30}$$

$$K_1=\frac{p_c}{\rho_c\nu_c u_c} \tag{9-1-31}$$

$$U(X,\ Y)|_{X=0}=0 \tag{9-1-32}$$

$$U(X,\ Y)|_{Y=\delta_Y}=U(X) \tag{9-1-33}$$

$$\left.\frac{\partial U(X,\ Y)}{\partial Y}\right|_{Y=\delta_Y}=0 \tag{9-1-34}$$

$$\frac{\partial^2U(X,\ Y)}{\partial Y^2}<0 \tag{9-1-35}$$

理想流体运动区：

$$U(X)\frac{\mathrm{d}U(X)}{\mathrm{d}X}=-K_2\frac{\mathrm{d}p_0}{\rho_0\mathrm{d}X} \tag{9-1-36}$$

$$U(X)|_{X=0}=1 \tag{9-1-37}$$

式中：

$$K_2=\frac{p_c}{\rho_c u_c^2} \tag{9-1-38}$$

压力只是 x 的函数，故

$$p_0(X)\big|_{X=0}=1 \tag{9-1-39}$$

9.1.2.3 理想与实际流区速度分布关系

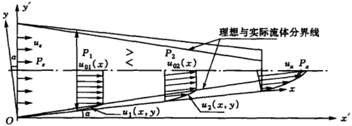


图 9-3 各不同断面流量守恒示意图

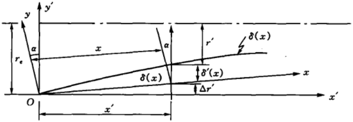


图 9-4 两坐标系中变量之间关系示意图

进口断面流量

$$Q_c=\pi r_c^2u_c \tag{1}$$

通过理想流体运动流过断面上的流量

$$Q_{理}=\pi(r_c-x'\tan\alpha-\delta'(x))^2u(x) \tag{2}$$

通过实际流体运动区任一断面上的流量

$$Q_{实}=2\pi(r_c-x'\tan\alpha)\delta'(x)\frac{1}{2}\int_0^{\delta'(x)}u(x,y)\mathrm{d}y' \tag{3}$$

根据流量守恒原则，则有

$$\pi r_c^2u_c=\pi(r_c-x'\tan\alpha-\delta'(x))^2u(x)+\pi(r_c-x'\tan\alpha)\delta'(x)\int_0^{\delta'(x)}u(x,y)\mathrm{d}y' \tag{4}$$

$$x'=x\cos\alpha,\quad \delta(x)=\delta'(x)\cos\alpha,\quad y\cos\alpha=y'$$

将式(4)中的$(r_c-x'\tan\alpha-\delta'(x))^2$展开为

$$(r_c-x'\tan\alpha-\delta'(x))^2=r_c^2+x'^2\tan^2\alpha+\delta^2(x)-2r_cx'\tan\alpha-2r_c\delta'(x)+2x'\delta'(x)\tan\alpha \tag{5}$$

将式(5)代回式(4)，并将x'变成x，则有

$$\pi r_c^2u_c=\pi[r_c^2+x^2\sin^2\alpha+\sec^2\alpha\delta^2(x)-2r_cx\sin\alpha-2r_c\delta'(x)\sec\alpha+2x\delta(x)\tan\alpha]u(x)$$
$$+\pi(r_c-x\sin\alpha)\delta(x)\int_0^{\delta(x)}u(x,y)\mathrm{d}y \tag{6}$$

将式(6)无因次化，除以 $\pi l^2 u_c$，并且由于$\frac{\delta(x)}{l}=\delta_Y$，则

$$R^2=\left(R^2+X^2\sin^2\alpha+\sec^2\alpha\delta_Y^2-2RX\sin\alpha-2R\delta_Y\sec\alpha+2X\delta_Y\tan\alpha\right)U(X)$$
$$+(R-X\sin\alpha)\delta_Y\int_0^{\delta_Y}U(X,Y)\mathrm{d}Y \tag{9-1-40}$$

9.1.2.4 实际流体运动区控制方程应用形式

因为压力变化只是 x 的函数，所以将理想流体运动控制方程代入式(9-1-29)，消掉含压力变化项，则有

$$ReU\frac{\partial U}{\partial X}=ReU(X)\frac{\mathrm{d}U(X)}{\mathrm{d}X}+\nu\left(\frac{\partial^2U(X,\ Y)}{\partial X^2}+\frac{\partial^2U(X,\ Y)}{\partial Y^2}\right) \tag{9-1-41}$$

9.1.2.5 理想与实际流体运动区速度分布

根据边界条件要求，实际流体运动区速度分布为

$$U(X,\ Y)=\eta X^{\frac{1}{2}}(2Y-Y^2) \tag{9-1-42}$$

理想流体运动区速度分布为

$$U(X)=1+\eta X^{\frac{1}{2}}(2\delta_Y-\delta_Y^2) \tag{9-1-43}$$

9.1.2.6 确定参变量 η 与边界层厚度 $\delta(x)$

将式(9-1-42)与式(9-1-43)代入式(9-1-41)，得

$$\frac{1}{2}Re\eta^2(2Y-Y^2)^2=Re[1+\eta X^{\frac{1}{2}}(\delta_Y-\delta_Y^2)]\eta X^{-\frac{1}{2}}(2\delta_Y-\delta_Y^2)+\nu_0\left[-\frac{1}{4}\eta(2Y-Y^2)X^{-\frac{3}{2}}-2\eta X^{\frac{1}{2}}\right] \tag{9-1-41$'$}$$

$$\frac{1}{2}Re\eta^2(2Y-Y^2)^2=\frac{Re}{2}[\eta X^{-\frac{1}{2}}(2\delta_Y-\delta_Y^2)+\eta^2(\delta_Y-\delta_Y^2)^2]+\nu_0\left[\frac{1}{4}\eta(2Y-Y^2)X^{-\frac{3}{2}}+2X^{\frac{1}{2}}\eta\right] \tag{9-1-41$''$}$$

式中 $\nu_0=1$。当 $X=0$ 时，将出现∞项，去掉，合理化，则有

$$\eta=\frac{2X^{\frac{1}{2}}}{1+0.5Re[(2\delta_Y-\delta_Y^2)^2-(2Y-Y^2)^2]} \tag{9-1-44}$$

将式(9-1-42)与式(9-1-43)代入式(9-1-40)，则有

$$R^2=(R^2+X^2\sin^2\alpha+\sec^2\alpha\delta_Y^2-2RX\sin\alpha-2R\delta_Y\sec\alpha+2X\delta_Y\tan\alpha)[1+\eta X^{\frac{1}{2}}(\delta_Y-\delta_Y^2)]$$
$$+(R-X\sin\alpha)\eta X^{\frac{1}{2}}\delta_Y^2 \tag{9-1-40$'$}$$

$$R^2=R^2+X^2\sin^2\alpha+\sec^2\alpha\delta_Y^2-2RX\sin\alpha-2R\delta_Y\sec\alpha+2X\delta_Y\tan\alpha+\{[R^2+X^2\sin^2\alpha+$$
$$\sec^2\alpha\delta_Y^2-2RX\sin\alpha-2R\delta_Y\sec\alpha+2X\delta_Y\tan\alpha]X^{\frac{1}{2}}(\delta_Y-\delta_Y^2)+(R-X\sin\alpha)X^{\frac{1}{2}}\delta_Y^2\}\eta \tag{9-1-40$''$}$$

因为$\frac{\delta(x)}{l}=\delta_Y\ll1$，$\delta_Y^2$ 可以忽略不计，则有

$$\eta=\frac{2RX\sin\alpha-X^2\sin^2\alpha+2(R\sec\alpha-X\tan\alpha)\delta_Y}{(R^2+X^2\sin^2\alpha-2RX\sin\alpha)X^{\frac{1}{2}}\delta_Y} \tag{9-1-45}$$

由式(9-1-44)及式(9-1-45)，则有

$$\frac{2X^{\frac{1}{2}}}{1+0.5Re[(2\delta_Y-\delta_Y^2)^2-(2Y-Y^2)^2]}=\frac{2RX\sin\alpha-X^2\sin^2\alpha+2(R\sec\alpha-X\tan\alpha)\delta_Y}{(R^2+X^2\sin^2\alpha-2RX\sin\alpha)X^{\frac{1}{2}}\delta_Y} \tag{9-1-46}$$

利用边界条件，对式(9-1-46)分析，将 $Y=\delta_Y$，代入式(9-1-46)，得

$$\frac{2X^{\frac{1}{2}}}{1}=\frac{2RX\sin\alpha-X^2\sin^2\alpha+2(R\sec\alpha-X\tan\alpha)\delta_Y}{(R^2+X^2\sin^2\alpha-2RX\sin\alpha)X^{\frac{1}{2}}\delta_Y} \tag{9-1-46$'$}$$

整理为

$$\delta_Y=\frac{2RX\sin\alpha-X^2\sin\alpha}{2[X(R^2+X^2\sin^2\alpha-2RX\sin\alpha)-(R\sin\alpha-X\tan\alpha)]} \tag{9-1-46$''$}$$

根据边界条件要求，$X=0$，$\delta_Y=0$，而且不能为负值。由此可知当 $X=0$ 时，则有

$$\delta_Y=\frac{0}{-2R\sin\alpha} \tag{9-1-46$'''$}$$

不合理，根据边界条件要求，修正为

$$\delta_Y=\frac{2RX\sin\alpha-X^2\sin^2\alpha}{1+2[X(R^2+X^2\sin^2\alpha-2RX\sin\alpha)-(R\sin\alpha-X\tan\alpha)]} \tag{9-1-47}$$

式(9-1-47)与式(9-1-44)是两个独立方程，有了它们，则可以计算速度分布。

9.1.2.7　实际流体运动速度修正

根据边界条件 $U(X,\ Y)|_{Y=\delta_Y}=U(X)$，对式(9-1-42)进一步修正：在应用层流运动微分方程时，当 $X=0$ 时定为 $U(X,\ Y)|_{X=0}=0$，这样就得到速度仅相差一个来流速度 u_c，所以边界层内的速度分布应修正为

$$U(X,\ Y)=1+\eta X^{\frac{1}{2}}(2Y-Y^2) \tag{9-1-48}$$

例 9-1　如图 9-5 所示，一收缩形喷管长 $l'=0.2$m，收缩角 α 为 5°，进口流速 $u_c=0.05$m/s，水温 $t=15$℃，求喷嘴出口处边界层厚度 $\delta(x)$为多少？出口处边界层厚度之半处流体速度为多少？

解　准备工作：水运动黏性系数 $\nu_c=1.007\times10^{-6}\mathrm{m^2/s}$，$\sin5°=0.08716$，$\cos5°=0.99617$，$\tan5°=0.08741$，$l=\frac{l'}{\cos\alpha}=\frac{0.2}{0.99617}=0.2008$m，$R=\frac{r_c}{l}=\frac{0.02}{0.2008}=0.0996$

层流运动判定

$$Re'=\frac{u_cD}{\nu_c}=\frac{0.05\times0.04}{1.007}\times10^6=1986<2320$$

计算用准数

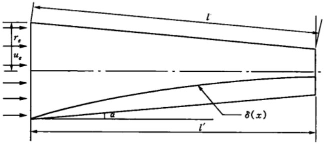


图 9-5 例 9-1 用图

$$Re=\frac{0.05\times 0.2008}{1.007}\times 10^6=9970$$

计算出口处边界层$\delta(x)\mid_{X=1}$处的厚度。由式(9-1-47)，有

$$\begin{aligned}
\delta_Y&=\frac{2RX\sin\alpha-X^2\sin^2\alpha}{1+2[X(R^2+X^2\sin^2\alpha-2RX\sin\alpha)-(R\sin\alpha-X\tan\alpha)]}\\
&=\frac{2\times 0.0996\times 0.08716-0.08716^2}{1+2[(0.0996^2+0.8716^2-2\times 0.0996\times 0.08716)-(0.0996\times 0.08716-0.08741)]}\\
&=\frac{0.00976}{1+2[(0.0099+0.0076-0.01736)-(0.00868-0.08741)]}\\
&=\frac{0.00976}{1+2[(0.0175-0.01736)+0.07873]}=\frac{0.00976}{1+2(0.00014+0.07873)}\\
&=\frac{0.00976}{1+2\times 0.07887}=\frac{0.00976}{1+0.15774}=\frac{0.00976}{1.15774}=0.0084
\end{aligned}$$

$$\delta(x)\mid_{X=1}=0.0084\times 0.2008=0.0017\text{m}$$

确定 $x=l$，$y=\dfrac{\delta(x)}{2}$处的速度。由式(9-1-48)，有

$$\begin{aligned}
U\mid_{\substack{X=1\\ Y=\frac{\delta_Y}{2}}}&=1+\eta X^{\frac{1}{2}}(2Y-Y^2)=1+\eta\left[\delta_Y-\left(\frac{\delta_Y}{2}\right)^2\right]=1+\eta(0.0084-0.0042^2)\\
&=1+\eta(0.0084-0.0000176)
\end{aligned}\tag{9-1-48}'$$

计算参变量 η。由式(9-1-44)，有

$$\begin{aligned}
\eta&=\frac{2X^{\frac{1}{2}}}{1+0.5Re[(2\delta_Y-\delta_Y^2)^2-(2Y-Y^2)^2]}\\
&=\frac{2}{1+0.5\times 9970[(2\times 0.0084-0.0084^2)^2-(0.0084-0.0042^2)^2]}\\
&=\frac{2}{1+0.5\times 9970(0.01673^2-0.00838^2)}=\frac{2}{1+0.5\times 9970(0.00028-0.00007)}\\
&=\frac{2}{1+0.5\times 9970\times 0.00021}=\frac{2}{1+1.04685}=\frac{2}{2.04685}=0.9771
\end{aligned}\tag{9-1-44}'$$

$$U=1+0.9771\times 0.00838\tag{9-1-48}''$$

$$u=u_c(1+0.00836)=0.05\times 1.00836=0.0504\text{m/s}$$

9.2　不可压缩湍流无压力变化平板边界层

定常流、无质量力、不可压缩湍流无压力变化平板边界层流动情况，如图 9-6 所示。

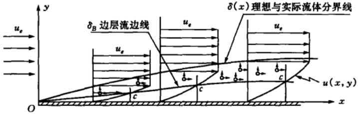


图 9-6　不可压缩湍流无压力变化平板边界层

9.2.1　运动控制方程与边界条件

(1) 动量方程

依式(4-4-11)并结合本问题，可得其运动控制方程为

$$\frac{\nu}{2}\left[(1-\varphi^{\frac{2}{3}})\frac{\partial^2 u}{\partial y \partial x}-(\varphi^{\frac{2}{3}}-1)\frac{1}{u}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right]=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+6\varphi^{\frac{2}{3}}\nu t\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} \tag{9-2-1}$$

(2) 边界条件

$$u(x,\ y)\Big|_{\substack{x=0\\y=0}}=0 \tag{9-2-2}$$

$$u(x,\ y)\Big|_{y=\delta(x)}=u_c \tag{9-2-3}$$

$$\left.\frac{\partial u(x,\ y)}{\partial y}\right|_{y=\delta(x)}=0 \tag{9-2-4}$$

$$\frac{\partial^2 u(x,\ y)}{\partial y^2}<0 \tag{9-2-5}$$

(3) 运动控制方程与边界条件无因次化

方法 1　取

$$\frac{u(x,\ y)}{u_c}=U,\quad \frac{x}{l}=X,\quad \frac{y}{\delta(x)}=Y,\quad \frac{\nu}{\nu_c}=\nu_0$$

将比值代入方程与边界条件，得

$$\frac{\nu_0}{2}\left[(1-\varphi^{\frac{2}{3}})\frac{\nu_c u_c}{l^2}\frac{\partial^2 U}{\partial X \partial Y}-(\varphi^{\frac{2}{3}}-1)\frac{\nu_c u_c}{l^2}\frac{1}{U}\frac{\partial U}{\partial X}\frac{\partial U}{\partial Y}\right]\frac{l}{\delta(x)}$$

$$=\nu_0\left[\frac{\nu_c u_c}{l^2}\frac{\partial^2 U}{\partial X^2}+\left(\frac{l}{\delta(x)}\right)^2\frac{\nu_c u_c}{l^2}\frac{\partial^2 U}{\partial Y^2}\right]+\frac{6tu_c^2}{l^3}\varphi^{\frac{2}{3}}\left(\frac{l}{\delta(x)}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} \tag{9-2-1$'$}$$

全式除以$\frac{\nu_c u_c}{l^2}$，则有

$$\frac{\nu_0}{2}\left[(1-\varphi^{\frac{2}{3}})\frac{\partial^2 U}{\partial X \partial Y}-(\varphi^{\frac{2}{3}}-1)\frac{1}{U}\frac{\partial U}{\partial X}\frac{\partial U}{\partial Y}\right]\frac{l}{\delta(x)}$$

$$=\nu_0\left[\frac{\partial^2 U}{\partial X^2}+\left(\frac{l}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial Y^2}\right]+\nu_0 K_1\varphi^{\frac{2}{3}}\left(\frac{l}{\delta(x)}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} \tag{9-2-6}$$

式中：

$$K_1=\frac{6tu_c}{l} \tag{9-2-7}$$

$$U(X,\ Y)|_{X=0,\,Y=0}=0 \tag{9-2-8}$$

$$U(X,\ Y)|_{Y=1}=1 \tag{9-2-9}$$

$$\left.\frac{\partial U(X,\ Y)}{\partial Y}\right|_{Y=1}=0 \tag{9-2-10}$$

$$\frac{\partial^2 U(X,\ Y)}{\partial Y^2}<0 \tag{9-2-11}$$

方法 2　其他比值同方法 1，只将$\frac{y}{l}=Y$代入方程与边界条件

$$\nu_0\left[(1-\varphi^{\frac{2}{3}})\frac{\partial^2 U}{\partial X\partial Y}-(\varphi^{\frac{2}{3}}-1)\frac{1}{U}\frac{\partial U}{\partial X}\frac{\partial U}{\partial Y}\right]=\nu_0\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)+\nu_0 K_1\varphi^{\frac{2}{3}}\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} \tag{9-2-12}$$

$$U(X,\ Y)|_{X=0,\,Y=0}=0 \tag{9-2-13}$$

$$U(X,\ Y)|_{Y=\delta_Y}=1 \tag{9-2-14}$$

9.2.2　确定边界层厚度 $\delta(x)$与速度分布

根据边界条件要求，选择速度分布为

$$U=\eta\,\mathrm{arccot}X(2Y-Y^2) \tag{9-2-15}$$

为确定边界层厚度，将其代入式(9-2-6)，得

$$-2(1-\varphi^{\frac{2}{3}})\frac{1-Y}{1+X^2}\frac{l}{\delta(x)}\eta$$

$$=2\eta(2Y-Y^2)\frac{X}{(1+X^2)^2}-2\eta\,\mathrm{arccot}X\left(\frac{l}{\delta(x)}\right)^2-4K_1\varphi^{\frac{2}{3}}\mathrm{arccot}^2X(1-Y)\eta^2 \tag{9-2-6'}$$

因为要求边界层厚度 $\delta(x)$，故取 $Y=1$，则有

$$\frac{\delta(x)}{l}=(1+X^2)\sqrt{\frac{\mathrm{arccot}X}{X}} \tag{9-2-16}$$

当 $X=0$ 时，出现奇点，去掉奇点，则改为

$$\frac{\delta(x)}{l}=(1+X^2)\sqrt{\frac{\mathrm{arccot}X}{1+X}} \tag{9-2-17}$$

由式(9-2-17)则可以定出边界层厚度 $\delta(x)$。

要确定边界层内的速度分布，除以上知道边界层厚度 $\delta(x)$外，还要确定参变量 η，为此，将式(9-2-15)代入式(9-2-12)，得

$$\frac{1}{2}(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 U}{\partial X\partial Y}+\frac{1}{U}\frac{\partial U}{\partial X}\frac{\partial U}{\partial Y}\right)=-(1-\varphi^{\frac{2}{3}})\left(\frac{1-Y}{1+X^2}\right)\eta \tag{1}$$

$$\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}=2\left[(2Y-Y^2)\frac{X}{(1+X^2)^2}-\operatorname{arccot}X\right]\eta \tag{2}$$

$$K_1\varphi^{\frac{2}{3}}\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}=-4K_1\varphi^{\frac{2}{3}}\operatorname{arccot}^2X(1-Y)\eta^2 \tag{3}$$

将式(1)～式(3)代回式(9-2-12)，则有

$$\eta=\frac{2(2Y-Y^2)\dfrac{X}{(1+X^2)^2}+(1-\varphi^{\frac{2}{3}})\dfrac{1-Y}{1+X^2}-\operatorname{arccot}X}{4K_1\varphi^{\frac{2}{3}}(\operatorname{arccot}X)^2(1-Y)} \tag{9-2-18}$$

将 η 代入式(9-2-15)，则可以确定速度分布；但有一个问题，它不满足边界条件，当 $Y=\delta_Y$ 时，速度为1，即来流速度 u_c。当 $Y=\delta_Y$ 时，参变量 $\eta=\eta_1$，则为

$$\eta_1=\frac{2(2\delta_Y-\delta_Y^2)\dfrac{X}{(1+X^2)^2}+(1-\varphi^{\frac{2}{3}})\dfrac{1-\delta_Y}{1+X^2}-\operatorname{arccot}X}{4K_1\varphi^{\frac{2}{3}}(\operatorname{arccot}X)^2(1-\delta_Y)} \tag{9-2-19}$$

根据边界条件，$Y=\delta_Y$ 时 $U=1$，确定边界层速度分布，将式(9-2-15)写成

$$U=\xi\eta\operatorname{arccot}X(2Y-Y^2) \tag{9-2-15$'$}$$

当 $Y=\delta_Y$ 时，为

$$1=\xi\eta_1\operatorname{arccot}X(2\delta_Y-\delta_Y^2) \tag{9-2-15$''$}$$

解出 ξ，代入式(9-2-15)′，则得

$$\xi=\frac{1}{\eta_1(\operatorname{arccot}X)(2\delta_Y-\delta_Y^2)} \tag{9-2-15$'''$}$$

$$U=\frac{\eta(2Y-Y^2)}{\eta_1(2\delta_Y-\delta_Y^2)} \tag{9-2-20}$$

若将 η 与 η_1 的具体表达式代入式(9-2-20)，则有

$$U=\frac{\left[2(2Y-Y^2)\dfrac{X}{(1+X^2)^2}+(1-\varphi^{\frac{2}{3}})\dfrac{1-Y}{1+X^2}-\operatorname{arccot}X\right](1-Y)(2Y-Y^2)}{\left[2(2\delta_Y-\delta_Y^2)\dfrac{X}{(1+X^2)^2}+(1-\varphi^{\frac{2}{3}})\dfrac{1-\delta_Y}{1+X^2}-\operatorname{arccot}X\right](1-\delta_Y)(2\delta_Y-\delta_Y^2)} \tag{9-2-21}$$

式(9-2-21)满足边界条件要求，而且当 $X=0$，$Y=0$ 时，$U=0$；当 $Y=\delta_Y$ 时，$U=1$，即为来流速度 u_c，而在边界层内即 $Y<\delta_Y$ 时，速度 $u<u_c$。所以它是合理式。

9.2.3 边层流厚度 $Y_B(x)$

首先确定边界层内断面上平均速度($\overline{U}$)，根据式(9-2-20)，可以计算为

$$\overline{U}=\frac{\eta}{\eta_1}\frac{1}{2}\int_0^{\delta_Y}(2Y-Y^2)\mathrm{d}Y=\frac{\eta}{\eta_1}\frac{1}{2}\left(\frac{2Y^2}{2}-\frac{Y^3}{3}\right)\Bigg|_0^{\delta_Y}=\frac{\eta_B}{\eta_1}\frac{\delta_Y^2}{2}\left(1-\frac{\delta_Y}{3}\right) \tag{9-2-22}$$

将式(9-2-22)代回式(9-2-20)，则有

$$\frac{\delta_Y^2}{2}\left(1-\frac{\delta_Y}{3}\right)=2Y_B-Y_B^2 \tag{a}$$

式(a)中，Y_B 就是对应 X 处的边层流厚度，将它解出，得

$$Y_B=1-\sqrt{1-\frac{\delta_Y^2}{2}\left(1-\frac{\delta_Y}{3}\right)} \tag{9-2-23}$$

因为 $\delta_Y=\frac{\delta(x)}{l}$，它已由式(9-2-17)计算，所以边层流厚 Y_B 也是可以计算的。

9.2.4 边层流界面上涡旋强度与速度

$$\omega_{z^-}=\frac{1}{2}\frac{\partial u}{\partial y}=\frac{1}{2}\frac{u_c}{\delta(x)}\frac{\partial U}{\partial Y}=\frac{1}{2}\frac{u_c}{\delta(x)}\frac{\eta}{\eta_1}2(1-Y_B)=\frac{\eta_B}{\eta_1}\frac{u_c}{\delta(x)}(1-Y_B) \tag{9-2-24}$$

涡旋转向是顺时针。

涡旋在与流向垂直方向的速度 u_y^* 为

$$u_y^*=12\nu t(\omega_{z^-})^2v_0^{-1}=12\nu t\left[\frac{\eta_B}{\eta_1}\frac{u_c}{\delta(x)}(1-Y_B)\right]^2\left[\frac{\eta_B}{\eta_1}\frac{\delta_Y^2}{2}\left(1-\frac{\delta_Y}{3}\right)u_c\right]^{-1}$$

$$=\frac{12\nu(\delta(x)-y_B)}{u_y^*}\frac{\frac{\eta_B}{\eta_1}u_c(1-Y_B)^2}{\delta(x)^2\frac{\delta_Y^2}{2}\left(1-\frac{\delta_Y}{3}\right)}=\sqrt{\frac{72\nu(\delta(x)-y_B)u_c\eta_B(1-Y_B)^2}{\eta_1\delta(x)^2\delta_Y^2(1-\delta_Y)}} \tag{9-2-25}$$

涡旋直径

$$d_s=2r_s=2\sqrt{\frac{10\nu}{\omega_{z^-}}} \tag{9-2-26}$$

9.2.5 涡旋体积分数 φ

根据涡旋体积分数 φ 的定义

$$\varphi=\frac{\pi[秒]u_y^*}{48\delta(x)} \tag{9-2-27}$$

9.2.6 公式计算表

公式计算表见表 9-1。

表 9-1

X	$\operatorname{arccot}X$	$\delta_Y=X^2\sqrt{\frac{\operatorname{arccot}X}{1+X}}$	$Y_B=1-\sqrt{1-\frac{\delta_Y^2}{2}\left(1-\frac{\delta_Y}{3}\right)}$
0.2	1.413	0.043	0.0005
0.4	1.257	0.152	0.006
0.6	1.100	0.248	0.029
0.8	0.942	0.463	0.047
1.0	0.795	0.630	0.082

例 9-2　空气温度为 15℃，流过平板板长为 1m，气流速度 80m/s，求 $X=0.8$ 处边界层厚度 $\delta(x)$，边层流厚度 y_B，该断面边界层内平均速度 v_0，边层流界面上涡旋强度 ω_{z^-}，直径 d_s，及其在垂直于壁面方向的速度 u_y^*，最后确定该处涡旋体积分数 φ 为多少？

解　$X=0.8$，即 $\frac{x}{l}=X=0.8$，$l=1\text{m}$，即 $x=0.8\text{m}$ 处。求该断面上边界层厚度 $\delta(x)$，$\frac{y}{l}=Y$，$\frac{\delta(x)}{l}=\delta_Y$，由表 9-1 查得 $X=0.8$ 处，$\delta_Y=0.463$，即 $\delta(x)=0.463\text{m}$。

该断面上边层流厚度 y_B，$\frac{y_B}{l}=Y_B$，由表 9-1 中查得当 $X=0.8$ 处 $Y_B=0.047$，$y_B=0.047l=0.047\text{m}$。

该处边界层内断面上平均速度 v_0 由式(9-2-22)计算：

$$v_0=u_c\frac{\eta_B}{\eta_1}\frac{\delta_Y^2}{2}\left(1-\frac{\delta_Y}{3}\right)\tag{9-2-28}$$

表 9-2

X	$\frac{\eta_B}{\eta_1}=\frac{\left[2(2Y_B-Y_B^2)\frac{X}{(1+X^2)^2}+(1-\varphi^{\frac{2}{3}})\frac{1-Y_B}{1+X^2}-\text{arccot}X\right](1-Y_B)}{\left[2(2\delta_Y-\delta_Y^2)\frac{X}{(1+X^2)^2}+(1-\varphi^{\frac{2}{3}})\frac{1-\delta_Y}{1+X^2}-\text{arccot}X\right](1-\delta_Y)}$
0.2	$\frac{\eta_B}{\eta_1}=\frac{[0.961(1-\varphi^{\frac{2}{3}})-1.4126]0.957}{[0.92(1-\varphi^{\frac{2}{3}})-1.383]0.9995}$
0.4	$\frac{\eta_B}{\eta_1}=\frac{[0.857(1-\varphi^{\frac{2}{3}})-1.24]0.848}{[0.731(1-\varphi^{\frac{2}{3}})-1.143]0.994}$
0.6	$\frac{\eta_B}{\eta_1}=\frac{[0.714(1-\varphi^{\frac{2}{3}})-1.063]0.752}{[0.553(1-\varphi^{\frac{2}{3}})-0.919]0.971}$
0.8	$\frac{\eta_B}{\eta_1}=\frac{[0.581(1-\varphi^{\frac{2}{3}})-0.887]0.537}{[0.327(1-\varphi^{\frac{2}{3}})-0.526]0.953}$
1.0	$\frac{\eta_B}{\eta_1}=\frac{[0.459(1-\varphi^{\frac{2}{3}})-0.64]0.37}{[0.185(1-\varphi^{\frac{2}{3}})-0.363]0.918}$

当 $X=0.8$ 处，$\frac{\eta_B}{\eta_1}$计算公式由表 9-2 查得，为

$$\frac{\eta_B}{\eta_1}=\frac{[0.581(1-\varphi^{\frac{2}{3}})-0.881]0.537}{[0.327(1-\varphi^{\frac{2}{3}})-0.526]0.953}$$

由于涡旋体积分数 φ 不知道，所以只能先假设，现设 $\varphi=0.1$，则

$$\frac{\eta_B}{\eta_1}=\frac{[0.581(1-0.1^{\frac{2}{3}})-0.887]0.537}{[0.327(1-0.1^{\frac{2}{3}})-0.526]0.953}=0.891$$

代入式(9-2-28)，得

$$v_0=u_c\frac{\eta_B}{\eta_1}\frac{\delta_Y^2}{2}\left(1-\frac{\delta_Y}{3}\right)=80\times0.891\times\frac{0.463^2}{2}\left(1-\frac{0.463}{3}\right)$$

$$=80\times0.891\times0.107\times0.846=6.452\text{m/s}$$

涡旋强度 ω_{z^-}：由式(9-2-24)，得

$$\omega_{z^-}=\frac{\eta_B}{\eta_1}\frac{u_c}{\delta(x)}(1-Y_B)=0.891\times\frac{80}{0.463}(1-0.047)$$

$$=0.891\times172.786\times0.953=146.71\text{rad/s}$$

涡旋直径 d_s：当温度为 15℃时，$\nu=14.66\times10^{-6}\text{m}^2/\text{s}$。从而

$$d_s=2r_s=2\sqrt{\frac{10\nu}{\omega_{z^-}}}=2\sqrt{\frac{10\times14.66\times10^{-6}}{146.71}}=\frac{2}{100}\sqrt{\frac{14.66}{1467.1}}$$

$$=\frac{2}{100}\sqrt{0.00999}=\frac{2\times0.0996}{100}=\frac{0.1999}{100}\approx\frac{0.2}{100}=0.002\text{m}=2\text{mm}$$

涡旋在垂直壁面方向分速度 u_y^*：由式(9-2-25)，得

$$u_y^*=\sqrt{\frac{72\nu(\delta(x)-y_B)u_c\eta_B(1-Y_B)^2}{\delta(x)^2\delta_Y^2(1-\delta_Y)\eta_1}}$$

$$=\sqrt{\frac{72\times14.66\times10^{-6}(0.463-0.047)80\times0.891(1-0.047)^2}{0.463^2\times0.463^2(1-0.463)}}$$

$$=\sqrt{\frac{0.001055\times0.416\times80\times0.891\times0.908}{0.046\times0.537}}=\sqrt{\frac{0.000439\times80\times0.809}{0.0247}}$$

$$=\sqrt{\frac{0.0284}{0.0247}}=\sqrt{1.150}=1.073\text{m/s}$$

该点处涡旋体积分数 φ：由式(9-2-27)，得

$$\varphi=\frac{[秒]\pi u_y^*}{48\delta(x)}=\frac{[秒]\pi\times1.073}{48\times0.463}=\frac{3.371}{22.224}=0.152$$

得到的 φ 与假设的 $\varphi=0.1$ 相差很大，故重新假设 $\varphi=0.15$，重新计算。

$$\frac{\eta_B}{\eta_1}=\frac{[0.581(1-0.15^{\frac{2}{3}})-0.881]0.537}{[0.327(1-0.15^{\frac{2}{3}})-0.526]0.953}=\frac{-0.1681}{-0.19669}=0.8546$$

$$v_0=u_c\frac{\eta_B}{\eta_1}\left(1-\frac{\delta_Y}{3}\right)\frac{\delta_Y^2}{2}=80\times0.8546\times0.846\times0.107=0.8546\times7.242=6.189\text{m/s}$$

$$\omega_{z^-}=\frac{\eta_B}{\eta_1}\frac{u_c}{\delta(x)}(1-Y_B)=0.854\times\frac{80}{0.463}(1-0.047)=0.8546\times172.786\times0.953$$

$$=0.8546\times164.665=140.723\text{rad/s}$$

$$u_y^*=\sqrt{\frac{72\nu(\delta(x)-y_B)u_c\eta_B(1-Y_B)^2}{\delta(x)^2\delta_Y^2(1-\delta_Y)\eta_1}}$$

$$=\sqrt{\frac{72\times14.66\times10^{-6}(0.463-0.047)80(1-0.047)^2\times0.8546}{0.463^2\times0.463^2(1-0.463)}}$$

$$=\sqrt{\frac{0.001055\times0.416\times80\times0.908\times0.8540}{0.046\times0.537}}=\sqrt{\frac{0.03188\times0.8546}{0.0247}}$$

$$=\sqrt{\frac{0.272}{0.0247}}=\sqrt{1.103}=1.05\text{m/s}$$

$$\varphi=\frac{[秒]\pi u_{y}^{*}}{48\delta(x)}=\frac{\pi\times1.05}{48\times0.463}=\frac{3.2987}{22.224}=0.1484$$

假设 $\varphi=0.15$，结果计算得 $\varphi=0.1484\approx0.15$，可以取 $\varphi=0.15$。

例 9-3　题目同例 9-2，要求绘出 $X=0.8$ 处边界层内断面速度分布曲线。

解　$l=1\text{m}$，故 $\delta(x)=\delta_Y$，由式(9-2-21)，得

$$U=\frac{\left[2(2Y-Y^2)\frac{X}{(1+X^2)^2}+(1-\varphi^{\frac{2}{3}})\frac{1-Y}{1+X^2}-\operatorname{arccot}X\right](1-Y)(2Y-Y^2)}{\left[2(2\delta_Y-\delta_Y^2)\frac{X}{(1+X^2)^2}+(1-\varphi^{\frac{2}{3}})\frac{1-\delta_Y}{1+X^2}-\operatorname{arccot}X\right](1-\delta_Y)(2\delta_Y-\delta_Y^2)}$$

$$=\frac{\left[2(2Y-Y^2)\frac{0.8}{(1+0.8^2)^2}+(1-0.15^{\frac{2}{3}})\frac{1-Y}{1+0.8^2}-\operatorname{arccot}0.8\right](1-Y)(2Y-Y^2)}{\left[2(2\times0.463-0.463^2)\frac{0.8}{(1+0.8^2)^2}+(1-0.15^{\frac{2}{3}})\frac{1-0.463}{1+0.8^2}-\operatorname{arccot}0.8\right](1-0.463)(2\times0.463-0.463^2)}$$

$$=\frac{\left[2(2Y-Y^2)0.276+0.85\,\frac{1-Y}{1.64}-0.942\right](1-Y)(2Y-Y^2)}{-0.237}\tag{Ⅰ}$$

设 $Y=0.4$ 时

$$U=\frac{\left[2(2\times0.8-0.16)0.276+0.85\times\frac{1-0.4}{1.64}-0.942\right](1-0.4)(2\times0.4-0.4^2)}{-0.237}$$

$$=\frac{-0.107}{-0.237}=0.45$$

换算速度 u

$$\frac{80}{0.463}=\frac{u}{0.45},\qquad u=77.754\text{m/s}$$

同理，当 $Y=0.3$，$U=0.44$

$$\frac{80}{0.463}=\frac{u}{0.44},\qquad u=76.026\text{m/s}$$

同理，当 $Y=0.2$ 时，$U=0.4$，$u=69.114\text{m/s}$。当 $Y=0.1$ 时，$U=0.268$，$u=46.307\text{m/s}$。将结果绘出曲线，如图 9-7 所示。

9.3　不可压缩湍流有压力变化平板边界层

问题发生在类似图 9-2 设备上，当管嘴长度比较短时，流体运动尚未达到充分发展；不过现在是湍流边界层，对此处理方法与层流情况类似，只是运用湍流运动微分方程。

9.3.1　运动控制方程与边界条件

湍流边界层内流体运动受湍流运动微分方程控制，其外是理想流体运动区，流体运动受理想流体运动微分方程控制。

(1) 边界层内流体运动区

依式(4-4-11)并结合本问题，可得其运动控制方程：

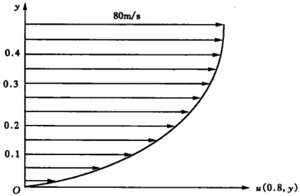


图 9-7 当 $X=0.8$ 处边界层内速度分布曲线

$$\frac{\nu}{2}\left[(1-\varphi^{\frac{2}{3}})\frac{\partial^2 u}{\partial y\partial x}-(\varphi^{\frac{2}{3}}-1)\frac{1}{u}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right]=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+6\varphi^{\frac{2}{3}}\nu t\ \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} \tag{9-3-1}$$

$$u(x,\ y)\big|_{x=0,\,y=0}=0 \tag{9-3-2}$$

$$u(x,\ y)\big|_{y=\delta(x)}=u_0(x) \tag{9-3-3}$$

$$\left.\frac{\partial u(x,\ y)}{\partial y}\right|_{y=\delta(x)}=0 \tag{9-3-4}$$

$$\frac{\partial^2 u(x,\ y)}{\partial y^2}<0 \tag{9-3-5}$$

(2) 理想流体运动区

依流体力学可知

$$u_0(x)\frac{\mathrm{d}u_0(x)}{\mathrm{d}x}=-\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{9-3-6}$$

$$u_0(x)\big|_{x=0,\,y=0}=u_c \tag{9-3-7}$$

压力的变化只与 x 有关，它与理想和实际流体运动区无关，所以它的边界条件为

$$p(x,\ y)\big|_{x=0,\,y=0}=p_c \tag{9-3-8}$$

式中，u_c，p_c——进口处流体压力与速度；

$u_0(x)$——理想流体运动区流体运动速度。

9.3.2 运动控制方程与边界条件无因次化

(1) 边界层内流体运动区

取

$$\frac{u(x,\ y)}{u_c}=U,\quad \frac{x}{l}=X,\quad \frac{y}{\delta(x)}=Y,\quad \frac{\nu}{\nu_c}=\nu_0,\quad \frac{p}{p_c}=p_0$$

将比值代入方程与边界条件中，则有

$$\frac{\nu_0}{2}(1-\varphi^{\frac{2}{3}})\frac{l}{\delta(x)}\left[\frac{\nu_c u_c}{l^2}\frac{\partial^2 U}{\partial Y \partial X}+\frac{\nu_c u_c}{l^2}\frac{1}{U}\frac{\partial U}{\partial X}\frac{\partial U}{\partial Y}\right]$$

$$=-\frac{p_c}{\rho_c l p_0}\frac{\mathrm{d}p}{\mathrm{d}X}+\nu_0\left[\frac{\nu_c u_c}{l^2}\frac{\partial^2 U}{\partial X^2}+\frac{l^2}{\delta(x)^2}\frac{\nu_c u_c}{l^2}\frac{\partial^2 U}{\partial Y^2}\right]+\frac{l^3}{\delta(x)^3}6\varphi^{\frac{2}{3}}\frac{\nu_c u_c^2}{l^3}t\,\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} \tag{9-3-1}'$$

全式除以$\frac{\nu_c u_c}{l^2}$，则有

$$\frac{\nu_0}{2}(1-\varphi^{\frac{2}{3}})\frac{l}{\delta(x)}\left(\frac{\partial^2 U}{\partial Y \partial X}+\frac{1}{U}\frac{\partial U}{\partial X}\frac{\partial U}{\partial Y}\right)=-K_1\frac{\mathrm{d}p_0}{\mathrm{d}X}+\nu_0\left[\left(\frac{\partial^2 U}{\partial X^2}+\frac{l^2}{\delta(x)^2}\frac{\partial^2 U}{\partial Y^2}\right)+\varphi^{\frac{2}{3}}K_2\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right] \tag{9-3-9}$$

式中：

$$K_1=\frac{p_c l}{\rho_c \nu_c u_c} \tag{9-3-10}$$

$$K_2=\frac{6tu_c}{\delta(x)} \tag{9-3-11}$$

$$U(X,\ Y)\big|_{X=0,\,Y=0}=0 \tag{9-3-12}$$

$$U(X,\ Y)\big|_{Y=\delta_Y}=U_0(X) \tag{9-3-13}$$

$$\left.\frac{\partial U(X,\ Y)}{\partial Y}\right|_{Y=\delta_Y}=0 \tag{9-3-14}$$

$$\frac{\partial^2 U(X,\ Y)}{\partial Y^2}<0 \tag{9-3-15}$$

(2) 理想流体运动区

取$\frac{u_0(x)}{u_c}=U_0(X)$，将比值代入式(9-3-6)与式(9-3-7)，有

$$\frac{u_c^2 U_0(X)}{l}\frac{\mathrm{d}U_0(X)}{\mathrm{d}X}=-\frac{p_c}{\rho_c\rho_0 l}\frac{\mathrm{d}p_0}{\mathrm{d}X} \tag{9-3-6}'$$

全式除以$\frac{u_c^2}{l}$，则有

$$U_0(X)\frac{\mathrm{d}U_0(X)}{\mathrm{d}X}=-K_3\frac{\mathrm{d}p_0}{\rho_0\mathrm{d}X} \tag{9-3-16}$$

式中：

$$K_3=\frac{p_c}{\rho_c u_c^2} \tag{9-3-17}$$

$$U_0(X)\big|_{X=0}=1 \tag{9-3-18}$$

9.3.3　建立流量守恒方程

因是不可压缩收缩管嘴，流量沿各断面是相等的。依此可以建立起理想流体运动区速度

分布与实际流体运动速度分布之间的关系式。

由于边界层厚度是倾斜壁面变化的，而断面变化是水平方向变化的。为把两者结合起来，将采用两个坐标系，如图 9-8 所示。

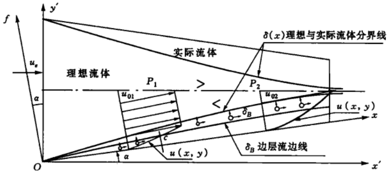


图 9-8　建立流量守恒方程用图

进口断面流量：

$$Q_c = \pi r_c^2 u_c \tag{1}$$

通过理想流体运动区断面流量：

$$Q_{理} = \pi(r_c - x'\tan\alpha - \delta'(x))^2 u(x) \tag{2}$$

通过实际流体运动区断面上流量：

$$Q_{实} = 2\pi(r_c - \tan\alpha)\delta(x)\,\frac{1}{2}\int_0^{\delta'(x)} u(x,y)\mathrm{d}y' \tag{3}$$

根据流量守恒原则，则有

$$\pi r_c^2 u_c = \pi(r_c - x'\tan\alpha - \delta'(x))^2 u(x) + \pi(r_c - x'\tan\alpha - \delta'(x))\int_0^{\delta'(x)} u(x,y)\mathrm{d}y' \tag{4}$$

将式(4)中$(r_c - x'\tan\alpha - \delta'(x))^2$ 展开

$$(r_c - x'\tan\alpha - \delta'(x))^2 = r_c^2 + x'^2\tan^2\alpha + \delta^2(x) - 2r_c x'\tan\alpha - 2r_c\delta'(x) + 2x'\delta'(x)\tan\alpha \tag{5}$$

因为 $x' = x\cos\alpha$，$\delta(x) = \delta'(x)\cos\alpha$，$y\cos\alpha = y'$，将它与式(5)均代回式(4)

$$\pi r_c^2 u_c = \pi[r_c^2 + x^2\sin^2\alpha + \sec^2\alpha\delta(x)^2 - 2r_c x\sin\alpha - 2r_c\delta'(x)\sec\alpha + 2x\delta(x)\tan\alpha]u(x) + \pi(r_c - x\sin\alpha)\delta(x)\int_0^{\delta(x)} u(x,y)\mathrm{d}y \tag{6}$$

将式(6)无因次化，全式除以 $\pi l^2 u_c$，而$\frac{\delta(x)}{l} = \delta_Y$，$\frac{r_c}{l} = R$，则有

$$R^2 = (R^2 + X^2\sin^2\alpha + \sec^2\alpha\delta_Y^2 - 2RX\sin\alpha - 2R\delta_Y\sec\alpha + 2X\delta_Y\tan\alpha)U(X) + (R - X\sin\alpha)\delta_Y\int_0^{\delta_Y} U(X,Y)\mathrm{d}Y \tag{9-3-19}$$

9.3.4　确定速度分布

这里说确定速度分布，就是确定理想流体运动区速度分布、实际流体运动区速度分布。首先根据边界条件确定实际流区速度分布为

$$U=\eta X^{\frac{1}{2}}(2Y-Y^2) \tag{9-3-20}$$

式中，η 是待定参变量。

将式(9-3-20)代入式(9-3-9)，则有

$$\nu_0(1-\varphi^{\frac{2}{3}})\frac{l}{\delta(x)}\eta(1-Y)X^{\frac{1}{2}}=ReU_0(X)\frac{\mathrm{d}U_0(X)}{\mathrm{d}X}-\nu\eta\left[\frac{1}{4}(2Y-Y^2)X^{-\frac{3}{2}}\right]$$

$$-4\eta^2K_2\varphi^{\frac{2}{3}}\left(\frac{l}{\delta(x)}\right)^3(1-Y)X \tag{9-3-9$'$}$$

通过式(9-3-9)′，求理想流体运动区速度分布，为此令 $Y=1$，则有

$$Re\frac{\mathrm{d}U_0(X)^2}{\mathrm{d}X}=\frac{\nu_0\eta}{4}X^{-\frac{3}{2}}+2\left(\frac{l}{\delta(x)}\right)^2\eta X^{\frac{1}{2}} \tag{9-3-9$''$}$$

$$U_0^2(X)=\frac{1}{Re}\left[-\frac{\nu_0}{2}\eta X^{-\frac{1}{2}}+\frac{4}{3}\left(\frac{l}{\delta(x)}\right)^2\eta X^{\frac{3}{2}}\right]+C \tag{a}$$

去掉当 $X=0$ 速度造成无穷大项，则式(a)为

$$U_0^2(X)=\frac{4}{3Re}\left(\frac{l}{\delta(x)}\right)^2\eta X^{\frac{3}{2}}+C \tag{b}$$

当 $X=0$ 时，$U_0(X)=1$，则 $C=1$，从而

$$U_0(X)=\sqrt{\frac{4}{3Re}\left(\frac{l}{\delta(x)}\right)^2\eta X^{\frac{3}{2}}+1} \tag{9-3-21}$$

实际流体运动速度 $U(X,\ Y)$，理想流体运动速度 $U_0(X)$，均如式(9-3-20)与式(9-3-21)表示出来；但要应用还必须知道 η 与 $\delta(x)$，下面就解决这个问题。

9.3.5　理想流体与实际流体运动分界线 $\delta(x)$

利用动能活动微量 E_v 与损失能量活动微量相平衡的原则，求 $\delta(x)$ 的表达式。依图9-9建立如下方程式。

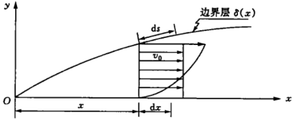


图 9-9　能量运动微量平衡示意图

$$\rho AxU_0^2\mathrm{d}s=\frac{\lambda}{4}\rho Ax\ \frac{x}{r_0}v_0\mathrm{d}x \tag{c}$$

$$\sqrt{1+y'^2}=\frac{\lambda}{4}\frac{x}{r_0} \tag{d}$$

将上式无因次化：

$$\sqrt{1+Y'^2}=\frac{\lambda}{4}\frac{x}{l}\frac{l}{r_0}=\frac{\lambda}{4}\frac{l}{r_0}X \tag{e}$$

$$Y'^2=\left(\frac{\lambda}{4}\right)^2\left(\frac{l}{r_0}\right)^2X^2-1 \tag{f}$$

$$Y'=\sqrt{\left(\frac{\lambda}{4}\right)^2\left(\frac{l}{r_0}\right)^2X^2-1} \tag{g}$$

对式(g)分析，当 $X=0$ 时，$Y'=0$，所以

$$Y'=\left(\frac{\lambda}{4}\right)^2\left(\frac{l}{r_0}\right)X \tag{h}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}X}=\frac{\lambda}{4}\frac{l}{r_0}X \tag{i}$$

积分，得

$$Y=\frac{\lambda}{8}\frac{l}{r_0}X^2+C \tag{j}$$

当 $X=0$ 时，$Y=\delta_Y=\frac{\delta(x)}{l}=0$，故 $C=0$，从而

$$Y=\delta_Y=\frac{\delta(x)}{l}=\frac{\lambda}{8}\frac{l}{r_0}X^2 \tag{9-3-22}$$

9.3.6 确定参变量 η

首先将式(9-3-19)依变量略小者去掉，有

$$R^2=[R^2+2\delta_Y(X\tan\alpha-R\sec\alpha)-2RX\sin\alpha]U_0(X)+(R-X\sin\alpha)\delta_Y\int_0^1U(X,Y)\mathrm{d}Y \tag{9-3-19$'$}$$

将式(9-3-21)、式(9-3-20)、式(9-3-22)代入式(9-3-19)′中，有

$$\begin{aligned}R^2=&\left[R^2+\frac{\lambda}{4}\frac{l}{r_0}X^2(X\tan\alpha-R\sec\alpha)-2RX\sin\alpha\right]\sqrt{\frac{4}{3Re}\left(\frac{l}{\delta(x)}\right)^2\eta X^{\frac{3}{2}}+1}\\&+(R-X\sin\alpha)\frac{\lambda}{8}\frac{l}{r_0}X^2X^{\frac{1}{2}}\eta\int_0^1(2Y-Y^2)\mathrm{d}Y\end{aligned} \tag{9-3-19$''$}$$

整理为

$$R^2=\left[R^2+\frac{\lambda}{4}\frac{l}{r_0}X^2(X\tan\alpha-R\sec\alpha)-2RX\sin\alpha\right]\sqrt{\frac{4l^2\eta X^{\frac{3}{2}}}{3Re\delta(x)^2}+1}+\frac{\lambda}{12}(R-X\sin\alpha)\frac{l}{r_0}X^{\frac{5}{2}}\eta \tag{9-3-23}$$

由上式可知，参变量 η 是 X 的隐函数，不好计算，应采取在设定 X 后，再设不同 η 值，以绘曲线与常数 R^2 相交的办法解决。

9.3.7 理想与实际流体速度对接

理想流体运动速度分布，由式(9-2-21)计算；实际流体运动速度分布，由式(9-3-20)计算；理想与实际流体分界线 $\delta(x)$，由式(9-3-22)计算；参变量 η，由式(9-3-23)计算。但是有一个问题还没有得到解决，就是理想流体速度与实际流体运动速度，在分界线上应该相等。也就是说，边界条件式(9-3-13)还没有用上。

因为理想流体运动控制微分方程与实际流体运动控制方程是不一样的，由各自导出的速度分布是不同的。要想实现它们在边界线上相等，必须利用式(9-3-13)的条件。

当在理想与实际流体运动分界线 $\delta(x)$上时，则实际流体运动速度分布式(9-3-20)应为

$$U_0(X)=\xi\eta X^{\frac{1}{2}}(2\delta_Y-\delta_Y^2) \tag{9-3-24}$$

式中，ξ 是衔接系数。为确定它，将式(9-3-20)代入式(9-3-24)，得

$$\sqrt{\frac{4l^2\eta X^{\frac{3}{2}}}{3Re\delta(x)^2}+1}=\xi\eta X^{\frac{1}{2}}(2\delta_Y-\delta_Y^2) \tag{9-3-25}$$

$$\xi=\sqrt{\frac{4l^2\eta X^{\frac{3}{2}}}{3Re\delta(x)^2}+1}\left[\frac{1}{\eta X^{\frac{1}{2}}(2\delta_Y-\delta_Y^2)}\right] \tag{9-3-26}$$

将式(9-3-26)代入式(9-3-20)

$$U(X,\ Y)=\sqrt{\frac{4l^2\eta X^{\frac{3}{2}}}{3Re\delta(x)^2}+1}\left[\frac{2Y-Y^2}{2\delta_Y-\delta_Y^2}\right] \tag{9-3-27}$$

式(9-3-27)是边界层内速度分布具体计算公式。

在不可压缩湍流管嘴分析过程中，首先是以实际流体运动控制方程为主导，而将理想流体速度 $U_0(X)$作为参变量引入实际流体运动控制方程，利用流量守恒原则，首先导出它的具体计算公式(9-3-23)，最后利用式(9-3-13)边界条件，用实际流体运动速度分布找到式(9-3-27)。

例 9-4 设有一收缩形管嘴，如图 9-10 所示，$l'=0.8\text{m}$，收缩角 $\alpha=3°$，进口处半径 $r_c=0.2\text{m}$，速度 $u_c=5\text{m/s}$，水运动黏性系数 $\nu=1.141\times10^{-6}\text{m}^2/\text{s}$，确定管嘴长之半处向上速度分布。($\lambda=0.02$)

解 准备工作：

$l=l'(1+\tan\alpha)=0.8(1+\tan3°)=0.8(1+0.05241)=0.8\times1.05241=0.8419$

$\tan\alpha=\tan3°=0.05241$

$\sin\alpha=\sin3°=0.05234$

$\sec\alpha=\dfrac{1}{\cos\alpha}=\dfrac{1}{\cos3°}=\dfrac{1}{0.99861}=1.0014$

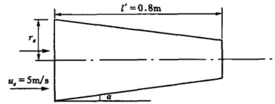


图 9-10　管嘴

$$R = \frac{r_c}{l} = \frac{0.2}{0.8419} = 0.2376$$

$$r_0 = r_c - x\tan\alpha = 0.2 - 0.4\times0.05241 = 0.2 - 0.01 = 0.19$$

$$Re = \frac{u_c l}{\nu_c} = \frac{5\times0.8419}{1.141}\times10^6 = 3.6\times10^6$$

计算理想与实际流体运动分界线 $\delta(x)$，依式(9-3-22)($X=0.5$ 处)

$$\delta(x) = \frac{\lambda}{8}\frac{l^2}{r_0}X^2 = \frac{0.02}{8}\times\frac{0.8419^2}{0.19}\times0.5^2 = 0.0025\times\frac{0.7088}{0.19}\times0.25$$

$$= 0.0025\times3.705\times0.25 = 0.0023\text{m}$$

计算 $X=0.5$ 处理想流体速度 $U_0(X)$，依式(9-3-21)

$$U_0(X) = \sqrt{\frac{4l^2\eta X^{\frac{3}{2}}}{3Re\delta(x)^2}+1} = \sqrt{\frac{4\times0.8419^2\times0.5^{1.5}\eta}{3\times3.6\times10^6\times0.0023^2}+1} = \sqrt{0.0118\eta+1} \tag{a}$$

式中含有参变量 η，还不能定量计算。

计算参变量 η，依式(9-3-23)

$$R^2 = \left[R^2 + \frac{\lambda}{4}\frac{l}{r_0}X^2(X\tan\alpha - R\sec\alpha) - 2RX\sin\alpha\right]\sqrt{0.0118\eta+1} + \frac{\lambda}{12}(R - X\sin\alpha)\frac{l}{r_0}X^{\frac{5}{2}}\eta \tag{9-3-23}$$

$$0.2376^2 = \left[0.2376^2 + \frac{0.02}{4}\times\frac{0.8419}{0.19}\times0.5^2(0.5\times0.05241 - 0.5\times1.0014) - 2\times0.5\times0.2376\right.$$
$$\left.\times0.05234\right]\sqrt{0.0118\eta+1} + \frac{0.02}{12}(0.2376 - 0.5\times0.05234)\frac{0.8419}{0.19}\times0.5^{2.5}\eta \tag{b}$$

$$0.0565 = [0.0565 + 0.005\times4.4311\times0.25(0.0262 - 0.5007) - 0.0124]\sqrt{0.0118\eta+1} +$$
$$0.0017(0.2376 - 0.0262)4.4311\times0.1768\eta \tag{b)$'$(}$$

$$0.0565 = [0.0565 + 0.0055(-0.4745) - 0.0124]\sqrt{0.0118\eta+1} + 0.0017\times0.2114\times$$
$$4.4311\times0.1768\eta \tag{b)$''$(}$$

$$0.0565 = (0.0565 - 0.015)\sqrt{0.0118\eta+1} + 0.00028\eta \tag{b)$'''$(}$$

$$0.0565 = 0.0415\sqrt{0.0118\eta+1} + 0.00028\eta \tag{b)$''''$(}$$

近似处理：

$$\frac{0.0565}{0.0415} = \sqrt{0.0118\eta+1} \tag{c}$$

$$\frac{1.8535-1}{0.0118}=\eta \tag{c}'$$

$$\eta=\frac{0.8535}{0.0118}=72.3 \tag{c}''$$

计算 $X=0.5$ 处理想流体速度：将式(c)″代入式(a)

$$U_0(X)\big|_{X=0.5}=\sqrt{0.0118\eta+1}=\sqrt{0.0118\times 72.3+1}=1.3546$$

$$u(x)=u_c\times 1.3546=5\times 1.3546=6.773\text{m/s}$$

计算 $X=0.5$ 处边界层速度分布：由式(9-3-27)

$$U(X,\ Y)\big|_{X=0.5}=1.3456\times\frac{2Y-Y^2}{2\delta_Y-\delta_Y^2} \tag{d}$$

已知

$$\delta_Y=\frac{\lambda}{8}\frac{l}{r_0}X^2=\frac{0.02}{8}\times\frac{0.8419}{0.19}\times 0.5^2=0.0025\times 4.431\times 0.25=0.0028 \tag{e}$$

$$u(x,\ y)\big|=u_c 1.3546\times\frac{2Y-Y^2}{2\times 0.0028-0.0028^2} \tag{f}$$

近似处理：

$$u(x,\ y)\big|_{\frac{l}{2}}=6.773\frac{Y}{\delta_Y} \tag{g}$$

设不同 Y，计算结果如下：

Y	0.0028	0.0025	0.002	0.0015	0.001
$u(x,\ y)$	6.773	6.054	4.838	3.628	2.419

9.4　可压缩湍流无压力变化平板边界层

由于可压缩流体密度 ρ 随速度与温度变化，所以，研究可压缩湍流无压力变化平板边界层、流体运动控制方程，除动量方程外，还有能量方程和气体状态方程。

9.4.1　运动控制方程与边界条件

由于要运用数学分析，所以，必须对所研究对象，选定坐标系与原点，如图 9-11 所示。

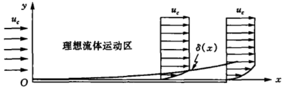


图 9-11　坐标系与原点选定示意图

(1) 动量方程

依式(4-5-16)，忽略分子能量和涡旋温度变化，结合本题得

$$\frac{\nu}{2}\rho\frac{\partial\rho}{\partial x}\frac{\partial u}{\partial y}+u\frac{\partial u}{\partial x}+\frac{\nu}{2}\left(\frac{1}{u}\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial x}=\nu\left[\frac{4}{3}\frac{\partial^2 u}{\partial x^2}+(1-\varphi^{\frac{2}{3}})\frac{\partial^2 u}{\partial y^2}+6\varphi^{\frac{2}{3}}t\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right] \tag{9-4-1}$$

(2) 能量方程

依式(4-5-41)，忽略分子能量和涡旋温度变化，结合本题，为

$$u^2\frac{\partial u}{\partial x}-\frac{u^3}{2}\frac{\partial\rho}{\rho\partial x}+\varphi\frac{\nu}{2}u\frac{\partial^2 u}{\partial x\partial y}+\frac{\nu}{2}\frac{\partial u}{\partial y}\left(\frac{u}{\rho}\frac{\partial\rho}{\partial x}+\frac{\partial u}{\partial x}\right)$$
$$=\frac{\lambda}{\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+u\left(\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}\right)-\frac{1}{2}\varphi^{\frac{2}{3}}\nu\frac{\partial u}{u\partial y}\left(\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}+2\nu t\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right) \tag{9-4-2}$$

(3) 连续性方程

$$\rho_c u_c=\rho u=\text{常数} \tag{9-4-3}$$

$$\frac{\partial\rho}{\rho\partial x}=-\frac{1}{u}\frac{\partial u}{\partial x} \tag{9-4-4}$$

(4) 气体状态方程

$$\frac{p}{\rho}=RT,\quad p=R\rho T$$

$\frac{p}{R}=\rho T$，因压力为常数，则

$$\frac{1}{T}\frac{\partial T}{\partial x}=-\frac{1}{\rho}\frac{\partial\rho}{\partial x} \tag{9-4-5}$$

(5)边界条件

$$u(x,\ y)\big|_{\substack{x=0\\y=0}}=0 \tag{9-4-6}$$

$$u(x,\ y)\big|_{y=\delta(x)}=u_c(x) \tag{9-4-7}$$

$$\left.\frac{\partial u(x,\ y)}{\partial y}\right|_{y=\delta(x)}=0 \tag{9-4-8}$$

$$\frac{\partial^2 u(x,\ y)}{\partial y^2}<0 \tag{9-4-9}$$

当来流温度 T_c 比平板边壁 T_w 高时，则边界条件为

$$T(x,\ y)\big|_{x=0,y=0}=T_c \tag{9-4-10}$$

$$T(x,\ y)\big|_{y=\delta_T(x)}=T_c \tag{9-4-11}$$

$$\left.\frac{\partial T(x,\ y)}{\partial y}\right|_{y=\delta_T(x)}=0 \tag{9-4-12}$$

$$\frac{\partial^2 T(x,\ y)}{\partial y^2}<0 \tag{9-4-13}$$

$$T(x,\ y)\big|_{y=0}=T_w \tag{9-4-14}$$

$\delta_T(x)$是理想流体运动区与实际流体运动区从温度角度划分的分界线。

9.4.2 控制方程与边界条件无因次化

(1) 无因次质量守恒

取

$$\frac{\rho}{\rho_c}=\rho_0,\ \frac{u}{u_c}=U,\ \frac{x}{l}=X$$

将比值代入式(9-4-4)

$$\frac{\partial\rho_0}{\rho_0\partial X}=-\frac{1}{U}\frac{\partial U}{\partial X} \tag{9-4-15}$$

(2) 气体状态方程无因次化

取

$$\frac{T}{T_c}=T_0$$

其他比值不变，将它们代入式(9-4-5)

$$\frac{\partial T_0}{T_0\partial X}=-\frac{\partial\rho_0}{\rho_0\partial X} \tag{9-4-16}$$

(3) 动量方程无因次化

取

$$\frac{y}{\delta(x)}=Y,\quad \frac{\nu}{\nu_c}=\nu_0$$

其他比值不变，将其代入式(9-4-1)，$\delta(x)$是理想流体运动区与实际流体运动区分界线。

$$\frac{\nu_c u_c}{l^2}\frac{l}{\delta(x)}\frac{\nu_0}{2}\frac{\partial\rho_0}{\rho_0\partial X}\frac{\partial U}{\partial Y}+\frac{u_c^2}{l}U\frac{\partial U}{\partial X}+\frac{\nu_c u_c}{l^2}\frac{\nu_0}{2}\frac{l}{\delta(x)}\frac{1}{U}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}$$

$$=\frac{\nu_c u_c}{l^2}\nu_0\left[\frac{4}{3}\frac{\partial^2 U}{\partial X^2}+(1-\varphi^{\frac{2}{3}})\frac{l}{\delta(x)}\frac{\partial^2 U}{\partial Y^2}\right]+\frac{6\nu_c t u_c^2}{l^3}\varphi^{\frac{2}{3}}\nu_0\left(\frac{l}{\delta(x)}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} \tag{9-4-1$'$}$$

全式除以$\frac{\nu_c u_c}{l^2}$，则为

$$\frac{\nu_0}{2}\frac{l}{\delta(x)}\frac{\partial\rho_0}{\rho_0\partial X}\frac{\partial U}{\partial Y}+ReU\frac{\partial U}{\partial X}+\frac{\nu_0}{2}\frac{l}{\delta(x)}\frac{1}{U}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}$$

$$=\nu_0\left[\frac{4}{3}\frac{\partial^2 U}{\partial X^2}+(1-\varphi^{\frac{2}{3}})\left(\frac{l}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial Y^2}\right]+K_1\nu_0\left(\frac{l}{\delta(x)}\right)^2\varphi^{\frac{2}{3}}\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} \tag{9-4-1$''$}$$

将式(9-4-15)代入式(9-4-1)″，则为

$$ReU\frac{\partial U}{\partial X}=\nu_0\left[\frac{4}{3}\frac{\partial^2 U}{\partial X^2}+(1-\varphi^{\frac{2}{3}})\left(\frac{l}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial Y^2}\right]+K_1\nu_0\varphi^{\frac{2}{3}}\left(\frac{l}{\delta(x)}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} \tag{9-4-17}$$

式中：

$$Re=\frac{u_c l}{\nu_c} \tag{9-4-18}$$

$$K_1=\frac{6u_c t}{l} \tag{9-4-19}$$

$$U(X,\ Y)\big|_{X=0,\,Y=0}=0 \tag{9-4-20}$$

$$U(X,\ Y)\big|_{Y=1}=1 \tag{9-4-21}$$

$$\frac{\partial U(X,\ Y)}{\partial Y}\bigg|_{Y=1}=0 \tag{9-4-22}$$

$$\frac{\partial^2 U(X,\ Y)}{\partial Y^2}<0 \tag{9-4-23}$$

(4) 能量方程无因次化

取

$$\frac{y}{\delta_T(x)}=Y$$

$\delta_T(x)$是理想流体温度区与温度边界层分界线，或者说它是温度边界层厚度。其他比值同前，代入式(9-4-2)，则有

$$\frac{u_c^3}{l}U^2\frac{\partial U}{\partial X}-\frac{u_c^3}{2l}\frac{U^3}{\rho_0}\frac{\partial\rho_0}{\partial X}+\varphi\frac{\nu_c}{2}\frac{u_c^2}{l^2}\nu_0 U\frac{\partial^2 U}{\partial X\partial Y}\frac{l}{\delta_T(x)}+\frac{\nu_0}{2}\frac{l}{\delta_T(x)}\frac{\partial U}{\partial Y}\left(\frac{u_c}{l^2}\frac{u_c^2}{\rho_0}\frac{\partial\rho_0}{\partial X}+\frac{\nu_0 u_c^2}{l^2}\frac{\partial U}{\partial X}\right)$$
$$=\frac{\lambda_c T_c}{\rho_c l^2}\frac{\lambda_0}{\rho_0}\left[\frac{\partial^2 T_0}{\partial X^2}+\left(\frac{l}{\delta(x)}\right)^2\frac{\partial^2 T_0}{\partial Y^2}\right]+\frac{u_c^2\nu_c}{l^2}\frac{4}{3}\nu_0 U\frac{\partial^2 U}{\partial X^2}-\frac{1}{2}\varphi^{\frac{2}{3}}\nu_0\frac{\partial U}{\partial Y}\frac{l}{\delta_T(x)}\cdot$$
$$\left[\frac{4}{3}\nu_0\frac{\nu_c^2 u_c}{l^3}\frac{\partial^2 U}{\partial X^2}+\frac{2\nu_c^2\nu_0 u_c^2}{l^4}\left(\frac{l}{\delta_T(x)}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right] \tag{9-4-2$'$}$$

全式除以$\frac{\nu_c u_c^2}{l^2}$，则有

$$ReU^2\frac{\partial U}{\partial X}-\frac{Re}{2}U^3\frac{\partial\rho_0}{\rho_0\partial X}+\frac{\varphi}{2}\nu_0\left(\frac{l}{\delta_T(x)}\right)U\frac{\partial^2 U}{\partial X\partial Y}+\frac{\nu_0}{2}\frac{l}{\delta_T(x)}\frac{\partial U}{\partial Y}\left(\frac{U}{\rho_0}\frac{\partial\rho_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$
$$=K_2\frac{\lambda_0}{\rho_0}\left[\frac{\partial^2 T_0}{\partial X^2}+\left(\frac{l}{\delta(x)}\right)^2\frac{\partial^2 T_0}{\partial Y^2}\right]+\frac{4}{3}\nu_0 U\frac{\partial^2 U}{\partial X^2}+\frac{1}{2}\nu_0\varphi^{\frac{2}{3}}\frac{l}{\delta_T(x)}\frac{1}{U}\frac{\partial U}{\partial Y}$$
$$\left[\frac{4}{3}\frac{1}{Re}\frac{\partial^2 U}{\partial X^2}+K_3\left(\frac{l}{\delta_T(x)}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right] \tag{9-4-2$''$}$$

将式(9-4-15)与式(9-4-3)无因次化后，代入式(9-4-2)″，则有

$$ReU^2\frac{\partial U}{\partial X}+\frac{Re}{2}U^2\frac{\partial U}{\partial X}+\frac{\varphi}{2}\nu_0\frac{l}{\delta_T(x)}U\frac{\partial^2 U}{\partial X\partial Y}+\frac{\nu_0}{2}\frac{l}{\delta_T(x)}\frac{\partial U}{\partial Y}\left(\frac{\partial U}{\partial X}-\frac{U}{U}\frac{\partial U}{\partial X}\right)$$
$$=K_2\lambda_0 U\left[\frac{\partial^2 T_0}{\partial X^2}+\left(\frac{l}{\delta(x)}\right)^2\frac{\partial^2 T_0}{\partial Y^2}\right]+\frac{4}{3}\nu_0 U\frac{\partial^2 U}{\partial X^2}+\frac{1}{2}\nu_0\varphi^{\frac{2}{3}}\frac{l}{\delta_T(x)}\frac{1}{U}\frac{\partial U}{\partial Y}$$
$$\left[\frac{4}{3}\frac{1}{Re}\frac{\partial^2 U}{\partial X^2}+K_3\left(\frac{l}{\delta_T(x)}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right] \tag{9-4-24}$$

式中：

$$K_2=\frac{\lambda_c T_c}{\rho_c \nu_c u_c^2} \tag{9-4-25}$$

$$K_3=\frac{2\nu_c t}{l^2} \tag{9-4-26}$$

$$T_0(X,\ Y)\big|_{X=0,\ Y=0}=1 \tag{9-4-27}$$

$$T_0(X,\ Y)\big|_{Y=1}=1 \tag{9-4-28}$$

$$\left.\frac{\partial T_0(X,\ Y)}{\partial Y}\right|_{Y=1}=0 \tag{9-4-29}$$

$$T_0(X,\ Y)\big|_{Y=0}=T_w \tag{9-4-30}$$

$$\frac{\partial^2 T_0(X,\ Y)}{\partial Y^2}<0 \tag{9-4-31}$$

9.4.3 确定边界层内速度分布

根据边界条件与式(9-4-17)的要求，暂选速度分布

$$U=\eta \operatorname{arccot} X(2Y-Y^2) \tag{9-4-32}$$

式中，η 是参变量；$\delta(x)$是边界层厚度；$Y=\frac{y}{\delta(x)}$。式(9-4-32)只有确定 η 与$\delta(x)$才可应用。

(1) 建立边界层厚度 $\delta(x)$的计算公式

涡流边界层厚度是涡旋运动所能达到的极限位置。由于受阻力，其运动速度逐渐衰减，当到达边界层边缘时，u^*为零。

根据物体运动阻力，按雷诺、阿连、牛顿三个区进行计算，首先按雷诺区计算。

$$\frac{\rho\pi d_s^3}{6}\frac{\mathrm{d}u^*}{\mathrm{d}t}=-3\pi\mu d_s u^* \tag{a}$$

$$\frac{\mathrm{D}u^*}{\mathrm{D}t}=-\frac{18\nu u^*}{d_s^2} \tag{b}$$

$$\frac{\partial u^*}{\partial t}-u^*\frac{\partial u^*}{\partial y}=-18\nu\frac{u^*}{d_s^2} \tag{c}$$

将上式无因次化，为此取$\frac{y}{\delta(x)}=Y$，$t=\frac{d_s}{u_c}t_0$，$\frac{u^*}{u_c}=U^*$，代入式(c)，整理，得

$$\frac{\partial U^*}{\partial t_0}-\frac{d_s}{\delta(x)}U^*\frac{\partial U^*}{\partial Y}=-\frac{18\nu}{d_s u_c}U^* \tag{d}$$

令

$$K_1=\frac{18\nu}{d_s u_c} \tag{e}$$

$$\frac{\partial U^*}{\partial t_0}-\frac{d_s}{\delta(x)}U^*\frac{\partial U^*}{\partial Y}=-K_1U^* \tag{9-4-33}$$

根据边界条件，选

$$U^{*}=X\mathrm{e}^{-t_0 Y} \tag{9-4-34}$$

代入式(9-4-33)，整理为

$$\frac{d_s}{\delta(x)}t_0 X\mathrm{e}^{-t_0 Y}=Y-K_1 \tag{9-4-33$'$}$$

因为要求 $\delta(x)$，$Y=1$，则

$$\delta(x)=\frac{d_s t_0 X\mathrm{e}^{-t_0}}{1-K_1} \tag{9-4-35}$$

当 $t_0=0$ 时，涡旋在边层流界面上产生，而边层流界面位置如以前有关各节计算应为 $Y_B=0.183$，$y_B=0.183\delta(x)$。据此，式(9-4-35)应该改为

$$\delta(x)=\frac{d_s t_0 X\mathrm{e}^{-t_0}}{0.817(1-K_1)} \tag{9-4-36}$$

因为 d_s 是涡旋直径，不易求得，故可以将时间无因次化，取 $t=\frac{l}{u_c}t_0$，则可得

$$\delta(x)=\frac{l t_0 X\mathrm{e}^{-t_0}}{0.817(1-K_1)} \tag{9-4-37}$$

按阿连区计算

$$\rho\frac{\pi d_s^3}{6}\frac{\mathrm{d}u^{*}}{\mathrm{d}t}=-1.25\pi\sqrt{\mu\rho d_s^3}\,u^{*1.5} \tag{f}$$

$$\frac{\mathrm{D}u^{*}}{\mathrm{D}t}=-7.5\mu^{\frac{1}{2}}\frac{u^{*1.5}}{\rho^{\frac{1}{2}}d_s^{1.5}}=-7.5\cdot\frac{\nu^{\frac{1}{2}}u^{*1.5}}{d_s^{1.5}}$$

$$\frac{\partial u^{*}}{\partial t}-u^{*}\frac{\partial u^{*}}{\partial y}=-7.5\cdot\frac{\nu^{\frac{1}{2}}}{d_s^{1.5}}u^{*1.5} \tag{g}$$

无因次化

$$\frac{\partial U^{*}}{\partial t_0}-\frac{d_s}{\delta(x)}U^{*}\frac{\partial U^{*}}{\partial Y}=-7.5\cdot\frac{U^{*1.5}}{\sqrt{K_2}} \tag{9-4-38}$$

式中：

$$K_2=\frac{d_s u_c}{\nu} \tag{h}$$

根据边界条件，将式(9-4-34)代入式(h)

$$\delta(x)=\frac{d_s X t_0\mathrm{e}^{-t_0 Y}}{Y-7.5\cdot\frac{1}{\sqrt{K_2}}X^{0.5}\mathrm{e}^{-0.5t_0 Y}} \tag{i}$$

取 $Y=1$

$$\delta(x)=\frac{d_s X t_0 \mathrm{e}^{-t_0 Y}}{1-7.5\left(\frac{X}{K_2}\right)^{\frac{1}{2}}\mathrm{e}^{-\frac{t_0}{2}}} \tag{j}$$

当 $t_0=0$ 时，$\delta(x)=y_B$，则有

$$\delta(x)=\frac{d_s X t_0 \mathrm{e}^{-t_0}}{0.817\left[1-7.5\left(\frac{X}{K_2}\right)^{\frac{1}{2}}\mathrm{e}^{-\frac{t_0}{2}}\right]} \tag{9-4-39}$$

当无因次化时间时，取 l，则有

$$\delta(x)=\frac{l X t_0 \mathrm{e}^{-t_0}}{0.817\left[1-7.5\left(\frac{X}{K_2}\right)^{\frac{1}{2}}\mathrm{e}^{-\frac{t_0}{2}}\right]} \tag{9-4-40}$$

按牛顿区计算

$$R=0.055\pi\rho d_s^2 u^*$$

$$\frac{\mathrm{d}u^*}{\mathrm{d}t}=-0.33\cdot\frac{u^{*2}}{d_s} \tag{k}$$

$$\frac{\partial u^*}{\partial t}-u^*\frac{\partial u^*}{\partial y}=-0.33\cdot\frac{u^{*2}}{d_s} \tag{l}$$

将上式无因次化

$$\frac{\partial U^*}{\partial t_0}-\frac{d_s}{\delta(x)}U^*\frac{\partial U^*}{\partial Y}=-0.33\cdot U^{*2} \tag{9-4-41}$$

将式(9-4-34)代入，整理得

$$\delta(x)=\frac{d_s t_0 X \mathrm{e}^{-t_0 Y}}{Y-0.33X\mathrm{e}^{-t_0 Y}} \tag{m}$$

取 $Y=1$

$$\delta(x)=\frac{d_s t_0 X \mathrm{e}^{-t_0}}{1-0.33X\mathrm{e}^{-t_0}} \tag{n}$$

引入 $t_0=0$ 边界条件

$$\delta(x)=\frac{d_s t_0 X \mathrm{e}^{-t_0}}{0.817(1-0.33X\mathrm{e}^{-t_0})} \tag{9-4-42}$$

同理，也可以得

$$\delta(x)=\frac{l t_0 X \mathrm{e}^{-t_0}}{0.817(1-0.33X\mathrm{e}^{-t_0})} \tag{9-4-43}$$

(2) 确定参变量 $\eta(X, Y)$

将式(9-4-32)代入式(9-4-17)

$$\frac{-Re(\operatorname{arccot}X)(2Y-Y^2)^2}{1+X^2}\eta^2=\nu_0\left[\frac{4}{3}\frac{2X}{(1+X^2)^2}(2Y-Y^2)-2(1-\varphi^{\frac{2}{3}})\left(\frac{l}{\delta(x)}\right)^2\operatorname{arccot}X\right]\eta$$

$$-4K_1\nu_0^2\varphi^{\frac{2}{3}}\left(\frac{l}{\delta(x)}\right)^3(1-Y)\eta^2(\operatorname{arccot}X)^2 \tag{o}$$

式中，$\nu_0=1$。化简，得

$$\eta(X,\ Y)=\frac{2(1-\varphi^{\frac{2}{3}})\left(\frac{l}{\delta(x)}\right)^2(1+X^2)^2\operatorname{arccot}X-\frac{8}{3}X(2Y-Y^2)}{Re(2Y-Y^2)\operatorname{arccot}X-4K_1\varphi^{\frac{2}{3}}\left(\frac{l}{\delta(x)}\right)^3(1-Y)(1+X^2)^2(\operatorname{arccot}X)^2} \tag{9-4-44}$$

当 $Y=1$ 时，则为

$$\eta(X,\ 1)=\frac{2(1-\varphi^{\frac{2}{3}})(1+X^2)^2\left(\frac{l}{\delta(x)}\right)^2\operatorname{arccot}X-\frac{8}{3}(1+X^2)^2X}{Re(\operatorname{arccot}X)(1+X^2)} \tag{9-4-45}$$

近似处理

$$\eta(X,\ Y)=\frac{\eta(X,\ Y)}{\eta(X,\ 1)}=\frac{Re(1+X^2)(2Y-Y^2)\operatorname{arccot}X-4(1+X^2)^2K_1\varphi^{\frac{2}{3}}\left(\frac{l}{\delta(x)}\right)^3(1-Y)\operatorname{arccot}X}{Re(1+X^2)\operatorname{arccot}X} \tag{9-4-46}$$

综合分析，最后得速度分布为

$$U=\eta(X,\ Y)(2Y-Y^2) \tag{9-4-47}$$

式中，$\eta(X,\ Y)$是参变量，由式(9-4-46)计算。

例 9-5 来流空气速度 $u_c=80\text{m/s}$，温度 $t_c^0=15℃$，运动黏性系数 $\nu_c=14.66\text{m}^2/\text{s}$，准定常流时间 $t=0.01\text{s}$，板长 $l=1.2\text{m}$，确定 $x=0.6$ 处断面速度分布。

解 计算准备工作：

$$K_1=\frac{6u_ct}{l}=\frac{6\times80\times0.01}{1.2}=4$$

$$Re=\frac{u_cl}{\nu_c}=\frac{80\times1.2\times10^6}{14.66}=6.548\times10^6$$

设 $\varphi=0.15$，$\varphi^{\frac{2}{3}}=0.282$，$\operatorname{arccot}0.5=0.393$。计算当 $X=0.5$ 处，边界层厚度 $\delta(x)$，由式(9-4-43)

$$\delta(x)=\frac{lXt_0l^{-t_0}}{0.817(1-0.33Xl^{-t_0})}=\frac{1.2\times0.5\times0.01l^{-0.01}}{0.817(1-0.33\times0.5l^{-0.01})}$$

$$=\frac{0.6\times0.01\times0.99}{0.817(1-0.33\times0.5\times0.99)}=\frac{0.00594}{0.684}=0.0087\text{m}$$

计算 $X=0.5$ 处断面速度分布，由式(9-4-46)

$\eta(0.5,\ 0.8)=$

$$\frac{6.548\times10^6(1+0.5^2)(2\times0.8-0.8^2)0.393-4(1+0.5^2)^2\times4\times0.282\left(\frac{1.2}{0.0087}\right)^3(1-0.8)\times0.393^2}{6.548\times10^6(1+0.5^2)\times0.393}$$

$$=\frac{0.472\times 6.548\times 10^6-0.054\times 2624132}{0.491\times 0.548\times 10^6}$$

$$=\frac{(3.096-0.148)\times 10^6}{3.217\times 10^6}=\frac{2.948}{3.217}=0.916$$

$$u(0.5,\ 0.8)=80\times\eta(0.5,\ 0.8)(2Y-Y^2)=80\times 0.916(2\times 0.8-0.8^2)$$

$$=73.311\times 0.96=70.378\text{m/s}$$

$$\eta(0.5,\ 0.6)=\frac{3.217\times 10^6(2\times 0.6-0.6^2)-714331.668\times(1-0.6)}{3.217\times 10^6}$$

$$=\frac{(2.702-0.286)\times 10^6}{3.217\times 10^6}=\frac{2.416}{3.217}=0.751$$

$$u(0.5,\ 0.6)=u_c\eta(0.5,\ 0.6)(2Y-Y^2)=80\times 0.751(2\times 0.6-(0.6)^2)=50.467\text{m/s}$$

$$\eta(0.5,\ 0.4)=\frac{3.217\times 10^6\times(0.8-0.16)-714331\times(1-0.4)}{3.217\times 10^6}$$

$$=\frac{(2.059-0.429)\times 10^6}{3.217\times 10^6}=\frac{1.63}{3.217}=0.507$$

$$u(0.5,\ 0.4)=80\times 0.507(0.8-0.16)=40.56\times 0.64=25.958\text{m/s}$$

$$\eta(0.5,\ 0.2)=\frac{3.217\times 10^6(0.4-0.04)-714331\times(1-0.2)}{3.217\times 10^6}=\frac{(-0.572)\times 10^6}{3.217\times 10^6}$$

$$=\frac{0.586}{3.217}=0.182$$

$$u(0.5,\ 0.2)=80\times 0.182(0.4-0.04)=5.247\text{m/s}$$

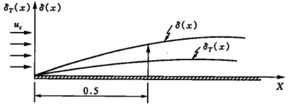


图 9-12　例 9-5 与例 9-6 计算用图

9.4.4 确定温度分布

设来流温度大于板面温度，即 $t_c^0>t_w^0$。据此，选

$$T_0=\eta_T\frac{2X}{1+X}(2Y-Y^2) \tag{9-4-48}$$

式中，$Y_T=\frac{y}{\delta_T(x)}$，$\delta_T(X)$是温度边界层厚度；$\eta$ 是参变量，它的计算公式与速度公式中的 η 形式一样，只是 $\delta(x)$变为 $\delta_T(x)$，Y 变为 Y_T，即

$$\eta_T(X,\ Y)=\frac{Re(1+X^2)(2Y_T-Y_T^2)\operatorname{arccot}X-4(1+X^2)K_1\varphi^{\frac{2}{3}}\left(\frac{l}{\delta_T(x)}\right)^3(1-Y_T)(\operatorname{arccot}X)^2}{Re(1+X^2)\operatorname{arccot}X}\tag{9-4-49}$$

将式(9-4-48)与式(9-4-32)代入式(9-4-24)，则有

$$-1.5Re\eta^3(2Y-Y^2)\frac{(\operatorname{arccot}X)^2}{1+X^2}+\frac{\varphi}{2}\nu_0\frac{l}{\delta(x)}\eta\frac{2(1-Y)}{1+X^2}$$

$$=\frac{-4K_2\lambda_0\eta^2(2Y-Y^2)}{1+X}\operatorname{arccot}X\left[\frac{2Y_T-Y_T^2}{(1+X)^2}+\left(\frac{l}{\delta_T(x)}\right)^2X\right]+\frac{8}{3}\eta^2(2Y-Y^2)\cdot$$

$$\eta^2\frac{X\operatorname{arccot}X}{(1+X^2)^2}+\frac{1}{2}\nu_0\varphi^{\frac{2}{3}}\frac{l}{\delta(x)}\frac{2(1-Y)}{2Y-Y^2}\left[\frac{4}{3}\frac{1}{Re}\frac{\partial^2U}{\partial X^2}+K_3\left(\frac{l}{\delta(x)}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2U}{\partial Y^2}\right]\tag{9-4-24$'$}$$

因为目的是确定温度边界层厚度 $\delta_T(x)$，所以分别取

$\eta(X,\ Y)=\eta(X,\ 1)=1$，$\eta_T(X,\ Y_T)=\eta_T(X,\ 1)=1$，$Y=1$，$Y_T=1$，$\nu_0=1$，$\lambda_0=1$，则

$$1.5Re\frac{\operatorname{arccot}X}{1+X^2}=-4K_2\left[\frac{1}{(1+X)^3}+\frac{X}{1+X}\left(\frac{l}{\delta_T(x)}\right)^2\right]+\frac{8}{3}\frac{X\operatorname{arccot}X}{(1+X^2)^2}\tag{9-4-24$''$}$$

$$\frac{4K_2X}{1+X}\left(\frac{l}{\delta_T(x)}\right)^2=1.5Re\frac{\operatorname{arccot}X}{1+X^2}+\frac{8}{3}\frac{X}{(1+X^2)^2}-\frac{4K_2}{(1+X)^3}\tag{9-4-24$'''$}$$

近似处理

$$\delta_T(x)=l\sqrt{\frac{4K_2X(1+X^2)}{1.5Re(1+X)\operatorname{arccot}X}}\tag{9-4-50}$$

例 9-6 题目同例 9-5，确定 $X=0.5$ 处温度边界层厚度 $\delta_T(0.5)$为多少?

解 当温度 $t_c^0=15℃$，$\lambda_c=2.55\times10^{-2}\mathrm{W/(m\cdot K)}$，$\rho_c=1.22\mathrm{kg/m^3}$，$\nu_c=14.66\mathrm{m^2/s}$。由式(9-4-25)，计算

$$K_2=\frac{\lambda_cT_c}{\rho_c\nu_cu_c^2}=\frac{0.0255\times(273+15)\times10^6}{1.22\times4.66\times80^2}=\frac{7.344\times10^6}{0.114\times10^6}=64.421$$

由式(9-4-50)

$$\delta_T(0.6)=0.6\sqrt{\frac{4\times64.421\times0.5(1+0.5^2)}{1.5Re(1+0.5)0.393}}=0.6\sqrt{\frac{161.525}{5.79\times10^6}}$$

$$=0.6\sqrt{0.000021}=0.6\times0.0052=0.0031\mathrm{m}$$

温度分布计算公式为

$$T_0=\eta_T(2Y_T-Y_T^2)\tag{9-4-51}$$

$$\eta_T=\frac{Re(1+X^2)(2Y_T-Y_T^2)\operatorname{arccot}X-4(1+X^2)^2K_1\varphi^{\frac{2}{3}}\left(\frac{l}{\delta_T(x)}\right)^3(1-Y_T)(\operatorname{arccot}X)^2}{Re(1+X^2)\operatorname{arccot}X}\tag{9-4-52}$$

$$T_0=\frac{t^0}{t_c^0}-\frac{t_w^0}{t_c^0} \tag{9-4-53}$$

9.5 可压缩湍流有压力变化平板边界层

9.5.1 流场情况分析

可压缩湍流有压力变化平板边界层，一般发生在沿流程有断面变化设备结构上。如高压气罐外伸收缩短管内，就会呈现这种边界层。

短管中心部分是理想流体运动区，靠壁面是实际流体运动区，各自速度分布不同，在交界面又相等。从数学分析上来看，两个速度函数定义域不同，在相衔接的面上相等。

与速度场同时存在的，还有温度场。它分为两个区域：中心区的温度只是 x 的函数；边壁区则是 x，y 的函数。两者之间也有一个分界面为 $\delta_T(x)$，也就是温度边界层 $t^0(x)$与 $t^0(x,\ y)$两个温度函数有着不同的定义域。

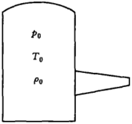


图 9-13　气罐

从数学分析角度来看，必须首先找到速度边界层厚度 $\delta(x)$的公式，然后再找到温度边界层厚度 $\delta_T(x)$，这样讨论问题时，则有了定义域范围。

问题涉及的变量函数有速度、温度、压力、准定常流时间 t、涡旋体积分数 φ、速度边界层厚度 $\delta(x)$、温度边界层 $\delta_T(x)$，要同时解得方程组不够，只好将有些问题暂时视为参变量，待以后条件具备时解决。

问题控制方程组包括：动量方程，能量方程，气体状态方程，连续性方程。

在解决问题过程中，要用到双坐标系，如图 9-14 所示，还要遇到一个函数两个表达式，各有各的应用地方。

9.5.2 运动控制方程与边界条件

(1) 控制方程与边界条件

实际流体动量方程：结合本问题，依式(4-5-16)，忽略分子能量和涡旋温度变化，则为

$$\begin{aligned}&\frac{1}{2}\nu\ \frac{1}{\rho}\frac{\partial\rho}{\partial x}\frac{\partial u}{\partial y}+u\ \frac{\partial u}{\partial x}+\frac{1}{2}\nu\left(\frac{1}{u}\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial x}\\&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left[\frac{4}{3}\frac{\partial^2 u}{\partial x^2}+(1-\varphi^{\frac{2}{3}})\frac{\partial^2 u}{\partial y^2}+6\varphi^{\frac{2}{3}}t\ \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right]\end{aligned} \tag{9-5-1}$$

理想流体运动区动量方程：

$$u_0\frac{\mathrm{d}u_0}{\mathrm{d}x}=-\frac{\mathrm{d}p}{\rho\mathrm{d}x} \tag{9-5-2}$$

能量方程：结合本问题，依式(4-5-41)，忽略分子能量、质量力和涡旋温度变化，则为

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{v^2}{2}\right)-\left(\frac{v^2}{2}\right)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\frac{\varphi}{2}\nu\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial u}{\partial y}\right)+\frac{\nu}{2}\frac{\partial u}{\partial y}\left(\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\nabla\cdot\boldsymbol{V}\right)$$
$$=\frac{\lambda}{\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+u\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}\right)-\frac{1}{2}\varphi^{\frac{2}{3}}\nu\frac{1}{u}\frac{\partial u}{\partial y}\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}+2\nu t\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right) \tag{9-5-3}$$

整理为

$$u^2\frac{\partial u}{\partial x}-\frac{u^3}{2}\frac{\partial \rho}{\rho\partial x}+\frac{\varphi}{2}\nu u\frac{\partial^2 u}{\partial x\partial y}+\frac{\nu}{2}\frac{\partial u}{\partial y}\left(\frac{u}{\rho}\frac{\partial \rho}{\partial x}+\frac{\partial u}{\partial x}\right)$$
$$=\frac{\lambda}{\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+u\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}\right)-\frac{1}{2}\varphi^{\frac{2}{3}}\frac{\nu}{u}\frac{\partial u}{\partial y}\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}+2\nu t\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right) \tag{9-5-4}$$

质量守恒方程：

研究的问题是喷嘴收缩或膨胀形的，因此结合断面变化特点，其形式为

$$\rho_c u_c r_c^2=\rho u(r_c\pm x\tan\alpha)^2 \tag{9-5-5}$$

其微分形式为

$$2u\rho\tan\alpha=(r_c\pm x\tan\alpha)^2\left(u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}\right) \tag{9-5-6}$$

气体状态方程：

$$p=R\rho T \tag{9-5-7}$$

其微分方程，应考虑到$\frac{\partial p}{\partial y}=0$，则有

$$\frac{\partial T}{\partial x}\frac{\partial \rho}{\partial y}+\frac{\partial \rho}{\partial x}\frac{\partial T}{\partial y}+\rho\frac{\partial^2 T}{\partial x\partial y}+T\frac{\partial^2 \rho}{\partial x\partial y}=0 \tag{9-5-8}$$

理想流体运动区：

质量守恒微分方程：

$$\frac{1}{\rho}\frac{\partial \rho}{\partial x}=-\frac{1}{u_0}\frac{\partial u_0}{\partial x} \tag*{(9-5-5)′}$$

理想气体状态方程：

$$\frac{\partial p}{\partial x}=R\left(\rho\frac{\partial T}{\partial x}+T\frac{\partial \rho}{\partial x}\right) \tag*{(9-5-6)′}$$

边界层内(实际流体运动区)：

质量守恒微分方程：

$$\frac{1}{\rho}\frac{\partial^2 \rho}{\partial y\partial x}+\frac{1}{u}\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial \rho}{\partial y}+\frac{1}{u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} \tag*{(9-5-7)′}$$

理想气体状态微分方程：

$$\frac{\partial T}{\partial x}\frac{\partial \rho}{\partial y}+\frac{\partial \rho}{\partial x}\frac{\partial T}{\partial y}+\rho\frac{\partial^2 T}{\partial x\partial y}+T\frac{\partial^2 \rho}{\partial x\partial y}=0 \tag*{(9-5-8)′}$$

实际流体运动区速度与温度边界条件：

$$u(x,\ y)\big|_{x=0,y=0}=0 \tag{9-5-9}$$

$$u(x,\ y)\big|_{y=\delta(x)}=u_0(x) \tag{9-5-10}$$

$$\left.\frac{\partial u(x,\ y)}{\partial y}\right|_{y=\delta(x)}=0 \tag{9-5-11}$$

$$\frac{\partial^2 u(x,\ y)}{\partial y^2}<0 \tag{9-5-12}$$

设来流温度 $t_c^0>$平板壁面温度 t_w^0，故取

$$t(x,\ y)\big|_{x=0,y=0}=t_c \tag{9-5-13}$$

$$t(x,\ y)\big|_{y=\delta_T(x)}=t(x) \tag{9-5-14}$$

$$t(x,\ y)\big|_{y=0}=t_w \tag{9-5-15}$$

$$\left.\frac{\partial t(x,\ y)}{\partial y}\right|_{y=\delta_T(x)}=\xi \tag{9-5-16}$$

$$\frac{\partial^2 t(x,\ y)}{\partial y^2}<0 \tag{9-5-17}$$

$$p(x)\big|_{x=0}=p_c \tag*{(9-5-17)$'$}$$

(2) 无因次化方程与边界条件

取

$$\frac{u}{u_c}=U,\ \frac{x}{r_c}=X,\ \frac{y}{\delta(x)}=Y,\ \frac{p}{p_c}=p_0,\ \frac{\rho}{\rho_c}=\rho_0,\ \frac{t}{t_c}=T_0,\ \frac{y}{\delta_T(x)}=Y_T$$

式中，l 是板长；$\delta(x)$是速度边界层厚度；$\delta_T(x)$是温度边界层厚度；u_c 是来流速度；p_c 是来流压力；ρ_c 是来流密度；t_c^0 是来流温度；t_w^0 是板面温度。

将有关比值代入动量方程式(9-5-1)

$$\begin{aligned}&\frac{1}{2}\frac{\nu_c u_c}{r_c^2}\nu_0\frac{l_c}{\delta(x)}\frac{\partial\rho_0}{\rho_0\partial X}\frac{\partial U}{\partial Y}+\frac{u_c^2}{l}U\frac{\partial U}{\partial X}+\frac{1}{2}\frac{\nu_c u_c}{r_c^2}\frac{r_c}{\delta(x)}\nu_0\left(\frac{1}{U}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}\right)\\&=-\frac{p_c}{\rho_c l}\frac{1}{\rho_0}\frac{\partial p_0}{\partial X}+\frac{\nu_c u_c}{l_c^2}\nu_0\left[\frac{4}{3}\frac{\partial^2 U}{\partial X^2}+\frac{\nu_c u_c}{r_c^2}\nu_0(1-\varphi^{\frac{2}{3}})\left(\frac{r_c}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial Y^2}+\frac{6t\nu_c u_c^2}{r_c^3}\left(\frac{l_c}{\delta(x)}\right)^3\varphi^{\frac{2}{3}}\nu_0\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right]\end{aligned} \tag*{(9-5-1)$'$}$$

全式除以$\frac{\nu_c u_c}{l^2}$，则为

$$\begin{aligned}&\frac{1}{2}\nu_0\frac{r_c}{\delta(x)}\frac{\partial\rho_0}{\rho_0\partial X}\frac{\partial U}{\partial Y}+ReU\frac{\partial U}{\partial X}+\frac{1}{2}\nu_0\frac{r_c}{\delta(x)}\frac{1}{U}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}\\&=-K_1\frac{1}{\rho_0}\frac{\partial p_0}{\partial X}+\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}+\nu_0(1-\varphi^{\frac{2}{3}})\left(\frac{r_c}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial Y^2}+K_2\varphi^{\frac{2}{3}}\nu_0\left(\frac{l_c}{\delta(x)}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\end{aligned} \tag{9-5-18}$$

式中：

$$Re=\frac{u_c l}{\nu_c} \tag{9-5-19}$$

$$K_1=\frac{p_c r_c}{\rho_c u_c \nu_c} \tag{9-5-20}$$

$$K_2=\frac{6tu_c}{r_c} \tag{9-5-21}$$

将有关比值代入能量方程式(9-5-4)，则有

$$\frac{u_c^3}{l}U^2\frac{\partial U}{\partial X}-\frac{1}{2}\frac{u_c^3}{l}U^3\frac{\partial \rho_0}{\rho_0\partial X}+\frac{\varphi}{2}\frac{\nu_c u_c^2}{l^2}\frac{l}{\delta(x)}\frac{\partial^2 U}{\partial X\partial Y}+\frac{\nu_c u_c^2}{l^2}\frac{\nu_0}{2}\frac{l}{\delta(x)}\frac{\partial U}{\partial Y}\left(\frac{U}{\rho_0}\frac{\partial \rho_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$=\frac{\lambda_c T_c}{\rho_c l^2}\frac{\lambda_0}{\rho_0}\left[\frac{\partial^2 T_0}{\partial X^2}+\left(\frac{l}{\delta(x)}\right)^2\frac{\partial^2 T_0}{\partial Y^2}\right]-\frac{u_c p_c}{\rho_c l}\frac{U}{\rho_0}\frac{\partial p_0}{\partial X}+\frac{\nu_c u_c^2}{l^2}\nu_0\frac{4}{3}U\frac{\partial^2 U}{\partial X^2}$$

$$-\frac{1}{2}\varphi^{\frac{2}{3}}\frac{\nu_0}{U}\frac{\partial U}{\partial Y}\left[-\frac{\nu_c p_c}{\rho_c l^2}\frac{l}{\delta(x)}\frac{\partial p_0}{\partial X}+\frac{\nu_c^2 u_c}{l^3}\nu_0\frac{l}{\delta(x)}\frac{\partial^2 U}{\partial X^2}+\frac{2t\nu_c^2}{l^4}\nu_0\left(\frac{l}{\delta(x)}\right)^4\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right] \tag{9-5-4$'$}$$

除以$\frac{\nu_c u_c^2}{l^2}$，则为

$$ReU^2\frac{\partial U}{\partial X}-\frac{1}{2}Re\frac{U^3\partial \rho_0}{\rho_0\partial X}+\frac{\varphi}{2}\frac{r_c}{\delta(x)}\frac{\partial^2 U}{\partial X\partial Y}+\frac{1}{2}\nu_0\frac{r_c}{\delta(x)}\frac{\partial U}{\partial Y}\left(\frac{U}{\rho_0}\frac{\partial \rho_0}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$=K_3\frac{\lambda_0}{\rho_0}\left[\frac{\partial^2 T_0}{\partial X^2}+\left(\frac{r_c}{\delta_T(x)}\right)^2\frac{\partial^2 T_0}{\partial Y^2}\right]-K_4\frac{U}{\rho_0}\frac{\partial p_0}{\partial X}+\frac{4}{3}\nu_0 U\frac{\partial^2 U}{\partial X^2}-\frac{1}{2}\varphi^{\frac{2}{3}}\frac{\nu_0}{U}\frac{\partial U}{\partial Y}\cdot$$

$$\left[-K_5\frac{r_c}{\delta(x)}\frac{\partial p_0}{\rho_0\partial X}+\frac{\nu_0}{Re}\frac{r_c}{\delta(x)}\frac{\partial^2 U}{\partial X^2}+K_6\nu_0\left(\frac{l}{\delta(x)}\right)^4\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right] \tag{9-5-22}$$

式中：

$$K_3=\frac{\lambda_c T_c}{\rho_c \nu_c u_c^2} \tag{9-5-23}$$

$$K_4=\frac{p_c r_c}{\rho_c \nu_c u_c} \tag{9-5-24}$$

$$K_5=\frac{p_c}{\rho_c u_c^2} \tag{9-5-25}$$

$$K_6=\frac{2t\nu_c}{r_c^2} \tag{9-5-26}$$

理想流体运动区微分方程无因次化。取$\frac{u_0}{u_c}=U_0$，其他比值不变，运动微分方程：由式(9-5-2)可得

$$U_0\frac{dU_0}{dX}=-K_3\frac{dp_0}{\rho_0 dX} \tag{9-5-27}$$

质量守恒无因次化方程。代数方程：由式(9-5-5)得

$$1=\rho_0 U(1\pm x\tan\alpha)^2 \tag{9-5-28}$$

微分形式：

$$\frac{\pm 2\tan\alpha}{(1\mp\tan\alpha)^2}-\frac{1}{U}\frac{\partial U}{\partial X}=\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial X} \tag{9-5-29}$$

无因次理想气体状态方程。代数形式：

$$p_0=\rho_0 T_0 \tag{9-5-30}$$

微分形式：

$$\frac{\partial p_0}{\partial X}=-\rho_0\frac{\partial T_0}{\partial X}+T_0\frac{\partial \rho_0}{\partial X} \tag{9-5-31}$$

$$\frac{\partial T_0}{\partial X}\frac{\partial \rho_0}{\partial Y}+\frac{\partial \rho_0}{\partial X}\frac{\partial T_0}{\partial Y}+\rho_0\frac{\partial^2 T_0}{\partial X\partial Y}+T_0\frac{\partial^2 \rho_0}{\partial X\partial Y}=0 \tag{9-5-32}$$

无因次方程是解决问题的工具，代数式解决变量换值，微分方程解决函数分布问题。

边界条件无因次化：

$$U(X,\ Y)|_{X=0,\,Y=0}=0 \tag{9-5-33}$$

$$U(X,\ Y)|_{Y=1}=U(X) \tag{9-5-34}$$

$$\left.\frac{\partial U(X,\ Y)}{\partial Y}\right|_{Y=1}=\xi(X) \tag{9-5-35}$$

$$\frac{\partial^2 U(X,\ Y)}{\partial Y^2}<0 \tag{9-5-36}$$

$$T(X,\ Y)|_{X=0,\,Y=0}=1 \tag{9-5-37}$$

$$T(X,\ Y)|_{Y_T=1}=T(X) \tag{9-5-38}$$

$$T(X,\ Y)|_{Y=0}=T_w \tag{9-5-39}$$

$$\left.\frac{\partial T(X,\ Y)}{\partial Y}\right|_{Y=1}=\xi(X) \tag{9-5-40}$$

$$\frac{\partial^2 T(X,\ Y)}{\partial Y^2}<0 \tag{9-5-41}$$

$$p_0(X)|_{X=0}=1 \tag{9-5-42}$$

9.5.3 确定流速分布

根据式(9-5-28)，将动量方程式(9-5-18)写成

$$\nu_0\frac{r_c}{\delta(x)}\frac{1}{U}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X}+ReU\frac{\partial U}{\partial X}=ReU_0\frac{\mathrm{d}U_0}{\mathrm{d}X}+\frac{4}{3}\nu_0\frac{\partial^2 U}{\partial X^2}+\nu_0(1-\varphi^{\frac{2}{3}})\left(\frac{r_c}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial Y^2}$$
$$+\nu_0\left(\frac{r_c}{\delta(x)}\right)^3\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2} \tag{9-5-43}$$

根据边界条件选不同的速度分布，分别代入式(9-5-43)，确定其中一个合理的速度分布为

$$U=X^{\frac{1}{2}}(2Y-Y^{\frac{3}{2}}) \tag{9-5-44}$$

9.5.4 确定速度边界层厚度 $\delta(x)$

将式(9-5-44)代入式(9-5-43)，并取 $Y=1$，则有对应的动量方程为

$$\frac{1}{2}\frac{r_c}{\delta(x)}+\frac{1}{2}Re=ReU_0\frac{\mathrm{d}U_0}{\mathrm{d}X}-\frac{1}{3}X^{-\frac{3}{2}}-\frac{4}{3}(1-\varphi^{\frac{2}{3}})\left(\frac{r_c}{\delta(x)}\right)^2-\frac{3}{8}K_2\varphi^{\frac{2}{3}}X\left(\frac{r_c}{\delta(x)}\right)^3 \tag{9-5-45}$$

建立断面质量流守恒方程：进口处质量流量

$$Q_{进}=\rho\pi r_c^2u_c \tag{a}$$

通过理想流体运动区断面质量流量

$$Q_{理}=\pi\rho[(r_c-x'\tan\alpha)-\delta(x)]^2u(x) \tag{b}$$

通过边界区断面上质量流量

$$\begin{aligned}Q_{层}&=\rho_{层}2\pi(r_c-x\sin\alpha)\frac{\delta(x)}{2}u_c\int_0^1X^{\frac{1}{2}}(2Y-Y^{\frac{3}{2}})\mathrm{d}Y\\&=\frac{3}{5}\rho_{层}\pi(r_c-x\sin\alpha)u_c\delta(x)X^{\frac{1}{2}}\end{aligned} \tag{c}$$

$$\rho_cu_cr_c^2=\rho_{理}\pi(r_c-x\sin\alpha-\delta(x))^2u(x)+\frac{3}{5}\rho_{层}\pi(r_c-x\sin\alpha)u_c\delta(x)X^{\frac{1}{2}} \tag{9-5-46}$$

无因次化

$$\begin{aligned}1=&\frac{\rho_{理}}{\rho_c}\left[1+X^2\sin^2\alpha+\left(\frac{\delta(x)}{r_c}\right)^2+2X\frac{\delta(x)}{r_c}\sin\alpha-2X\sin\alpha-2\frac{\delta(x)}{r_c}\right]U_0(X)\\&+\frac{\rho_{层}}{\rho_c}\frac{3}{5}(1-X\sin\alpha)\frac{\delta(x)}{r_c}X^{\frac{1}{2}}\end{aligned} \tag{9-5-46$'$}$$

任取一个断面，在理想流体运动区，其密度 $\rho_{理}$ 与进口处密度 ρ_c 进行比较。首先从能量守恒来看

$$p_c+\frac{1}{2}\rho_cu_c^2=p_{理}+\frac{1}{2}\rho_{理}u_{理}^2+h_w \tag{d}$$

压力 $p_c>p_{理}$，用以克服阻力和转化动能 $u_{理}$，因此 $u_{理}>u_c$。

从质量守恒来看

$$\rho_cu_cA_c=\rho_{理}u_{理}A_{理} \tag{e}$$

$A_{理}<A_c$，必然 $u_{理}>u_c$。

从气体状态方程来看，因为已知 $T_c>T_w$(壁面温度)

$$\frac{p_c}{\rho_c}=RT_c,\qquad\frac{p_{理}}{\rho_{理}}=RT_{理} \tag{f}$$

这样，只有 p_c 下降的量值一定小于 ρ_c 下降的量值，才能保证 $T_c>T_w$。结果 $\rho_c>\rho_{理}$。

密度由 ρ_c 变小到 $\rho_{理}$，从式(e)来讲，必然增加速度 $u_{理}$，结果速度 $u_{理}$ 增大，压力下降、断面缩小、密度变小(气体膨胀)。

$\frac{\rho_{理}}{\rho_c}<1$，而只有取 1，才能体现密度转化为速度的那部分速度量值。

在同一个断面上，$p_{理}$ 与 $p_{层}$(边界层内压力)是一样的。而已知 $T_c > T_w$，从气体状态方程来看

$$\frac{p}{\rho_{理}} = RT_{理},\quad \frac{p}{\rho_{层}} = RT_{层},\quad T_{理} > T_{层}$$

而 $T_{层} > T_w$，$\rho_{理} < \rho_{层}$。

从速度分布 $u_{理} > u_{层}$，引起从质量守恒来看 $\rho_{理} < \rho_{层}$，结果愈近壁面，$\rho_{理} \ll \rho_{层}$，致使 $\rho_{层} \approx \rho_c$，所以$\frac{\rho_{层}}{\rho_c} = 1$。分析结果则得

$$\begin{aligned}1 = &\left[1 + X^2\sin^2\alpha + \left(\frac{\delta(x)}{r_c}\right)^2 + 2X\frac{\delta(x)}{r_c}\sin\alpha - 2X\sin\alpha - 2\frac{\delta(x)}{r_c}\right]U_0(X)\\ &+ \frac{3}{5}(1 - X\sin\alpha)\frac{\delta(x)}{r_c}X^{\frac{1}{2}}\end{aligned} \tag{9-5-47}$$

因为$\frac{\delta(x)}{r_c} \ll 1$，$\sin\alpha \ll 1$，故去掉相对小量，则式(9-5-47)为

$$U_0(X) = \frac{1 - \frac{3}{5}\frac{\delta(x)}{r_c}X^{\frac{1}{2}}}{1 - 2\left(X\sin\alpha + \frac{\delta(x)}{r_c}\right)} \tag{9-5-48}$$

说明：式中 $X\sin\alpha$ 中的 X 是参变量，$\frac{\delta(x)}{r_c}$也是参变量，它们不参加微积分运算。

$$U_0(X)\frac{\mathrm{d}U_0(X)}{\mathrm{d}X} = \frac{-\frac{3}{10}\frac{\delta(x)}{r_c}X^{-\frac{1}{2}}}{\left[1 - 2\left(\frac{\delta(x)}{r_c} + X\sin\alpha\right)\right]^2} \tag{a}$$

将式(a)代入式(9-5-45)

$$\frac{1}{2}\frac{r_c}{\delta(x)} + \frac{1}{2}Re = -\frac{3}{10}Re\frac{\frac{\delta(x)}{r_c}X^{-\frac{1}{2}}}{\left[1 - 2\left(\frac{\delta(x)}{r_c} + X\sin\alpha\right)\right]^2} - \frac{3}{4}(1 - \varphi^{\frac{2}{3}})\left(\frac{r_c}{\delta(x)}\right)^2 - \frac{3}{8}K_2\varphi^{\frac{2}{3}}X\left(\frac{r_c}{\delta(x)}\right)^3 \tag{9-5-45$'$}$$

将$\left[1 - 2\left(\frac{\delta(x)}{r_c} + X\sin\alpha\right)\right]^2$简化为

$$\left[1 - 2\left(\frac{\delta(x)}{r_c} + X\sin\alpha\right)\right]^2 = \left[1 - 4\left(\frac{\delta(x)}{r_c} + X\sin\alpha\right)\right] \tag{b}$$

将式(b)代入式(9-5-45)′，然后全式乘以$\left[1 - 4\left(\frac{\delta(x)}{r_c} + X\sin\alpha\right)\right]$，则为

$$\begin{aligned}&\frac{1}{2}\frac{r_c}{\delta(x)}\left[1 - 4\left(\frac{\delta(x)}{r_c} + X\sin\alpha\right)\right] + \frac{1}{2}Re\left[1 - 4\left(\frac{\delta(x)}{r_c} + X\sin\alpha\right)\right]\\ &= -\frac{3}{10}Re\frac{\delta(x)}{r_c}X^{-\frac{1}{2}} - \frac{1}{3}X^{-\frac{3}{2}}\left[1 - 4\left(\frac{\delta(x)}{r_c} + X\sin\alpha\right)\right] - \frac{3}{4}(1 - \varphi^{\frac{2}{3}})\cdot\end{aligned}$$

$$\left[1-4\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)\right]-\frac{3}{8}K_2\varphi^{\frac{2}{3}}X\left(\frac{r_c}{\delta(x)}\right)^3\left[1-4\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)\right] \tag{9-5-45)''}$$

全式除以$\left(\frac{r_c}{\delta(x)}\right)^3$，则为

$$\frac{1}{2}\left(\frac{\delta(x)}{r_c}\right)^2\left[1-4\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)\right]+\frac{1}{2}Re\left(\frac{\delta(x)}{r_c}\right)^3\left[1-4\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)\right]$$
$$=-\frac{3}{10}Re\left(\frac{\delta(x)}{r_c}\right)^4X^{-\frac{1}{2}}-\frac{1}{3}X^{-\frac{3}{2}}\left(\frac{\delta(x)}{r_c}\right)^3\left[1-4\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)\right]-\frac{3}{4}(1-\varphi^{\frac{2}{3}})\frac{\delta(x)}{r_c}\cdot$$
$$\left[1-4\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)\right]-\frac{3}{8}K_2\varphi^{\frac{2}{3}}\left[1-4\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)\right]X \tag{9-5-45)'''}$$

化简并整理为

$$\left(-2Re+\frac{3}{10}ReX^{-\frac{1}{2}}-\frac{4}{3}X^{-\frac{3}{2}}\right)\left(\frac{\delta(x)}{r_c}\right)^4+\left(\frac{1}{2}Re-2-2ReX\sin\alpha+\frac{1}{3}X^{-\frac{3}{2}}-\frac{4}{3}X^{-\frac{1}{2}}\sin\alpha\right)\cdot$$
$$\left(\frac{\delta(x)}{r_c}\right)^3-[2X\sin\alpha+3(1-\varphi^{\frac{2}{3}})]\left(\frac{\delta(x)}{r_c}\right)^2+\left[\frac{3}{4}(1-\varphi^{\frac{2}{3}})-3(1-\varphi^{\frac{2}{3}})X\sin\alpha-\frac{3}{2}K_2\varphi^{\frac{2}{3}}X\right]\cdot$$
$$\frac{\delta(x)}{r_c}+\frac{3}{8}K_2\varphi^{\frac{2}{3}}X-3K_2\varphi^{\frac{2}{3}}X^2\sin\alpha=0 \tag{9-5-45)''''}$$

去掉式中不合理项，得

$$2Re\left(\frac{\delta(x)}{r_c}\right)^4+\left(2ReX\sin\alpha-\frac{1}{2}Re\right)\left(\frac{\delta(x)}{r_c}\right)^3+[2X\sin\alpha+3(1-\varphi^{\frac{2}{3}})]\left(\frac{\delta(x)}{r_c}\right)^2$$
$$+\left[3(1-\varphi^{\frac{2}{3}})X\sin\alpha+\frac{3}{2}K_2\varphi^{\frac{2}{3}}X-\frac{3}{4}(1-\varphi^{\frac{2}{3}})\right]\frac{\delta(x)}{r_c}+3K_2\varphi^{\frac{2}{3}}X^2\sin\alpha-\frac{3}{8}K_2\varphi^{\frac{2}{3}}X=0 \tag{9-5-49}$$

将式(9-5-49)写成

$$\left(\frac{\delta(x)}{r_c}\right)^4+(X\sin\alpha-0.25)\left(\frac{\delta(x)}{r_c}\right)^3+\left[\frac{X\sin\alpha}{Re}+\frac{1.5}{Re}(1-\varphi^{\frac{2}{3}})\right]\left(\frac{\delta(x)}{r_c}\right)^2+\left[\frac{1.5}{Re}(1-\varphi^{\frac{2}{3}})\cdot\right.$$
$$\left.X\sin\alpha+\frac{0.75}{Re}K_2\varphi^{\frac{2}{3}}X-\frac{0.375}{Re}(1-\varphi^{\frac{2}{3}})\right]\frac{\delta(x)}{r_c}+\frac{1.5}{Re}K_2\varphi^{\frac{2}{3}}X^2\sin\alpha-\frac{0.188}{Re}K_2\varphi^{\frac{2}{3}}X=0 \tag{9-5-49)'}$$

将式(9-5-49)′写成

$$\left(\frac{\delta(x)}{r_c}\right)^4+b\left(\frac{\delta(x)}{r_c}\right)^3+c\left(\frac{\delta(x)}{r_c}\right)^2+d\left(\frac{\delta(x)}{r_c}\right)+e=0 \tag{9-5-50}$$

式中：

$$b=X\sin\alpha-0.25 \tag{9-5-51}$$

$$c=\frac{1}{Re}[X\sin\alpha+1.5(1-\varphi^{\frac{2}{3}})] \tag{9-5-52}$$

$$d=\frac{1}{Re}[1.5(1-\varphi^{\frac{2}{3}})X\sin\alpha+0.75K_2\varphi^{\frac{2}{3}}X-0.375(1-\varphi^{\frac{2}{3}})] \tag{9-5-53}$$

$$e=\frac{1}{Re}(1.5K_2\varphi^{\frac{2}{3}}X^2\sin\alpha-0.188K_2\varphi^{\frac{2}{3}}X) \tag{9-5-54}$$

式(9-5-50)是含参变量的四次代数方程，当给定参变量后，则为常系数的四次代数方程。其解法采用 2000 年 5 月第八次印刷的高等教育出版社出版的《数学手册》第 90 页介绍的方法。按该法，其四个根为

$$\left(\frac{\delta(x)}{r_c}\right)^2+\frac{1}{2}\left(b+\sqrt{8y+b^2-4c}\right)\frac{\delta(x)}{r_c}+\left(y+\frac{by-d}{\sqrt{8y+b^2-4c}}\right)=0 \tag{Ⅰ}$$

$$\left(\frac{\delta(x)}{r_c}\right)^2+\frac{1}{2}\left(b-\sqrt{8y+b^2-4c}\right)\frac{\delta(x)}{r_c}+\left(y-\frac{by-d}{\sqrt{8y+b^2-4c}}\right)=0 \tag{Ⅱ}$$

式中，y 由下式计算

$$y^3-0.5cy^2+(0.25bd-e)y+e(4c-b^2)-d^2=0 \tag{Ⅲ}$$

例 9-7　由压气包引出收缩喷嘴，如图 9-13 所示。进口半径 $r_c=0.2\text{m}$，收缩角 $\alpha=3°$，气包中静压为 9 个标准大气压，在喷嘴进口断面上测得压力为 8 个标准大气压，温度 $t^0=30℃$，准定常流时间 $t=0.01\text{s}$，涡旋体积分数 $\varphi=0.15$(暂设)，试确定沿斜壁边界层厚度 $\delta(x)$的分布规律。

解　计算进口断面上气体密度 ρ_c：

$$\rho_c=\frac{p_c}{RT_c}=\frac{8\times1.013\times10^5}{(273+30)287.05}=9.319\text{kg/m}^3$$

计算进口处气流速度 u_c：应用理想流体伯努利方程

$$p_{静}=p_c+\rho_c\frac{u_c^2}{2} \tag{a}$$

$$u_c=\sqrt{\frac{2(p_{静}-p_c)}{\rho_c}}=\sqrt{\frac{2(9-8)1.013\times10^5}{9.318}}=148.94\text{m/s}$$

计算进口处雷诺数：

$$Re=\frac{u_cr_c}{\nu_c}=\frac{148.94\times0.2\times10^6}{16.04}=1.856\times10^6$$

计算无因次数 K_2，由式(9-5-21)

$$K_2=\frac{6u_ct}{r_c}=\frac{6\times148.94\times0.01}{0.2}=45$$

当 X 取确定值时，计算 b，c，d，e 各值。当 $X=0.02$ 时

$b=X\sin\alpha-0.25=0.02\times0.05241-0.25=-0.227$

$$c=\frac{1}{Re}[X\sin\alpha+1.5(1-\varphi^{\frac{2}{3}})]=\frac{1}{1.856\times10^6}[0.02\times0.05241+1.5(1-0.15^{\frac{2}{3}})]$$
$$=0.581\times10^{-6}$$

$$d=\frac{1}{Re}[1.5(1-\varphi^{\frac{2}{3}})X\sin\alpha+0.75K_2\varphi^{\frac{2}{3}}-0.375(1-\varphi^{\frac{2}{3}})]$$
$$=\frac{1}{1.856\times10^6}[1.5(1-0.15^{\frac{2}{3}}\times0.02\times0.5241+0.75\times45\times0.15^{\frac{2}{3}}-0.375(1-0.15^{\frac{2}{3}})]$$

$$=5.00214\times10^{-6}$$

$$e=\frac{1}{Re}(1.5K_2\varphi^{\frac{2}{3}}X^2\sin\alpha-0.188K_2\varphi^{\frac{2}{3}}X)$$

$$=\frac{1}{1.856\times10^6}(1.5\times45\times0.15^{\frac{2}{3}}\times0.02^2\times0.05241-0.188\times45\times0.15^{\frac{2}{3}}\times0.02)$$

$$=0.00011\times10^{-6}-0.0257\times10^{-6}=-0.0256\times10^{-6}$$

应用式(Ⅲ)计算 y 值：

$$y^3-0.5cy^2+(0.25bd-e)y+e(4c-b^2)-d^2=0$$

$$y^3-0.5\times0.581\times10^{-6}y^2+[0.25\times(-0.227)\times0.104\times10^{-6}+0.0256\times10^{-6}]y$$
$$-0.0256\times10^{-6}[4\times0.581\times10^{-6}-(-0.227)^2]-(0.104\times10^{-6})^2=0 \quad (\text{Ⅲ})'$$

去掉相对小量

$$y^3-0.2905\times10^{-6}y^2+0.0197\times10^{-6}y-0.0595\times10^{-6}=0 \quad (\text{Ⅲ})''$$

对式(Ⅲ)″分析

$$0.2905\times10^{-6}y^2=0.0197\times10^{-6}y$$

故简化为

$$y^3=0.0595\times10^{-6}$$
$$y=0.00399\approx0.004$$

将 $y=0.004$ 代入式(Ⅱ)

$$\left(\frac{\delta(x)}{r_c}\right)^2+\frac{1}{2}\left(-0.227-\sqrt{8\times0.004+(-0.227)^2-4\times0.581\times10^{-6}}\right)$$
$$+\left(0.004-\frac{(-0.227)\times0.004-0.104\times10^{-6}}{\sqrt{8\times0.004+(-0.227)^2-4\times0.581\times10^{-6}}}\right)=0 \quad (\text{Ⅱ})'$$

$$\left(\frac{\delta(x)}{r_c}\right)^2+\frac{1}{2}\left(0.227-\sqrt{0.0835}\right)\frac{\delta(x)}{r_c}+\left(0.004-\frac{-0.00091}{\sqrt{0.0835}}\right)=0 \quad (\text{Ⅱ})''$$

$$\left(\frac{\delta(x)}{r_c}\right)^2-0.258\cdot\frac{\delta(x)}{r_c}+0.00714=0 \quad (\text{Ⅱ})'''$$

$$\frac{\delta(x)}{r_c}=\frac{0.258-\sqrt{0.258^2-4\times0.00714}}{2}=\frac{0.258-0.195}{2}=0.0315$$

$$\delta(x)=0.2\times0.0315=0.0063\text{m}$$

当 X 取不同值时，分别进行计算 $\delta(x)$ 厚度，列入表 9-3。

表 9-3　$\delta(x)$ 厚度计算结果

X	0.02	0.05	0.1	0.5	1.0	2.0	3.0	4.0
$\delta(x)$/m	0.0063	0.0023	0.0024	0.0003	0.0061	0.6090	0.0097	无值

结论　依据反映具体流动状态的准数 Re，K_2，φ，可以计算出对应的边界层厚度 $\delta(x)$ 分布；边界层厚度是起伏波浪式曲面；喷嘴长度有一定限制，应在 $\frac{x}{r_c}<4$ 范围，超出此限，它已不是喷嘴，此法计算失效。

9.5.5 确定温度分布

因为流体运动速度是影响推动其他物理量变化的决定性的因素。既然流场分为理想流体与实际流体，则温度场也应分为理想流体运动区温度分布与实际流体运动区温度分布。

(1) 理想流体运动区温度分布

要找温度分布，必须由含温度的微分方程来完成。关键是建立理想流体运动区温度微分方程。

将式(9-5-28)代入式(9-5-27)

$$U_0\frac{\mathrm{d}U_0}{\mathrm{d}X}=-\frac{K_s\mathrm{d}p_0}{\rho_0\,\mathrm{d}X}=-K_5(1-X\sin\alpha)^2U_0\frac{\mathrm{d}p_0}{\mathrm{d}X}\tag{9-5-55}$$

将理想气体无因次状态方程(9-5-30)对 X 微分

$$\frac{\mathrm{d}p_0}{\mathrm{d}X}=\rho_0\frac{\mathrm{d}T_0}{\mathrm{d}X}+T_0\frac{\mathrm{d}\rho_0}{\mathrm{d}X}\tag{9-5-56}$$

因为

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}X}=\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{1}{U_0(1-X\sin\alpha)^2}\right)=\frac{1}{(1-X\sin\alpha)^2}\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{1}{U}\right)=-\frac{1}{(1-X\sin\alpha)^2U_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}X}\tag{9-5-57}$$

代入式(9-5-56)

$$\frac{\mathrm{d}p_0}{\mathrm{d}X}=\frac{1}{(1-X\sin\alpha)^2}\left(\frac{1}{U_0}\frac{\mathrm{d}T_0}{\mathrm{d}X}-\frac{T_0}{U_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}X}\right)\tag{9-5-58}$$

代入式(9-5-55)

$$U_0\frac{\mathrm{d}U_0}{\mathrm{d}X}=-K_5\left(\frac{\mathrm{d}T_0}{\mathrm{d}X}-\frac{T_0}{U_0}\frac{\mathrm{d}U_0}{\mathrm{d}X}\right)\tag{9-5-56$'$}$$

代入式(9-5-55)

$$\frac{\mathrm{d}T_0}{\mathrm{d}X}=\frac{T_0}{U_0}\frac{\mathrm{d}U_0}{\mathrm{d}X}-\frac{U_0}{K_5}\frac{\mathrm{d}U_0}{\mathrm{d}X}\tag{9-5-59}$$

将理想速度式(9-5-48)代入式(9-5-59)

$$\frac{\mathrm{d}T_0}{\mathrm{d}X}+\frac{0.3\frac{\delta(x)}{r_c}X^{-\frac{1}{2}}}{1-0.6\frac{\delta(x)}{r_c}X^{\frac{1}{2}}}T_0=\frac{0.3\frac{\delta(x)}{r_c}X^{-\frac{1}{2}}}{\left[1-2\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)\right]^2}\tag{9-5-60}$$

这是温度在理想流体运动区变化控制微分方程。将它写成下式

$$\frac{\mathrm{d}T_0}{\mathrm{d}X}+P(X)T_0=Q(X)\tag{9-5-60$'$}$$

式中：

$$P(X)=\frac{0.3\frac{\delta(x)}{r_c}X^{-\frac{1}{2}}}{1-0.6\frac{\delta(x)}{r_c}X^{\frac{1}{2}}}=\frac{aX^{-\frac{1}{2}}}{1-2aX^{\frac{1}{2}}}$$

$$Q(X)=\frac{0.3\dfrac{\delta(x)}{r_c}X^{-\frac{1}{2}}}{\left[1-2\left(\dfrac{\delta(x)}{r_c}+X\sin\alpha\right)\right]^2}=\eta X^{-\frac{1}{2}} \tag{a}$$

这是非齐次一阶线性微分方程，其解公式为

$$T_0=\mathrm{e}^{-\int P(X)\mathrm{d}X}\left[C_1+\int Q(X)\mathrm{e}^{\int P(X)\mathrm{d}X}\mathrm{d}X\right] \tag{9-5-61}$$

计算

$$\int P(X)\mathrm{d}X=\int\frac{aX^{-\frac{1}{2}}}{-2aX^{\frac{1}{2}}}\mathrm{d}X=a\int\frac{X^{-\frac{1}{2}}}{1-2aX^{\frac{1}{2}}}\mathrm{d}X$$

因为 $\mathrm{d}X=2X^{-\frac{1}{2}}\mathrm{d}(X^{\frac{1}{2}})$，代入上积分，则

$$a\int\frac{2X^{\frac{1}{2}}X^{-\frac{1}{2}}\mathrm{d}(X^{\frac{1}{2}})}{1-2aX^{\frac{1}{2}}}=2a\int\frac{\mathrm{d}(X^{\frac{1}{2}})}{1-2aX^{\frac{1}{2}}}=-2a\cdot\frac{1}{2a}\ln(1-2aX^{\frac{1}{2}})=-\ln(1-2aX^{\frac{1}{2}})$$

将其代入式(9-5-61)，则有

$$T_0=\mathrm{e}^{\ln(1-2aX^{\frac{1}{2}})}\left[C_1+\int Q(X)\mathrm{e}^{-\ln(1-2aX^{\frac{1}{2}})}\mathrm{d}X\right] \tag{9-5-61)$'$}$$

计算

$$\int Q(X)\mathrm{e}^{-\ln(1-2aX^{\frac{1}{2}})}\mathrm{d}X=\eta\int X^{-\frac{1}{2}}\mathrm{e}^{-\ln(1-2aX^{\frac{1}{2}})}\mathrm{d}X=2\eta\int\mathrm{e}^{-\ln(1-2aX^{\frac{1}{2}})}\mathrm{d}(X^{\frac{1}{2}})\approx 2\eta$$

$$\int\frac{\mathrm{d}X^{\frac{1}{2}}}{1+\ln(1-2aX^{\frac{1}{2}})}=2\eta\int\frac{\mathrm{d}X^{\frac{1}{2}}}{1+2\left(\dfrac{1-2aX^{\frac{1}{2}}-1}{1-2aX^{\frac{1}{2}}+1}\right)}=2\eta\int\frac{1-aX^{\frac{1}{2}}}{1-3aX^{\frac{1}{2}}}\mathrm{d}X^{\frac{1}{2}}$$

$$=2\eta\left(\int\frac{\mathrm{d}X^{\frac{1}{2}}}{1-3aX^{\frac{1}{2}}}-\int\frac{aX^{\frac{1}{2}}}{1-3aX^{\frac{1}{2}}}\mathrm{d}X^{\frac{1}{2}}\right)$$

$$=-2\eta\left\{\frac{1}{3a}\ln(1-3aX^{\frac{1}{2}})-\left[\frac{X^{\frac{1}{2}}}{3}+\frac{1}{9a}\ln(1-3aX^{\frac{1}{2}})\right]\right\} \tag{b}$$

将其代入式(9-5-61)′，则有

$$T_0=\mathrm{e}^{\ln(1-2aX^{\frac{1}{2}})}\left\{C_1+2\eta\left[\frac{2}{9a}\ln(1-3aX^{\frac{1}{2}})-\frac{X^{\frac{1}{2}}}{3}\right]\right\} \tag{9-5-62}$$

式中：

$$a=0.3\frac{\delta(x)}{r_c} \tag{c}$$

$$\eta=\frac{0.3\dfrac{\delta(x)}{r_c}}{\left[1-2\left(\dfrac{\delta(x)}{r_c}+X\sin\alpha\right)\right]^2} \tag{d}$$

根据边界条件，$X=0$，$T_0=1$，则 $C_1=1$，从而

$$T_0=e^{\ln(1-2aX^{\frac{1}{2}})}\left\{1+2\eta\left[\frac{2}{9a}\ln(1-3aX^{\frac{1}{2}})-\frac{X^{\frac{1}{2}}}{3}\right]\right\} \tag{9-5-62$'$}$$

例 9-8 题意同例 9-7 所述，此处只求 $X=0.5$，$X=2$ 两断面上理想流体运动区温度各为多少？

解 由例 9-7 可知，当 $X=0.5$ 时，$\dfrac{\delta(x)}{r_c}=0.0117$；$X=2$ 时，$\dfrac{\delta(x)}{r_c}=0.0452$。依此计算

$$e^{\ln(1-2aX^{\frac{1}{2}})}=e^{\ln(1-2\times0.3\times0.0117\times0.5^{\frac{1}{2}})}=e^{\ln(1-0.005)}=e^{\ln0.995}=l^{-0.005}=\frac{1}{1.005}=0.995$$

$$\eta=\frac{0.3\dfrac{\delta(x)}{r_c}}{\left[1-2\left(\dfrac{\delta(x)}{r_c}+X\sin\alpha\right)\right]^2}=\frac{0.3\times0.0117}{[1-2(0.0117+0.5\times0.05234)]^2}$$

$$=\frac{0.00351}{(1-0.0757)^2}=\frac{0.00351}{0.8543}=0.0041$$

$$\frac{2}{9a}=\frac{2}{9\times0.3\times0.0117}=\frac{2}{0.0316}=63.3112$$

$$\ln(1-3aX^{\frac{1}{2}})=\ln(1-3\times0.3\times0.0117(0.5)^{\frac{1}{2}})=\ln(1-0.0074)=-0.0074$$

将以上值代入式(9-5-62)$'$

$$T_0=e^{\ln(1-2aX^{\frac{1}{2}})}\left\{1+2\eta\left[\frac{2}{9a}\ln(1-3aX^{\frac{1}{2}})-\frac{X^{\frac{1}{2}}}{3}\right]\right\}$$

$$=0.995\left\{1+2\times0.0041\left[63.3112\times(-0.0074)-\frac{0.7071}{3}\right]\right\}$$

$$=0.995(1-2\times0.0041\times0.7042)=0.995\times0.99423=0.9883$$

$$T_0=\frac{T}{T_c}=0.9883$$

$$T=(273+30)\times0.9883=299.455\text{°K}$$

当 $X=2$ 时

$$e^{\ln(1-2aX^{\frac{1}{2}})}=e^{\ln(1-2\times0.3\times0.0452\times\sqrt{2})}=l^{\ln0.9616}=l^{-0.0392}=\frac{1}{1.04}=0.9612$$

$$\eta=\frac{0.3\times0.0452}{[1-2(0.0452+2\times0.05234)]^2}=\frac{0.0136}{0.76}=0.0179$$

$$\frac{2}{9a}=\frac{2}{9\times0.3\times0.0452}=\frac{2}{0.1224}=16.388$$

$$\ln(1-3aX^{\frac{1}{2}})=\ln(1-3\times 0.3\times 0.0452\sqrt{2})=\ln(1-0.0575)=-0.0592$$

将以上各值代入式(9-5-62)′，则为

$$\begin{aligned}T_0&=0.9612\left\{1+2\times 0.0179\left[16.388\times(-0.0592)-\frac{1.412}{3}\right]\right\}\\&=0.9612[1+0.0358\times(-1.4416)]=0.9612\times(1-0.0516)\\&=0.9612\times 0.9484=0.9116\end{aligned}$$

$$T=303\times 0.9116=276.2148^{\circ}\mathrm{K}$$

(2)边界层内温度分布

已设来流温度 t_c > 壁面温度 t_w。根据边界条件，及其受流体运动速度分布影响，选择边界层内温度分布为

$$T=1-X^{\alpha}(2Y_T-Y_T^{\frac{3}{2}}) \tag{9-5-63}$$

式中，$Y_T=\dfrac{y}{\delta_T(x)}$，$\delta_T(x)$是温度边界层厚度；$\alpha$ 是待定指数。确定边界层内温度分布就是确定 $\delta_T(x)$与 α。

① 确定 α 值。

将质量守恒式(9-5-30)代入式(9-5-32)，则气态方程为

$$\frac{\partial T}{\partial X}\frac{\partial}{\partial Y}\left(\frac{1}{U}\right)+\frac{\partial}{\partial X}\frac{1}{U}\frac{\partial T}{\partial Y}+\frac{1}{U}\frac{\partial^2 T}{\partial X\partial Y}+T_0\frac{\partial^2}{\partial X\partial Y}\left(\frac{1}{U}\right)=0 \tag{9-5-32′}$$

将上式展开后整理为

$$\frac{1}{U}\left(\frac{\partial T}{\partial Y}\frac{\partial U}{\partial X}+\frac{\partial T}{\partial X}\frac{\partial U}{\partial Y}+\frac{\partial^2 U}{\partial Y\partial X}\right)=\frac{\partial^2 T}{\partial X\partial Y}+\frac{2}{U^2}\frac{\partial U}{\partial Y}\frac{\partial U}{\partial X} \tag{9-5-64}$$

式中：

$$T=\frac{t-t_w}{t_c-t_w} \tag{9-5-65}$$

将速度分布式(9-5-44)与温度分布式(9-5-63)代入式(9-5-64)

$$\begin{aligned}&\frac{1}{X^{\frac{1}{2}}(2Y-Y^{\frac{3}{2}})}\left[-X^{\alpha}\left(2-\frac{3}{2}Y_T^{\frac{1}{2}}\right)\frac{1}{2}X^{-\frac{1}{2}}(2Y-Y^{\frac{3}{2}})-\alpha X^{\alpha-1}(2Y_T-Y_T^{\frac{3}{2}})X^{\frac{1}{2}}\left(2-\frac{3}{2}Y^{\frac{1}{2}}\right)+\right.\\&\left.\frac{1}{2}X^{-\frac{1}{2}}\left(2-\frac{3}{2}Y^{\frac{1}{2}}\right)\right]=-\alpha X^{\alpha-1}\left(2-\frac{3}{2}Y_T^{\frac{1}{2}}\right)+\frac{2}{X(2Y-Y^{\frac{3}{2}})^2}X^{\frac{1}{2}}\left(2-\frac{3}{2}Y^{\frac{1}{2}}\right)\frac{1}{2}X^{-\frac{1}{2}}(2Y-Y^{\frac{3}{2}})\end{aligned} \tag{9-5-64′}$$

将 $Y=1$，$Y_T=1$ 代入式(9-5-64)′，整理后为

$$X^{\alpha}=-0.5 \tag{9-5-64″}$$

取自然对数

$$\alpha\ln X=-\ln 0.5=0.693 \tag{9-5-64‴}$$

$$\alpha=\frac{0.693}{\ln X} \tag{9-5-64⁗}$$

α 应是常数，与 X 无关，可当 $X\to 0$，$\ln 0\to-\infty$，当 $X=1$，$\ln 1=0$，$\alpha=\infty$，均不合理。也就是说，由该式取得 α 与 X 无关无法实现。说明以前选的温度分布不合理，应改选为

$$T=1-X^{\alpha}(2Y_T-Y_T^{2.15}) \tag{9-5-66}$$

将式(9-5-66)与式(9-5-49)代入式(9-5-64)

$$\frac{1}{X^{\frac{1}{2}}(2Y-Y^{\frac{3}{2}})}\Big[-X^{\alpha}(2-2.15Y_T^{1.15})\frac{1}{2}X^{-\frac{1}{2}}(2Y-Y^{\frac{3}{2}})-\alpha X^{\alpha-1}(2Y_T-Y_T^{2.15})X^{\frac{1}{2}}\left(2-\frac{3}{2}Y^{\frac{1}{2}}\right)$$

$$+\frac{1}{2}X^{-\frac{1}{2}}\left(2-\frac{3}{2}Y^{\frac{1}{2}}\right)\Big]$$

$$=-\alpha X^{\alpha-1}(2-2.15Y^{1.15})+\frac{2}{X(2Y-Y^{\frac{3}{2}})^2}\Big[X^{\frac{1}{2}}\left(2-\frac{3}{2}Y^{\frac{1}{2}}\right)\frac{1}{2}X^{-\frac{1}{2}}(2Y-Y^{\frac{3}{2}})\Big] \tag{a)'$$

将 $Y=1$，$Y_T=1$ 代入上式，整理后为

$$0.075X^{\alpha-\frac{1}{2}}-0.5\alpha X^{\alpha-\frac{1}{2}}+0.25X^{-\frac{1}{2}}=0.15\alpha X^{\alpha-\frac{1}{2}}+0.05X^{-\frac{1}{2}} \tag{a)'$$

$$(0.075-0.65\alpha)X^{\alpha-\frac{1}{2}}=0.25X^{-\frac{1}{2}} \tag{a)''$$

$$(0.075-0.65\alpha)X^{\alpha}=0.25 \tag{a)'''$$

取对数

$$\ln(0.075-0.65\alpha)+\ln X^{\alpha}=\ln 0.25 \tag{a)''''$$

$$-\ln(0.65\alpha)+\ln X^{\alpha}=\ln 0.25-\ln 0.075-\ln\alpha+\ln X^{\alpha}=\ln 0.25-\ln 0.075+\ln 0.65$$

$$=\ln(0.25\times 0.65)-\ln 0.075=\ln\frac{0.1625}{0.075}=\ln 2.167 \tag{a)'''''$$

上式写成

$$\ln\frac{X^{\alpha}}{\alpha}=\ln 2.167 \tag{b}$$

$$\frac{X^{\alpha}}{\alpha}=2.167 \tag{b)'$$

因为 α 与 X 无关，与 Y 无关，它是一个常数，应与速度分布中 $X^{\frac{1}{2}}(2Y-Y^{\frac{3}{2}})$ 或 $\frac{1}{2}$ 近似。为达到此要求，取 $X=1$

$$\frac{1}{\alpha}=2.167,\ \alpha=0.461$$

故选定

$$T=1-X^{0.461}(2Y_T-Y_T^{2.15}) \tag{9-5-67}$$

② 确定温度边界层厚度 $\delta_T(x)$。

利用能量方程式(9-5-22)，确定温度边界层厚度 $\delta_T(x)$。为此，将质量守恒微分方程式(9-5-29)、边界层内速度分布与温度分布代入式(9-5-22)，同时也将理想流体运动微分方程式(9-5-27)代入，则为

$$ReU^2\frac{\partial U}{\partial X}+\frac{1}{2}Re\frac{U^2\partial U}{\partial X}+\frac{\varphi}{2}\frac{r_c}{\delta(x)}\frac{\partial^2 U}{\partial X\partial Y}+\frac{1}{2}\nu_0\frac{r_c}{\delta(x)}\frac{\partial U}{\partial Y}\left(-\frac{\partial U}{\partial X}+\frac{\partial U}{\partial X}\right)$$

$$=K_3\lambda_0\frac{1}{\rho_0}\left[\frac{\partial^2 T}{\partial X^2}+\left(\frac{r_c}{\delta_T(x)}\right)^2\frac{\partial^2 T}{\partial Y^2}\right]+Re\frac{U_0\mathrm{d}U_0}{\mathrm{d}X}U+\frac{4}{3}\nu_0 U\frac{\partial^2 U}{\partial X^2}-\frac{1}{2}\varphi^{\frac{2}{3}}\frac{\nu_0}{U}\frac{\partial U}{\partial Y}\cdot$$
$$\left[U_0\frac{\mathrm{d}U_0}{\mathrm{d}X}\frac{r_c}{\delta(x)}+\frac{\nu_0}{Re\delta(x)}\frac{r_c}{\partial X^2}\frac{\partial^2 U}{\partial X^2}+K_6\left(\frac{r_c}{\delta(x)}\right)^4\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right] \tag{9-5-68}$$

去掉相对小量，整理为

$$1.5ReU^2\frac{\partial U}{\partial X}+\frac{\varphi}{2}\frac{r_c}{\delta(x)}\frac{\partial^2 U}{\partial X\partial Y}$$
$$=\frac{K_3\lambda_0}{\rho_0}\left[\frac{\partial^2 T}{\partial X^2}+\left(\frac{r_c}{\delta_T(x)}\right)^2\frac{\partial^2 T}{\partial Y^2}\right]-ReUU_0\frac{\mathrm{d}U_0}{\mathrm{d}X}+\frac{4}{3}\nu_0 U\frac{\partial^2 U}{\partial X^2}-\frac{1}{2}\varphi^{\frac{2}{3}}\frac{\nu_0}{U}\frac{\partial U}{\partial Y}\cdot$$
$$\left[\frac{r_c}{\delta(x)}U_0\frac{\mathrm{d}U_0}{\mathrm{d}X}+K_6\left(\frac{r_c}{\delta(x)}\right)^4\frac{\partial U}{\partial Y}\frac{\partial^2 U}{\partial Y^2}\right] \tag{9-5-68$'$}$$

将边界层速度分布式(9-5-44)、温度分布式(9-5-67)与理想流体速度分布有关式(a)，一并代入式(9-5-68)′内，得

$$1.5ReX(2Y-Y^{\frac{3}{2}})\frac{1}{2}X^{-\frac{1}{2}}+\frac{\varphi}{2}\frac{r_c}{\delta(x)}\frac{1}{2}X^{-\frac{1}{2}}\left(2-\frac{3}{2}Y^{\frac{1}{2}}\right)$$
$$=K_3\lambda_0X^{\frac{1}{2}}(2Y-Y^{\frac{3}{2}})\left[0.248X^{-1.539}(2Y_T-Y_T^{2.15})+2.473\frac{r_c}{\delta_T(x)}X^{0.461}Y_T^{0.15}\right]$$
$$+ReX^{\frac{1}{2}}(2Y-Y^{\frac{3}{2}})\frac{-0.3\frac{\delta(x)}{r_c}X^{-\frac{1}{2}}}{\left[1-2\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)\right]^2}-\frac{1}{3}\nu_0X^{-1}(2Y-Y^{\frac{3}{2}})^2+\frac{1}{2}\varphi^{\frac{2}{3}}\frac{\nu_0\left(2-\frac{3}{2}Y^{\frac{1}{2}}\right)}{2Y-Y^{\frac{3}{2}}}\cdot$$
$$\left[\frac{r_c}{\delta(x)}\frac{0.3\frac{\delta(x)}{r_c}X^{-\frac{1}{2}}}{1-4\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)}+\nu_0K_6\left(\frac{r_c}{\delta(x)}\right)^4X\left(2-\frac{3}{2}Y^{\frac{1}{2}}\right)\frac{3}{4}Y^{-\frac{1}{2}}\right] \tag{9-5-68$''$}$$

取 $Y=1$，$Y_T=1$，则上式为

$$0.75ReX^{\frac{1}{2}}+\frac{\varphi}{8}\frac{r_c}{\delta(x)}X^{-\frac{1}{2}}=K_3\lambda_0\left[0.248X^{-1.039}+2.473\left(\frac{r_c}{\delta_T(x)}\right)^2X^{0.961}\right]-\frac{0.3Re\frac{\delta(x)}{r_c}}{1-4\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)}$$
$$-\frac{1}{3}\nu_0X^{-1}+\frac{1}{4}\varphi^{\frac{2}{3}}\nu_0\left[0.3\frac{r_c}{\delta(x)}X^{-\frac{1}{2}}+\frac{3}{8}K_6\nu_0\left(\frac{r_c}{\delta(x)}\right)^4X\right] \tag{9-5-68$'''$}$$

取 $\nu_0\approx1$，$\lambda_0\approx1$，去掉不合理项，如 $X^{-1}=\frac{1}{X}$，则

$$0.75ReX^{\frac{1}{2}}=2.473K_3\left(\frac{r_c}{\delta_T(x)}\right)^2X^{0.961}-\frac{0.3Re\frac{\delta(x)}{r_c}}{1-4\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)}+\frac{3}{32}\varphi^{\frac{2}{3}}K_6\left(\frac{r_c}{\delta(x)}\right)^4X \tag{9-5-68$''''$}$$

$$\delta_T(x)=r_c\sqrt{\frac{2.473K_3X^{0.961}}{Re\left[0.75X^{\frac{1}{2}}+\frac{0.3\frac{\delta(x)}{r_c}}{1-4\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)}-\frac{3}{32}\varphi^{\frac{2}{3}}K_6\left(\frac{r_c}{\delta(x)}\right)^4\right]}} \tag{9-5-69}$$

例 9-9　题意如例 9-7 所示，求 $X=0.5$，$X=1.0$，$X=2.0$ 三处的温度边界层 $\delta_T(x)$ 为多少？

解　当 $t_c=30$℃时，$\lambda_c=0.026\text{W}/(\text{m}\cdot\text{K})$

$$K_3=\frac{\lambda_c T_c}{\rho_c\nu_c u_c^2}=\frac{0.0267\times303\times10^6}{9.32\times16.04\times148.94^2}=2.44$$

$$K_6=\frac{2t\nu_c}{r_c^2}=\frac{2\times0.01\times16.04}{0.2^2\times10^6}=8.005\times10^{-6}$$

当 $X=0.5$ 时，$\frac{\delta(x)}{r_c}=0.0015$；当 $X=1.0$ 时，$\frac{\delta(x)}{r_c}=0.0305$；当 $X=2.0$ 时，$\frac{\delta(x)}{r_c}=0.045$。

通过实例运算，式(9-5-69)应改为

$$\delta_T(x)=r_c\sqrt{\frac{2.473K_3X^{0.961}}{Re\left[0.75X^{\frac{1}{2}}+\frac{0.3\frac{\delta(x)}{r_c}}{1-4\left(\frac{\delta(x)}{r_c}+X\sin\alpha\right)}\right]}} \tag{9-5-69$'$}$$

当 $X=0.5$ 时

$$\delta_T(x)=r_c\sqrt{\frac{2.473\times2.44\times0.5^{0.961}}{1.856\times10^6\left[0.75\times0.5^{\frac{1}{2}}+\frac{0.3\times0.0015}{1-4(0.0015+0.5\times0.05234)}\right]}}$$

$$=0.2\sqrt{\frac{3.464}{1.856\times10^6(0.5300+0.0041)}}=0.2\sqrt{\frac{3.464}{0.9912\times10^6}}$$

$$=0.2\times0.0019=0.00038\text{m}$$

当 $X=1$ 时

$$\delta_T(x)=0.2\sqrt{\frac{2.473\times2.44\times1}{1.856\times10^6\left[0.75+\frac{0.3\times0.0305}{1-4(0.0305+0.05234)}\right]}}$$

$$=0.2\sqrt{\frac{6.0341}{1.856\times10^6\left(0.75+\frac{0.00916}{0.6688}\right)}}=0.2\sqrt{\frac{6.0341}{1.856\times10^6\times0.764}}$$

$$=0.2\sqrt{0.00000462}=0.2\times0.00206=0.0004\text{m}$$

当 $X=2$ 时

$$\delta_T(x)=0.2\sqrt{\frac{6.0341\times2^{0.961}}{1.856\times10^6\left[0.75+\frac{0.3\times0.045}{1-4(0.045+2\times0.05234)}\right]}}$$

$$
\begin{aligned}
&= 0.2\sqrt{\frac{6.0341\times 1.947}{1.856\times 10^{6}\left(0.75+\dfrac{0.0135}{1-4\times 0.1497}\right)}}\\
&= 0.2\sqrt{\frac{11.7483}{1.856\times 10^{6}(0.75+0.0336)}}\\
&= 0.2\sqrt{\frac{11.7483}{1.856\times 0.7836\times 10^{6}}} = 0.2\sqrt{\frac{11.7483}{1.4544\times 10^{6}}}\\
&= 0.2\times 0.0028 = 0.00057\text{m}
\end{aligned}
$$

计算结果说明，温度边界层比流速边界层薄，因为边界层是湍流，热量传播快，这是合理的。

第 10 章　湍流自由射流

射流内容很广泛，不能进行全面讨论，自由射流是基本知识，而气体射流在工业上应用最广。本章就仅限于气体射流的分析与讨论。

10.1　湍流射流机理分析

湍流射流，是指它在射出前，在管道内是湍流，但是管中湍流特点对射出后的流动起着重要作用。射出后的流场，不存在着固体边界，不会形成边层流区，也就没有边层流界面，当然没有涡旋运动产生的可能性，所以流场不是湍流而是层流运动。

射出后的流股，必然带动周围的流体运动，它是用什么办法实现这个带动周围流体运动的呢？问题得从源头说起。流体在管道内受到壁面摩擦剪应力作用，这个力在流股射出后的瞬时内仍然起作用。它对原流股起着阻力作用，消耗流股动能，使流股最外层速度迅速下降，同时这个力对周围流体起着拉动作用，使它们也动起来，也就是通过这个力，原来流股的最外层动能把一部能量传递给周围流体，所以周围流体才得到动能，也跟着具有相应的速度运动起来。

当气体射入没有空间限制的静止气体中时，各点压力是一样的，剪应力消耗能量均由原流股动能下降来提供，使得流股沿轴向速度不断下降，直到某一距离速度为零。在剪应力的作用下，原流股把动能传递给周围流体的过程也是逐步进行的，随着沿轴向距离增加沿径上扩散范围也渐渐加大，也形成一个边界范围 $\delta(x)$。整个射流的流场内，质量是沿流程增加，但速度是下降的，最后沿轴向一定距离后动量为零。射流流场的情况如图 10-1 所示。

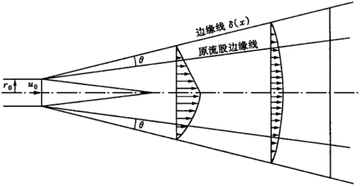


图 10-1　自由射流流场示意图

10.2 射流流场范围确定

湍流管道中，涡旋具有沿径向速度 u^* 与沿轴向速度 u_s。湍流流股射出的瞬时，它的涡旋运动速度仍然存在。它的合成速度是一个矢量，它的方向带动原流股向外扩散，其扩散角如图 10-2 所示，计算公式为

$$\theta = \arctan \frac{u^*}{u_s} \tag{10-2-1}$$

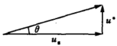


图 10-2 射流流股扩散角

式(10-2-1)说明湍流射流的特点，其具体计算方法详见例题。有 θ 就可以确定射流原流股的扩散边缘。如图 10-1 所示。湍流射流流场的边缘线确定方法：湍流管道壁面对流体有摩擦剪应力 $\Omega_{壁}$ 存在，它在流股射出的瞬时仍然起作用，对原流股继续起阻力作用，消耗流股的动能，对流股周围的流体有拉动作用，使其周围流体有相应的运动，其运动范围就是射流流场的边缘线。因此，它可以由下式确定

$$\Delta y = \frac{\mu v_x}{\Omega_{壁}} \tag{10-2-2}$$

式中：

$$v_x = \frac{v_0 r_0^2}{(r_0 + x\tan\theta)^2} \tag{10-2-3}$$

式(10-2-2)中，$\Omega_{壁}$ 是管道壁面剪应力，它可以由下式计算：

$$\Omega_{壁} = \frac{\frac{\mathrm{d}p}{\mathrm{d}x} r_0}{2(1 - \varphi^{\frac{2}{3}})} \tag{10-2-4}$$

将式(10-2-3)与式(10-2-4)代入式(10-2-2)

$$\Delta y = \frac{2\mu(1 - \varphi^{\frac{2}{3}}) v_0 r_0}{\frac{\mathrm{d}p}{\mathrm{d}x}(r_0 + x\tan\theta)^2} \tag{10-2-5}$$

式(10-2-5)就是被射出流股所带动的运动边界线。射出流股与被其带动的周围流体卷入运动时，沿径向断面速度分布为零的边缘线为

$$\delta(x) = r_0 + x\tan\theta + \frac{2\mu(1 - \varphi^{\frac{2}{3}}) v_0 r_0}{\frac{\mathrm{d}p}{\mathrm{d}x}(r_0 + x\tan\theta)^2} \tag{10-2-6}$$

射出流股长度确定：气体从管道射入等压静止流体中，其流形状是圆台状。依据动量改变等于作用力的原理计算其射流长度 l。管道壁面对流体运动有摩擦阻力，在射出的瞬时仍然起作用。圆台侧表面面积为

$$\pi l \sqrt{1 + \tan^2\theta}\,(2r_0 + l\tan\theta) \tag{a}$$

作用在侧表面上的阻力为

$$\pi l\sqrt{1+\tan^2\theta}(2r_0+l\tan\theta)\frac{\frac{\mathrm{d}p}{\mathrm{d}x}r_0}{2(1-\varphi^{\frac{2}{3}})} \tag{b}$$

令其与动量改变相等，则有

$$l\sqrt{1+\tan^2\theta}(2r_0+l\tan\theta)\frac{\frac{\mathrm{d}p}{\mathrm{d}x}r_0}{2(1-\varphi^{\frac{2}{3}})}=\rho r_0^2 v_0^2 \tag{c}$$

由式(c)得

$$\tan^2\theta l^2+2r_0 l-\frac{2\rho(1-\varphi^{\frac{2}{3}})v_0^2 r_0}{\sqrt{1+\tan^2\theta}\frac{\mathrm{d}p}{\mathrm{d}x}}=0 \tag{d}$$

由式(d)可得 l 的近似解

$$l=\frac{\rho v_0^2(1-\varphi^{\frac{2}{3}})}{\sqrt{1+\tan^2\theta}\frac{\mathrm{d}p}{\mathrm{d}x}} \tag{10-2-7}$$

精确解 l

$$l=\frac{-\frac{2r_0}{\tan\theta}+\sqrt{\left(\frac{2r_0}{\tan\theta}\right)^2+\frac{4\rho v_0^2(1-\varphi^{\frac{2}{3}})}{\tan\theta\sqrt{1+\tan^2\theta}\frac{\mathrm{d}p}{\mathrm{d}x}}}}{2} \tag{10-2-8}$$

10.3 射流沿轴与径向速度分布公式

从射出的流场是无法判别其为湍流还是层流射流。射流是湍流或是层流，要根据射出前的流态来决定，射前是层流，则称之为层流射流；射出前为湍流，则称之为湍流。

若从射出后的流场来看它的速度，无论沿径向还是沿轴向，均是逐渐变小的。由于流场中不存在边层流界面，不会产生涡旋运动，所以研究它的运动控制方程是层流方程。依式(4-4-11)并结合射流后流场特点，其控制微分方程为

$$u\frac{\partial u}{\partial x}=2\nu\frac{\partial^2 u}{\partial x^2}+\nu\frac{\partial^2 u}{\partial y^2} \tag{10-3-1}$$

以图 10-1 来分析，出口断面速度认为是均匀的，为 v_0，边界条件为

$$\left.\begin{array}{ll} u(x,\ y)|_{x=0}=v_0, & u(x,\ y)|_{y=\delta(x)}=0 \\ \left.\dfrac{\partial u}{\partial y}\right|_{y=0}=0, & \dfrac{\partial^2 u}{\partial y^2}<0 \end{array}\right\} \tag{10-3-2}$$

为进行数学分析，首先将上式无因次化，为此取

$$\frac{u}{v_0}=U,\ \frac{x}{r_0}=X,\ \frac{y}{\delta(x)}=Y$$

将它们代入式(10-3-1)与式(10-3-2)中，则有

$$U\frac{\partial U}{\partial X}=\frac{1}{Re}\left[2\frac{\partial^2 U}{\partial X^2}+\left(\frac{r_0}{\delta(x)}\right)^2\frac{\partial^2 U}{\partial Y^2}\right] \tag{10-3-3}$$

$$\left.\begin{array}{ll} U(X,\ Y)|_{X=0}=1, & U(X,\ Y)|_{Y=1}=0 \\ \left.\dfrac{\partial U}{\partial Y}\right|_{y=0}=0, & \dfrac{\partial^2 U}{\partial Y^2}<0 \end{array}\right\} \tag{10-3-4}$$

式(10-3-3)与式(10-3-4)就是研究射流后，流场速度分布的微分无因次方程。它是非线性偏微分方程。运用摄动法或其他方法未能解出。只能采用自创的变量分而不离的方法求解它们。设

$$U(X,\ Y)=U_1(X)U_2(Y) \tag{10-3-5}$$

将式(10-3-5)代入式(10-3-3)，则将偏微分方程变成常微分方程，不过它是两个常微分方程，为

$$U_1(X)U_2^2(Y)\frac{\mathrm{d}U(X)}{\mathrm{d}X}=\frac{1}{Re}\left[2U_2(Y)\frac{\mathrm{d}^2U(X)}{\mathrm{d}X^2}+\left(\frac{r_0}{\delta(x)}\right)^2U_2(X)\frac{\mathrm{d}^2U(Y)}{\mathrm{d}Y^2}\right] \tag{10-3-6}$$

运用逆向运算方法，首先根据边界条件，选 $U_2(Y)$的形函数为 $U_2(Y)=1-Y^2$，将它代入式(10-3-6)

$$\begin{aligned}(1-Y^2)^2U_1(X)\frac{\mathrm{d}U_1(X)}{\mathrm{d}X}&=\frac{1}{Re}\left[2(1-Y^2)\frac{\mathrm{d}^2U_1(X)}{\mathrm{d}X^2}-2\left(\frac{r_0}{\delta(x)}\right)^2U_1(X)\right]\\&=\frac{2}{Re}\left[(1-Y^2)\frac{\mathrm{d}^2U_1(X)}{\mathrm{d}X^2}-\left(\frac{r_0}{\delta(x)}\right)^2U_1(X)\right]\end{aligned} \tag{10-3-7}$$

式(10-3-7)是 $U_1(X)$的非线性常微分方程，对于 $U_1(X)$来说，无论 Y 取何值，式(10-3-7)均应成立。根据图 10-1 流场分析

$$U_1(X,\ Y)=U_1(X)U_2(Y)|_{Y=0}=U_1(X) \tag{10-3-8}$$

利用式(10-3-8)条件，选 $Y=0$，则式(10-3-7)变为

$$U_1(X)\left[\frac{\mathrm{d}U_1(X)}{\mathrm{d}X}+\left(\frac{r_0}{\delta(x)}\right)^2\right]=\frac{2}{\mathrm{Re}}\frac{\mathrm{d}^2U_1(X)}{\mathrm{d}X^2} \tag{10-3-9}$$

根据边界条件要求

$$U(X,\ Y)|_{X=0}=1 \tag{10-3-10}$$

依此选它的形函数为

$$U_1(X)=\frac{1}{2}\left[\frac{1}{1+X}+\frac{1}{(1+X)^a}\right] \tag{10-3-11}$$

式中，a 是待定常数。因为式(10-3-9)是二阶非线性的 $U_1(X)$常微分方程。按惯用的数学方法，积分后有两个常数，由两个边界条件来确定它们。现在，采用逆向运算方法，由边界条件选出 $U_1(X)$的表达式，其中含一个待定常数 a，它由式(10-3-9)来确定，为此将式(10-3-11)代入式(10-3-9)

$$\frac{1}{2}\left[\frac{1}{1+X}+\frac{1}{(1+X)^{a}}\right]\left[\frac{1}{(1+X)^{2}}+\frac{a}{(1+X)^{a+1}}+\left(\frac{r_0}{\delta(x)}\right)^{2}\right]=\frac{2}{Re}\left[\frac{-2}{(1+X)^{3}}-\frac{a(a+1)}{(1+X)^{a+2}}\right] \quad \text{(a)}$$

因为 a 是常数，无论 X 为何值，它都是不变的常数，所以 X 取零，则式(a)变为

$$\left[1+a+\left(\frac{r_0}{\delta(x)}\right)^{2}\right]=\frac{-2}{Re}(2+a^{2}+a) \quad \text{(b)}$$

将式(b)化简，取近似值，则 a 为

$$a=\frac{2+0.5Re\left[1+\left(\frac{r_0}{\delta(x)}\right)^{2}\right]}{1+0.5Re} \quad (10\text{-}3\text{-}12)$$

a 仍然是 $\delta(x)$ 有关的常数，因为在分析式(a)中的 X 是变量，而 $\delta(x)$ 是一个固定边缘。是不变的，一旦流场确定后，它就不变。也就是说，边缘已定的流场中研究 a 的值，最后它仍然是与 x 有关的变常数。

流场中速度分布公式为

$$U(X,\ Y)=\frac{1}{2}\left[\frac{1}{1+X}+\frac{1}{(1+X)^{a}}\right](1-Y^{2}) \quad (10\text{-}3\text{-}13)$$

变成有因次式

$$u(x,\ y)=0.5v_0\left[\frac{r_0}{r_0+x}+\left(\frac{r_0}{r_0+x}\right)^{a}\right]\left[1-\left(\frac{y}{\delta(x)}\right)^{2}\right] \quad (10\text{-}3\text{-}14)$$

式中 $\delta(x)$ 由式(10-2-6)计算，a 由式(10-3-12)计算。

第 11 章 柱坐标系流体运动微分方程

11.1 研究曲线坐标系的原因

寻找流场内的速度分布的目的，就是通过它来研究更深的、更实质性的问题。而流场内的速度分布特点与形成流场的边界条件和空间结构紧密相关，也就是说，不同的流场空间与边界条件，就会形成不同流场内速度分布。

要找到流场速度分布，必须运用数学分析。要运用数学分析，就得采用坐标系。是否可以用一个直角坐标系就可以解决各种不同流场速度分布呢？回答是能的。为研究人工笔直的渠道内速度分布，可以用直角坐标系进行研究，就可以找到其真正的速度分布；如采用柱坐标系，就不能找到其真正的速度分布。

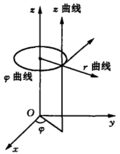


图 11-1 柱坐标系示意图

要找到不同的流场内的速度分布，必须结合其流场形成的结构特点，选择与其相适应的坐标系。坐标系是工具，是为解决不同流场的需要而创造出来的。本章以前多是采用直角坐标系研究问题，而实际工程中尚存在着不能用直角坐标系来研究的问题。所以在第 11 章讨论柱坐标系下流体运动微分方程，第 12 章讨论球坐标系下流体运动微分方程。每章均研究层流不可压缩运动微分方程组、层流可压缩运动微分方程组、湍流不可压缩运动微分方程组、湍流可压缩运动微分方程组。每组内，研究连续性微分方程、动量微分方程、动量矩微分方程、能量微分方程。柱坐标系示意图见图 11-1。

11.2 不可压缩层流微分方程组

11.2.1 连续性微分方程

利用不可压缩无坐标系质量守恒微分方程式(3-4-1)

$$\frac{\partial \rho}{\partial t}+\varphi^{\frac{2}{3}}\rho_s\nabla\cdot\boldsymbol{v}_s+\varphi_2\rho\nabla\cdot\boldsymbol{V}=0$$

当层流时 $\rho_s=0$，$v_s=0$，$\varphi=0$，$\varphi_2=1-\varphi^{\frac{2}{3}}$，则上式为

$$\frac{\partial \rho}{\partial t}+\rho\nabla\cdot\boldsymbol{V}=0 \tag{3-4-1)$'$}$$

当层流为不可压缩，$\rho=$ 常数，则为连续性微分方程

$$\nabla \cdot \boldsymbol{V}=0 \tag{11-2-1}$$

将式(11-2-1)变成柱坐标系对应形式，由曲线散度公式：

$$\nabla \cdot \boldsymbol{V}=\frac{1}{H_1H_2H_3}\left[\frac{\partial(V_1H_2H_3)}{\partial q_1}+\frac{\partial(V_2H_1H_3)}{\partial q_2}+\frac{\partial(V_3H_1H_2)}{\partial q_3}\right] \tag{a}$$

将柱坐标系 $H_1=1$，$H_2=r$，$H_3=1$ 代入式(a)

$$\nabla \cdot \boldsymbol{V}=\frac{1}{r}\left[\frac{\partial(u_r r)}{\partial r}+\frac{\partial v_\varphi}{\partial \varphi}+\frac{r\partial w_z}{\partial z}\right] \tag{11-2-2}$$

$$\nabla \cdot \boldsymbol{V}=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial v_\varphi}{r\partial \varphi}+\frac{\partial w_z}{\partial z} \tag{11-2-3}$$

式(11-2-3)就是柱坐标系不可压缩流层流连续性微分方程。

11.2.2 动量微分方程

由无坐标系不可压缩动量微分方程式(3-4-2)

$$\frac{\partial}{\partial t}(\varphi\rho_s \boldsymbol{v}_s+\varphi_1\rho \boldsymbol{v})+\varphi^{\frac{2}{3}}\rho_s\nabla\cdot \boldsymbol{V}_s\boldsymbol{V}_s+\varphi_2\rho\nabla\cdot \boldsymbol{V}\boldsymbol{V}=\boldsymbol{f}\rho+\nabla\cdot \boldsymbol{P}$$

当层流时，$\varphi=0$，$v_s=0$，$\rho_s=0$，$\varphi=1-\varphi^{\frac{2}{3}}$，并且为不可压缩流体，则上式为

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot \boldsymbol{P} \tag{11-2-4}$$

现将式(11-2-4)变成柱坐标系下对应方程：

$$\frac{\partial \boldsymbol{v}}{\partial t}+\left(u_r\frac{\partial}{\partial r}+v_\varphi\frac{\partial}{r\partial\varphi}+w_z\frac{\partial}{\partial z}\right)(u_r\boldsymbol{e}_r+v_\varphi\boldsymbol{e}_\varphi+w_z\boldsymbol{e}_z)$$
$$=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot[\boldsymbol{e}_r(p_{rr}\boldsymbol{e}_r+p_{\varphi r}\boldsymbol{e}_\varphi+p_{zr}\boldsymbol{e}_z)+\boldsymbol{e}_\varphi(p_{r\varphi}\boldsymbol{e}_r+p_{\varphi\varphi}\boldsymbol{e}_\varphi+p_{z\varphi}\boldsymbol{e}_z)+\boldsymbol{e}_z(p_{rz}\boldsymbol{e}_r+p_{\varphi z}\boldsymbol{e}_\varphi+p_{zz}\boldsymbol{e}_z)] \tag{11-2-4$'$}$$

$$\frac{\partial \boldsymbol{v}}{\partial t}+u_r\left(\boldsymbol{e}_r\frac{\partial u_r}{\partial r}+u_r\frac{\partial \boldsymbol{e}_r}{\partial r}+\boldsymbol{e}_\varphi\frac{\partial v_\varphi}{\partial r}+v_\varphi\frac{\partial \boldsymbol{e}_\varphi}{\partial r}+\boldsymbol{e}_z\frac{\partial w_z}{\partial r}+w_z\frac{\partial \boldsymbol{e}_z}{\partial r}\right)+v_\varphi\left(\boldsymbol{e}_r\frac{\partial u_r}{r\partial\theta}+u_r\frac{\partial \boldsymbol{e}_r}{r\partial\theta}\right.$$
$$\left.+\boldsymbol{e}_\varphi\frac{\partial v_\varphi}{r\partial\theta}+v_\varphi\frac{\partial \boldsymbol{e}_\varphi}{r\partial\theta}+\boldsymbol{e}_z\frac{\partial w_z}{r\partial\theta}+w_z\frac{\partial \boldsymbol{e}_z}{r\partial\theta}\right)$$
$$=\boldsymbol{f}+\frac{1}{\rho}\left(\boldsymbol{e}_r\frac{\partial p_{rr}}{\partial r}+\boldsymbol{e}_\varphi\frac{\partial p_{\varphi r}}{\partial r}+\boldsymbol{e}_z\frac{\partial p_{zr}}{\partial r}+\boldsymbol{e}_r\frac{\partial p_{r\varphi}}{r\partial\varphi}+\boldsymbol{e}_\varphi\frac{p_{r\varphi}}{r}+\boldsymbol{e}_\varphi\frac{\partial p_{\varphi\varphi}}{r\partial\varphi}-\boldsymbol{e}_r\frac{p_{\varphi\varphi}}{r}+\boldsymbol{e}_\varphi\frac{\partial p_{z\varphi}}{r\partial\varphi}\right.$$
$$\left.+\boldsymbol{e}_r\frac{\partial p_{rz}}{\partial z}+\boldsymbol{e}_\varphi\frac{\partial p_{\varphi z}}{\partial z}+\boldsymbol{e}_z\frac{\partial p_{zz}}{\partial z}\right) \tag{11-2-4$''$}$$

$$\frac{\partial \boldsymbol{v}}{\partial t}+u_r\left[\boldsymbol{e}_r\frac{\partial u_r}{\partial r}+u_r\left(-\boldsymbol{e}_\varphi\frac{\partial H_1}{r\partial\varphi}-\boldsymbol{e}_z\frac{\partial H_1}{\partial z}\right)+\boldsymbol{e}_\varphi\frac{\partial v_\varphi}{\partial r}+v_\varphi\frac{\boldsymbol{e}_r\partial H_1}{r\partial\varphi}+\boldsymbol{e}_z\frac{\partial w_z}{\partial r}+\boldsymbol{e}_z\frac{\partial H_1}{\partial z}\right]$$
$$v_\varphi\left(\boldsymbol{e}_r\frac{\partial u_r}{r\partial\theta}+u_r\frac{\boldsymbol{e}_r}{r}\frac{\partial H_2}{\partial r}+\boldsymbol{e}_\varphi\frac{\partial v_\varphi}{r\partial\theta}-\boldsymbol{e}_r\frac{v_\varphi}{r}+\boldsymbol{e}_z\frac{\partial w_z}{r\partial\theta}\right)$$
$$+w_\varphi\left[\boldsymbol{e}_r\frac{\partial u_r}{\partial z}+u_r\frac{\boldsymbol{e}_z\partial H_3}{\partial r}+\boldsymbol{e}_\varphi\frac{\partial v_\varphi}{\partial z}+v_\varphi\frac{\boldsymbol{e}_z}{r}\frac{\partial H_3}{\partial\theta}+\boldsymbol{e}_\varphi\frac{\partial w_z}{\partial z}+w_z\left(\frac{\boldsymbol{e}_r\partial H_3}{\partial r}-\boldsymbol{e}_\varphi\frac{\partial H_3}{r\partial\varphi}\right)\right]$$
$$=\boldsymbol{f}+\frac{1}{\rho}\left\{\boldsymbol{e}_r\frac{\partial}{\partial r}\left(-p+2\mu\frac{\partial u_r}{\partial r}\right)+\boldsymbol{e}_\varphi\frac{\partial}{\partial r}\mu\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial v_\varphi}{\partial r}-\frac{v_\varphi}{r}\right)+\frac{\partial\mu}{\partial r}\left(\frac{\partial u_r}{\partial z}+\frac{\partial w_z}{\partial r}\right)\right.$$

$$+e_r\frac{\partial\mu}{r\partial\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial u_r}{r\partial\varphi}-\frac{v_\varphi}{r}\right)+e_\varphi\frac{\mu}{r}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial u_r}{r\partial\varphi}-\frac{v_\varphi}{r}\right)+e_\varphi\frac{\partial}{r\partial\varphi}\left[-p+2\mu\left(\frac{\partial v_\varphi}{r\partial\varphi}+\frac{u_r}{r}\right)\right]$$

$$-e_r\frac{1}{r}\left[-p+2\mu\left(\frac{\partial v_\varphi}{r\partial\varphi}+\frac{u_r}{r}\right)\right]+e_\varphi\frac{\partial\mu}{r\partial\varphi}\left(\frac{\partial v_\varphi}{\partial z}+\frac{\partial w_z}{r\partial\varphi}\right)+e_r\frac{\partial\mu}{\partial z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)$$

$$+e_\varphi\frac{\partial\mu}{\partial z}\left(\frac{\partial w_z}{r\partial\varphi}+\frac{\partial v_\varphi}{\partial z}\right)+e_z\left[-p+2\mu\left(\frac{\partial w_z}{\partial z}\right)\right]\Bigg\}\tag{11-2-4}'''$$

$$\frac{\partial v}{\partial t}+u_r\left(e_r\frac{\partial u_r}{\partial r}+e_\varphi\frac{\partial v_\varphi}{\partial r}+e_z\frac{\partial w_z}{\partial r}\right)+v_\varphi\left(e_r\frac{\partial u_r}{r\partial\theta}+\frac{e_z}{r}u_r+e_\varphi\frac{\partial v_\varphi}{r\partial\theta}-\frac{e_r v_\varphi}{r}+e_z\frac{\partial w_z}{r\partial\theta}\right)$$

$$+w_z\left(e_r\frac{\partial\varphi_r}{\partial z}+e_\varphi\frac{\partial v_\varphi}{\partial z}+e_z\frac{\partial w_z}{\partial z}\right)$$

$$=f-\frac{1}{\rho}\left(\frac{\partial p}{\partial r}+\frac{r\partial\varphi}{\partial r}+\frac{\partial p}{\partial z}\right)+\nu\left[2\frac{\partial^2u_r}{\partial r^2}+\frac{\partial^2u_r}{r^2\partial\varphi^2}+\frac{\partial^2u_r}{\partial z^2}+\frac{\partial^2u_r}{r\partial\varphi\partial r}+\frac{\partial^2u_r}{\partial z\partial r}+2\left(\frac{\partial u_r}{r^2\partial\varphi}-\frac{u_r}{r^2}\right)+\frac{\partial^2v_\varphi}{\partial r}\right.$$

$$\left.+2\frac{\partial^2v_\varphi}{r^2\partial\varphi^2}+\frac{\partial^2v_\varphi}{\partial z^2}+\frac{\partial^2v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2v_\varphi}{r\partial\varphi\partial z}-3\frac{\partial^2v_\varphi}{r^2\partial\varphi}+\frac{\partial w_r}{\partial z^2}+\frac{\partial^2w_z}{r^2\partial\varphi^2}+2\frac{\partial^2w_z}{\partial z^2}+\frac{\partial^2w_z}{\partial r\partial z}+\frac{\partial^2w_z}{r\partial\varphi\partial z}\right]$$

$$\tag{11-2-4}''''$$

将式(11-2-4)''''按e_r，e_φ，e_z三个方向得出

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+v_\varphi\left(\frac{\partial u_r}{r\partial\varphi}+\frac{u_r}{r}\right)+w_\varphi\frac{\partial u_r}{\partial z}$$

$$=f_r-\frac{\partial p}{\rho\partial r}+\nu\left(2\frac{\partial^2u_r}{\partial r^2}+\frac{\partial^2u_r}{r^2\partial\varphi^2}+\frac{\partial^2u_r}{\partial z^2}+\frac{\partial^2u_r}{r\partial\varphi\partial r}+\frac{\partial^2u_r}{\partial z\partial r}+3\frac{\partial u_r}{r^2\partial\varphi}\right)\tag{11-2-5}$$

$$\frac{\partial v_\varphi}{\partial t}+u_r\frac{\partial v_\varphi}{\partial r}+v_\varphi\left(\frac{\partial v_\varphi}{r\partial\varphi}-\frac{v_\varphi}{r}\right)+w_z\frac{\partial v_\varphi}{\partial z}$$

$$=f_\varphi-\frac{\partial p}{\rho r\partial\varphi}+\nu\left(\frac{\partial^2v_\varphi}{\partial r^2}+2\frac{\partial^2v_\varphi}{r^2\partial\varphi^2}+\frac{\partial^2v_\varphi}{\partial z^2}+\frac{\partial^2v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2v_\varphi}{r\partial\varphi\partial z}-3\frac{\partial v_\varphi}{r^2\partial\varphi}\right)\tag{11-2-6}$$

$$\frac{\partial w_z}{\partial t}+u_r\frac{\partial w_z}{\partial r}+v_\varphi\frac{\partial w_z}{r\partial\varphi}+w_z\frac{\partial w_z}{\partial z}$$

$$=f_z-\frac{\partial p}{\rho\partial z}+\nu\left(\frac{\partial^2w_z}{\partial z^2}+\frac{\partial^2w_z}{r^2\partial\varphi^2}+2\frac{\partial^2w_z}{\partial z^2}+\frac{\partial^2w_z}{\partial r\partial z}+\frac{\partial^2w_z}{r\partial\varphi\partial z}\right)\tag{11-2-7}$$

式(11-2-5)至式(11-2-7)，是按坐标系下层流不可压缩流动动量微分方程，当定常流时去掉含$\frac{\partial}{\partial t}$项。

11.2.3 动量矩微分方程

由不可压缩流体无坐标系动量矩微分方程式(3-4-3)

$$\frac{\partial}{\partial t}\ [r\times\rho\ (\varphi V_s+\varphi_1V)]\ +\nabla\cdot V_s\ (r\times\varphi^{\frac{2}{3}}\rho V_s)\ +\nabla\cdot V\ (r\times\varphi_2\rho V)$$

$$=\ (r\times f)\ \rho+\nabla\cdot\ (r\times P)$$

当层流时 $\varphi=0$，$\varphi^{\frac{2}{3}}=0$，$\varphi_1=1-\varphi$，$\varphi_2=1-\varphi^{\frac{2}{3}}$，则上式为

$$\frac{\partial}{\partial t}(r\times\rho V)+\nabla\cdot V(r\times\rho V)=\ (r\times f)\ \rho+\nabla\cdot\ (r\times P)\tag{11-2-8}$$

除以 ρ，r 是参变量，则可写成

$$\boldsymbol{r}\times\frac{\partial \boldsymbol{V}}{\partial t}+\boldsymbol{r}\times(\boldsymbol{V}\cdot\nabla)=\boldsymbol{r}\times\boldsymbol{f}+\boldsymbol{r}\times\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{11-2-9}$$

利用式(11-2-5)至式(11-2-7)，则可写成

$$\begin{aligned}
&r_\varphi\frac{\partial w_z}{\partial t}-r_z\frac{\partial v_\varphi}{\partial t}+r_\varphi\left(u_r\frac{\partial w_z}{\partial r}+v_\varphi\frac{\partial w_z}{r\partial\varphi}+w_z\frac{\partial w_z}{\partial z}\right)-r_z\left[u_r\frac{\partial v_\varphi}{\partial r}+v_\varphi\left(\frac{\partial v_\varphi}{r\partial\varphi}-\frac{v_\varphi}{r}\right)+w_z\frac{\partial v_\varphi}{\partial z}\right]\\
&=r_\varphi f_z-r_zf_\varphi+\frac{1}{\rho}\left(r_z\frac{\partial p}{\partial z}-\frac{\partial p}{r\partial\varphi}\right)+\nu\left[r_\varphi\left(\frac{\partial^2 w_z}{\partial z^2}+\frac{\partial^2 w_z}{r^2\partial\varphi^2}+2\frac{\partial^2 w_z}{\partial z^2}+\frac{\partial^2 w_z}{\partial r\partial z}+\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)\right.\\
&\quad\left.-r_z\left(\frac{\partial^2 v_\varphi}{\partial r^2}+2\frac{\partial^2 v_\varphi}{r^2\partial\varphi^2}+\frac{\partial^2 v_\varphi}{\partial z^2}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}-\frac{\partial v_\varphi}{r^2\partial\varphi}\right)\right]
\end{aligned} \tag{11-2-10}$$

$$\begin{aligned}
&r_z\frac{\partial u_r}{\partial t}-r_r\frac{\partial w_z}{\partial t}+r_z\left[u_r\frac{\partial u_r}{\partial r}+v_\varphi\left(\frac{\partial u_r}{r\partial\varphi}+\frac{u_r}{r}\right)+w_\varphi\frac{\partial u_z}{\partial z}\right]-r_r\left(u_r\frac{\partial w_z}{\partial r}+v_\varphi\frac{\partial w_z}{r\partial\varphi}+w_z\frac{\partial w_z}{\partial z}\right)\\
&=r_zf_r-r_rf_z+\frac{1}{\rho}\left(\frac{\partial p}{\partial r}-\frac{\partial p}{\partial z}\right)+\nu\left[r_z\left(2\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial^2 u_r}{r^2\partial\varphi^2}+\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial^2 u_r}{r\partial\varphi\partial r}+\frac{\partial^2 u_r}{\partial z\partial r}\right.\right.\\
&\quad\left.\left.+2\frac{\partial u_r}{r^2\partial\varphi}+\frac{\partial u_r}{r^2\partial\varphi}-\frac{u_r}{r^2}\right)-r_r\left(\frac{\partial^2 w_z}{\partial z^2}+\frac{\partial^2 w_z}{r^2\partial\varphi^2}+2\frac{\partial^2 w_z}{\partial z^2}+\frac{\partial^2 w_z}{\partial r\partial z}+\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)\right]
\end{aligned} \tag{11-2-11}$$

$$\begin{aligned}
&r_r\frac{\partial v_\varphi}{\partial t}-r_\varphi\frac{\partial u_r}{\partial t}+r_r\left[u_r\frac{\partial v_\varphi}{\partial r}+v_\varphi\left(\frac{\partial v_\varphi}{r\partial\varphi}-\frac{v_\varphi}{r}\right)+w_z\frac{\partial v_\varphi}{\partial z}\right]-r_\varphi\left[u_r\frac{\partial u_r}{\partial r}+v_\varphi\left(\frac{\partial u_r}{r\partial\varphi}+\frac{u_r}{r}\right)+w_z\frac{\partial u_r}{\partial z}\right]\\
&=r_rf_\varphi-r_\varphi f_r+\frac{1}{\rho}\left(\frac{\partial p}{r\partial\varphi}-\frac{\partial p}{\partial r}\right)+\nu\left[r_r\left(\frac{\partial^2 v_\varphi}{\partial z^2}+2\frac{\partial^2 v_\varphi}{r^2\partial\varphi^2}+\frac{\partial^2 v_\varphi}{\partial z^2}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}-3\frac{\partial v_\varphi}{r^2\partial\varphi}\right)\right.\\
&\quad\left.-r_\varphi\left(2\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial^2 u_r}{r^2\partial\varphi^2}+\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial^2 u_r}{r\partial\varphi\partial r}+\frac{\partial^2 u_r}{\partial z\partial r}+3\frac{\partial u_r}{r^2\partial\varphi}\right)\right]
\end{aligned} \tag{11-2-12}$$

式(11-2-10)至式(11-2-12)，是柱坐标系下不可压缩流动量矩微分方程，当定常流时去掉含$\frac{\mathrm{D}}{\mathrm{D}t}$项。

11.2.4 能量微分方程

由不可压缩流体无坐标系能量微分方程式(3-4-4)

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\varphi\frac{v_s^2}{2}+\varphi_1\frac{v^2}{2}\right)+\nabla\cdot\left(\frac{v_s^2}{2}\boldsymbol{V}_s+\varphi_2\frac{v^2}{2}\boldsymbol{V}\right)\\
&=\frac{1}{\rho}\nabla(\lambda\nabla T)+q_{\mathrm{R}}+\boldsymbol{f}\cdot(\varphi\boldsymbol{V}_s+\varphi_1\boldsymbol{V})+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{V}_s+\varphi_2\boldsymbol{V})
\end{aligned}$$

当层流时，$\boldsymbol{V}_s=0$，$v_s=0$，$\varphi=0$，$\varphi^{\frac{2}{3}}=0$，$\varphi_1=1-\varphi$，$\varphi_2=1-\varphi^{\frac{2}{3}}$。则式(3-4-4)变为

$$\frac{\partial}{\partial t}\left(\frac{v_s^2}{2}\right)+(\boldsymbol{V}\cdot\nabla)\frac{v^2}{2}=\frac{1}{\rho}\nabla(\lambda\nabla T)+q_{\mathrm{R}}+\boldsymbol{f}\cdot\boldsymbol{V}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot\boldsymbol{V} \tag{11-2-13}$$

将式(11-2-13)转变为柱坐标系下对应方程

$$\frac{\partial}{\partial t}\left(\frac{v^2}{2}\right)+u_r\frac{\partial}{\partial r}\left(\frac{v^2}{2}\right)+v_\varphi\frac{\partial}{r\partial\varphi}\left(\frac{v^2}{2}\right)+w_z\frac{\partial}{\partial z}\left(\frac{v_s^2}{2}\right)$$

$$
\begin{aligned}
&=\frac{\lambda}{\rho}\left(\frac{\partial^2 T}{\partial r^2}+\frac{\partial^2 T}{r^2\partial\varphi^2}+\frac{\partial^2 T}{\partial z^2}\right)+q_R+f_r u_r+f_\varphi v_\varphi+f_z w_z-\frac{1}{\rho}\left(\frac{\partial p}{\partial r}u_r+\frac{\partial p}{r\partial\varphi}v_\varphi+\frac{\partial p}{\partial z}w_z\right)\\
&\quad+\nu\left[u_r\left(2\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial^2 u_r}{r^2\partial\varphi^2}+\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial^2 u_r}{r\partial\varphi\partial r}+\frac{\partial^2 u_r}{\partial z\partial r}+3\frac{\partial u_r}{r^2\partial\varphi}\right)+v_\varphi\left(\frac{\partial^2 v_\varphi}{\partial r^2}+2\frac{\partial^2 v_\varphi}{r^2\partial\varphi^2}+\frac{\partial^2 v_\varphi}{\partial z^2}\right.\right.\\
&\quad\left.\left.+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}-\frac{\partial v_\varphi}{r^2\partial\varphi}\right)+w_z\left(\frac{\partial^2 w_z}{\partial z^2}+\frac{\partial^2 w_z}{r^2\partial\varphi^2}+2\frac{\partial^2 w_z}{\partial z^2}+\frac{\partial^2 w_z}{\partial r\partial z}+\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)\right]
\end{aligned}
\quad (11\text{-}2\text{-}14)
$$

式(11-2-14)是柱坐标系下不可压缩层流运动能量微分方程。当定常流时，去掉含$\frac{\partial}{\partial t}$项。式中：

$$v^2=u_r^2+v_\varphi^2+w_z^2$$

11.3 可压缩层流微分方程组

11.3.1 连续性微分方程

由无坐标系可压缩流体运动质量守恒微分方程式(3-7-1)

$$\varphi\frac{\mathrm{D}\rho_s}{\mathrm{D}t}+(1-\varphi)\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho_s\nabla\cdot\boldsymbol{V}_s+\rho\nabla\cdot\boldsymbol{V}=0$$

当层流时，$\rho_s=0$，$\boldsymbol{V}_s=0$，$\varphi=0$，则

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\nabla\cdot\boldsymbol{V}=0 \quad (11\text{-}3\text{-}1)$$

将式(11-3-1)转变为柱坐标系下对应方程

$$\frac{\partial\rho}{\partial t}+u_r\frac{\partial\rho}{\partial r}+v_\varphi\frac{\partial\rho}{r\partial\varphi}+w_z\frac{\partial\rho}{\partial z}+\rho\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial v_\varphi}{r\partial\varphi}+\frac{\partial w_z}{\partial z}\right)=0 \quad (11\text{-}3\text{-}2)$$

11.3.2 动量微分方程

由无坐标系下可压缩流体运动动量微分方程式(3-7-2)

$$\varphi\frac{\mathrm{D}(\rho_s\boldsymbol{V}_s)}{\mathrm{D}t}+\varphi_1\frac{\mathrm{D}(\rho\boldsymbol{V})}{\mathrm{D}t}+\rho_s\boldsymbol{V}_s(\nabla\cdot\boldsymbol{V}_s)+\rho\boldsymbol{V}(\nabla\cdot\boldsymbol{V})=\boldsymbol{f}[\varphi\rho_s+(1-\varphi)\rho]+\nabla\cdot\boldsymbol{P}$$

当层流时，$\rho_s=0$，$\varphi=0$，$\varphi_1=1-\varphi$，则

$$\frac{\mathrm{D}(\rho\boldsymbol{V})}{\mathrm{D}t}+\rho\boldsymbol{V}(\nabla\cdot\boldsymbol{V})=\boldsymbol{f}\rho+\nabla\cdot\boldsymbol{P} \quad (11\text{-}3\text{-}3)$$

将式(11-3-1)代入式(11-3-3)，则有

$$\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \quad (11\text{-}3\text{-}4)$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}(u_r\boldsymbol{e}_r+v_\varphi\boldsymbol{e}_\varphi+w_z\boldsymbol{e}_z)\\
&=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot[(p_{rr}\boldsymbol{e}_r+p_{r\varphi}\boldsymbol{e}_\varphi+p_{rz}\boldsymbol{e}_z)\boldsymbol{e}_r+(p_{\varphi r}\boldsymbol{e}_r+p_{\varphi\varphi}\boldsymbol{e}_\varphi+p_{\varphi z}\boldsymbol{e}_z)\boldsymbol{e}_\varphi+(p_{zr}\boldsymbol{e}_r+p_{z\varphi}\boldsymbol{e}_\varphi+p_{zz}\boldsymbol{e}_z)\boldsymbol{e}_z]
\end{aligned}
\quad (11\text{-}3\text{-}4)'
$$

$$\frac{\partial(u_r\boldsymbol{e}_r)}{\partial t}+u_r\frac{\partial(u_r\boldsymbol{e}_r)}{\partial r}+v_\varphi\frac{\partial(u_r\boldsymbol{e}_r)}{r\partial\varphi}+w_z\frac{\partial(u_r\boldsymbol{e}_r)}{\partial z}+\frac{\partial(v_\varphi\boldsymbol{e}_\varphi)}{\partial t}+u_r\frac{\partial(v_\varphi\boldsymbol{e}_\varphi)}{\partial r}+v_\varphi\frac{\partial(v_\varphi\boldsymbol{e}_\varphi)}{r\partial\varphi}+$$
$$w_z\frac{\partial(v_\varphi\boldsymbol{e}_\varphi)}{\partial z}+\frac{\partial(w_z\boldsymbol{e}_z)}{\partial t}+u_r\frac{\partial(u_z\boldsymbol{e}_z)}{\partial r}+v_\varphi\frac{\partial(w_z\boldsymbol{e}_z)}{r\partial\varphi}+w_z\frac{\partial(w_z\boldsymbol{e}_z)}{\partial z}$$
$$=\boldsymbol{f}+\frac{1}{\rho}\left[\frac{\partial}{\partial r}(p_{rr}\boldsymbol{e}_r+p_{r\varphi}\boldsymbol{e}_\varphi+p_{rz}\boldsymbol{e}_z)+\frac{\partial}{r\partial\varphi}(p_{r\varphi}\boldsymbol{e}_r+p_{\varphi\varphi}\boldsymbol{e}_\varphi+p_{\varphi z}\boldsymbol{e}_z)+\frac{\partial}{\partial z}(p_{rz}\boldsymbol{e}_r+p_{\varphi z}\boldsymbol{e}_\varphi+p_{zz}\boldsymbol{e}_z)\right]$$
$$(11\text{-}3\text{-}4)''$$

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+v_\varphi\left(\frac{\partial u_r}{r\partial\varphi}+\frac{u_r}{r}\right)+w_z\frac{\partial u_r}{\partial z}+\frac{\partial v_\varphi}{\partial t}+u_r\frac{\partial v_\varphi}{\partial r}+v_\varphi\left(\frac{\partial v_\varphi}{r\partial\varphi}-\frac{v_\varphi}{r}\right)+w_z\frac{\partial v_\varphi}{\partial z}+\frac{\partial w_z}{\partial t}$$
$$+u_r\frac{\partial w_z}{\partial t}+u_r\frac{\partial w_z}{\partial r}+v_\varphi\frac{\partial w_z}{r\partial\varphi}+w_z\frac{\partial w_z}{\partial z}$$
$$=\boldsymbol{f}-\frac{\partial p}{\rho\partial r}+\nu\left[\frac{4}{3}\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial^2 u_r}{r^2\partial\varphi^2}+\frac{\partial^2 u_r}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2 u_r}{\partial z\partial r}+\frac{\partial^2 u_r}{r\partial\varphi\partial r}+\frac{\partial u_r}{r^2\partial\varphi}\right)-\frac{2}{3}\left(\frac{\partial u_r}{r\partial z}+\frac{u_r}{r^2}\right)\right.$$
$$+\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{4}{3}\frac{\partial^2 v_\varphi}{r^2\partial\varphi^2}+\frac{\partial^2 v_\varphi}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}\right)-\frac{7}{3}\frac{\partial v_\varphi}{r^2\partial\varphi}+\frac{\partial^2 w_z}{\partial r^2}+\frac{\partial^2 w_z}{r^2\partial\varphi^2}+\frac{4}{3}\frac{\partial^2 w_z}{\partial z^2}$$
$$\left.+\frac{1}{3}\left(\frac{\partial^2 w_z}{\partial r\partial z}+\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)+\frac{2}{3}\frac{\partial w_z}{r\partial z}\right]\qquad(11\text{-}3\text{-}4)'''$$

将式(11-3-4)‴按$\boldsymbol{e}_r$，$\boldsymbol{e}_\varphi$，$\boldsymbol{e}_z$三个方向列出

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+v_\varphi\left(\frac{\partial u_r}{r\partial\varphi}+\frac{u_r}{r}\right)+w_z\frac{\partial u_r}{\partial z}$$
$$=f_r-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{4}{3}\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial^2 u_r}{r^2\partial\varphi^2}+\frac{\partial^2 u_r}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2 u_r}{\partial z\partial r}+\frac{\partial^2 u_r}{r\partial\varphi\partial r}+3\frac{\partial u_r}{r^2\partial\varphi}\right)-\frac{2}{3}\left(\frac{\partial u_r}{r\partial z}+\frac{u_r}{r^2}\right)\right]$$
$$(11\text{-}3\text{-}5)$$

$$\frac{\partial v_\varphi}{\partial t}+u_r\frac{\partial v_\varphi}{\partial r}+v_\varphi\left(\frac{\partial v_\varphi}{r\partial\varphi}-\frac{v_\varphi}{r}\right)+w_z\frac{\partial v_\varphi}{\partial z}$$
$$=f_\varphi-\frac{\partial p}{\rho r\partial\varphi}+\nu\left[\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{4}{3}\frac{\partial^2 v_\varphi}{r^2\partial\varphi^2}+\frac{\partial^2 v_\varphi}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}\right)-\frac{7}{3}\frac{\partial v_\varphi}{r^2\partial\varphi}\right]\qquad(11\text{-}3\text{-}6)$$

$$\frac{\partial w_z}{\partial t}+u_r\frac{\partial w_z}{\partial r}+v_\varphi\frac{\partial w_z}{r\partial\varphi}+w_z\frac{\partial w_z}{\partial z}$$
$$=f_z-\frac{\partial p}{\rho\partial z}+\nu\left[\frac{\partial^2 w_z}{\partial r^2}+\frac{\partial^2 w_z}{r^2\partial\varphi^2}+\frac{4}{3}\frac{\partial^2 w_z}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2 w_z}{\partial r\partial z}+\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)+\frac{2}{3}\frac{\partial w_z}{r\partial z}\right]\qquad(11\text{-}3\text{-}7)$$

以上式(11-3-5)至式(11-3-7)，是柱坐标系下可压缩层流运动动量微分方程。

11.3.3 动量矩微分方程

由无坐标系可压缩流体动量矩微分方程式(3-7-3)

$$\varphi\frac{\mathrm{D}(\boldsymbol{r}\times\rho_s\boldsymbol{V}_s)}{\mathrm{D}t}+(\boldsymbol{r}\times\rho\boldsymbol{V}_s)\nabla\cdot\boldsymbol{V}_s+\varphi_1\frac{\mathrm{D}(\boldsymbol{r}\times\rho\boldsymbol{V})}{\mathrm{D}t}+(\boldsymbol{r}\times\rho\boldsymbol{V})\nabla\cdot\boldsymbol{V}$$
$$=\boldsymbol{r}\times[\varphi\rho_s+(1-\varphi)\rho]+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P})$$

当层流时，$\rho_s=0$，$V_s=0$，$\varphi=0$，$\varphi^{\frac{2}{3}}=0$，$\varphi_1=1-\varphi$，则

$$\frac{D(r\times\rho V)}{Dt}+(r\times\rho V)\nabla\cdot V=r\times f\rho+\nabla\cdot(r\times P)\tag{11-3-8}$$

将式(11-3-1)代入式(11-3-8)

$$\rho\frac{D(r\times V)}{Dt}+(r\times V)\frac{D\rho}{Dt}-(r\times V)\frac{D\rho}{Dt}=\rho(r\times f)+\nabla\cdot(r\times P)\tag{11-3-8$'$}$$

因为 r 是参变量，并将全式除以 ρ，则为

$$r\times\frac{DV}{Dt}=r\times f+r\times\frac{1}{\rho}\nabla\cdot P\tag{11-3-9}$$

利用层流可压缩柱坐标系下的动量方程，则可得

$$\begin{aligned}&r_\varphi\left(\frac{\partial w_z}{\partial t}+u_r\frac{\partial w_z}{\partial r}+v_\varphi\frac{\partial w_z}{r\partial\varphi}+w_z\frac{\partial w_z}{\partial z}\right)-r_z\left[\frac{\partial v_\varphi}{\partial t}+u_r\frac{\partial v_\varphi}{\partial r}+v_\varphi\left(\frac{\partial v_\varphi}{r\partial\varphi}-\frac{v_\varphi}{r}\right)+w_z\frac{\partial v_\varphi}{\partial z}\right]\\&=r_\varphi f_z-r_zf_\varphi+\frac{1}{\rho}\left(\frac{\partial p}{\partial z}-\frac{\partial p}{r\partial\varphi}\right)+\nu\left\{r_\varphi\left[\frac{\partial^2w_z}{\partial r^2}+\frac{\partial^2w_z}{r^2\partial\varphi^2}+\frac{4}{3}\frac{\partial^2w_z}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2w_z}{\partial r\partial z}+\frac{\partial^2w_z}{r\partial\varphi\partial z}\right)\right.\right.\\&\quad\left.\left.+\frac{2}{3}\frac{\partial w_\varphi}{r\partial z}\right]-r_z\left[\frac{\partial^2v_\varphi}{\partial r^2}+\frac{4}{3}\frac{\partial^2v_\varphi}{r^2\partial\varphi^2}+\frac{\partial^2v_\varphi}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2v_\varphi}{r\partial\varphi\partial z}\right)-\frac{7}{3}\frac{\partial v_\varphi}{r^2\partial\varphi}\right]\right\}\end{aligned}\tag{11-3-10}$$

$$\begin{aligned}&r_z\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+v_\varphi\left(\frac{\partial u_r}{r\partial\varphi}+\frac{u_r}{r}\right)+w_z\frac{\partial u_z}{\partial z}\right]-r_r\left(\frac{\partial w_z}{\partial t}+u_r\frac{\partial w_z}{\partial r}+v_\varphi\frac{\partial w_z}{r\partial\varphi}+w_z\frac{\partial w_z}{\partial z}\right)\\&=r_zf_r-r_rf_z+\frac{1}{\rho}\left(\frac{\partial p}{\partial r}-\frac{\partial p}{\partial z}\right)+\nu\left\{r_z\left[\frac{4}{3}\frac{\partial^2u_r}{\partial r^2}+\frac{\partial^2u_r}{r^2\partial\varphi^2}+\frac{\partial^2u_r}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2u_r}{\partial z\partial r}+\frac{\partial^2u_r}{r\partial\varphi\partial r}+\frac{\partial u_r}{r^2\partial\varphi}\right)\right.\right.\\&\quad\left.\left.-\frac{2}{3}\left(\frac{\partial u_r}{r\partial z}-\frac{u_r}{r^2}\right)\right]-r_r\left[\frac{\partial^2w_z}{\partial r^2}+\frac{\partial^2w_z}{r^2\partial\varphi^2}+\frac{4}{3}\frac{\partial^2w_z}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2w_z}{\partial r\partial z}+\frac{\partial^2w_z}{r\partial\varphi\partial z}\right)+\frac{2}{3}\frac{\partial w_z}{r\partial z}\right]\right\}\end{aligned}\tag{11-3-11}$$

$$\begin{aligned}&r_r\left[\frac{\partial v_\varphi}{\partial t}+u_r\frac{\partial v_\varphi}{\partial r}+v_\varphi\left(\frac{\partial v_\varphi}{r\partial\varphi}-\frac{v_\varphi}{r}\right)+w_z\frac{\partial v_\varphi}{\partial z}\right]-r_\varphi\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+v_\varphi\left(\frac{\partial u_r}{r\partial\varphi}+\frac{u_r}{r}\right)+w_z\frac{\partial u_r}{\partial z}\right]\\&=r_rf_\varphi-r_\varphi f_r+\frac{1}{\rho}\left(\frac{\partial p}{r\partial\varphi}-\frac{\partial p}{\partial r}\right)+\nu\left\{r_r\left[\frac{\partial^2v_\varphi}{\partial r^2}+\frac{4}{3}\frac{\partial^2v_\varphi}{r^2\partial\varphi^2}+\frac{\partial^2v_\varphi}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2v_\varphi}{r\partial\varphi\partial z}\right)\right.\right.\\&\quad\left.\left.-\frac{7}{3}\frac{\partial v_\varphi}{r^2\partial\varphi}\right]-r_\varphi\left[\frac{4}{3}\frac{\partial^2u_r}{\partial r^2}+\frac{\partial^2u_r}{r^2\partial\varphi^2}+\frac{\partial^2u_r}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2u_r}{\partial z\partial r}+\frac{\partial^2u_r}{r\partial\varphi\partial r}+\frac{\partial u_r}{r^2\partial\varphi}\right)-\frac{2}{3}\left(\frac{\partial u_r}{r\partial z}+\frac{u_r}{r^2}\right)\right]\right\}\end{aligned}\tag{11-3-12}$$

式(11-3-10)至式(11-3-12)，是柱坐标系下层流可压缩动量矩微分方程。

11.3.4 能量微分方程

由无坐标系下可压缩流体能量微分方程式(3-7-4)

$$\begin{aligned}&\varphi\frac{D}{Dt}\left[\rho_s\left(e_s+\frac{v_s^2}{2}\right)\right]+\rho_s\left(e_s+\frac{v_s^2}{2}\right)\nabla\cdot V_s+\varphi_1\frac{D}{Dt}\left[\rho\left(e+\frac{v^2}{2}\right)\right]+\rho\left(e+\frac{v^2}{2}\right)\nabla\cdot V\\&=\nabla\cdot(\lambda\nabla T)+q_R\rho+f(\varphi\rho_sV_s+\varphi_1\rho V)+\nabla\cdot P\cdot(\varphi^{\frac{2}{3}}V_s+\varphi_2V)\end{aligned}$$

当层流时，$\rho_s=0$，$V_s=0$，$v_s=0$，$\varphi=0$，$\varphi^{\frac{2}{3}}=0$，$\varphi_1=(1-\varphi)$，$\varphi_2=1-\varphi^{\frac{2}{3}}$，则

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho\left(e+\frac{v^2}{2}\right)\right]+\rho\left(e+\frac{v^2}{2}\right)\nabla\cdot\boldsymbol{V}=\nabla\cdot(\lambda\nabla T)+q_{\mathrm{R}}\rho+\boldsymbol{f}\cdot\rho\boldsymbol{V}+\nabla\cdot\boldsymbol{P}\cdot\boldsymbol{V} \tag{11-3-13}$$

将式(11-3-1)代入式(11-3-13)，并除以 ρ 后，得

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(e+\frac{v^2}{2}\right)=\frac{\lambda}{\rho}\nabla\cdot(\nabla T)+q+\boldsymbol{f}\cdot\boldsymbol{V}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot\boldsymbol{V} \tag{11-3-14}$$

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}\left(C_{PT}+\frac{v^2}{2}\right)=&\frac{\lambda}{\rho}\left(\frac{\partial^2T}{\partial r^2}+\frac{\partial^2T}{r^2\partial\varphi^2}+\frac{\partial^2T}{\partial z^2}\right)+q+f_ru_r+f_\varphi v_\varphi+f_zw_z-\frac{1}{\rho}\left(\frac{\partial p}{\partial r}u_r+\frac{\partial p}{r\partial\varphi}v_\varphi+\frac{\partial p}{\partial z}w_z\right)\\&+\nu\left\{\left[\frac{4}{3}\frac{\partial^2u_r}{\partial r^2}+\frac{\partial^2u_r}{r^2\partial\varphi^2}+\frac{\partial^2u_r}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2u_r}{\partial z\partial r}+\frac{\partial^2u_r}{r\partial\varphi\partial r}+\frac{\partial u_r}{r^2\partial\varphi}\right)-\frac{2}{3}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{u_r}{r^2}\right)\right]u_r\right.\\&+\left[\frac{\partial^2v_\varphi}{\partial r^2}+\frac{4}{3}\frac{\partial^2v_\varphi}{r^2\partial\varphi^2}+\frac{\partial^2v_\varphi}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2v_\varphi}{r\partial\varphi\partial z}\right)-\frac{7}{3}\frac{\partial v_\varphi}{r^2\partial\varphi}\right]v_\varphi\\&\left.+\left[\frac{\partial^2w_z}{\partial r^2}+\frac{\partial^2w_z}{r^2\partial\varphi^2}+\frac{4}{3}\frac{\partial^2w_z}{\partial z^2}+\frac{1}{3}\left(\frac{\partial^2w_z}{\partial r\partial z}+\frac{\partial^2w_z}{r\partial\varphi\partial z}\right)+\frac{2}{3}\frac{\partial w_z}{r\partial z}\right]w_z\right\}\end{aligned} \tag{11-3-15}$$

式(11-3-15)就是柱坐标系层流可压缩能量微分方程。

11.4　不可压缩湍流微分方程组

11.4.1　连续性微分方程

依方程式(3-4-1)：

$$\frac{\partial\rho}{\partial t}+(\varphi^{\frac{2}{3}}\nabla\cdot\rho_s\boldsymbol{V}_s+\varphi_2\nabla\boldsymbol{V})\rho=0$$

因为不可压缩，$\rho=$常量，则

$$\varphi^{\frac{2}{3}}\nabla\cdot\boldsymbol{V}_s+\varphi_2\nabla\boldsymbol{V}=0 \tag{11-4-1}$$

变成柱坐标系下对应方程

$$\varphi^{\frac{2}{3}}\left(\frac{u_{sr}}{r}+\frac{\partial u_{rr}}{\partial r}+\frac{\partial v_{s\varphi}}{r\partial\varphi}+\frac{\partial w_z}{\partial z}\right)+(1-\varphi^{\frac{2}{3}})\left(\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+\frac{\partial v_\varphi}{r\partial\varphi}+\frac{\partial w_z}{\partial z}\right)=0 \tag{11-4-2}$$

$$\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+\frac{\partial v_\varphi}{r\partial\varphi}+\frac{\partial w_z}{\partial z}+\varphi^{\frac{2}{3}}\left[\frac{1}{r}(u_{sr}-u_r)+\frac{\partial}{\partial r}(u_{sr}-u_r)+\frac{\partial}{r\partial\varphi}(v_{s\varphi}-v_\varphi)+\frac{\partial}{\partial z}(w_{sz}-w_z)\right]=0 \tag{11-4-2'}$$

变成以连续相变量表示

$$\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+\frac{\partial v_\varphi}{r\partial\varphi}+\frac{\partial w_z}{\partial z}+\frac{\nu}{2}\varphi^{\frac{2}{3}}\left[\frac{1}{r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{\partial}{\partial r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\right.$$

$$+\frac{\partial}{r\partial\varphi}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{\partial}{\partial z}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\Bigg]=0 \qquad (11\text{-}4\text{-}2)''$$

连续性微分方程主要功用是化简其他微分方程。

11.4.2 动量微分方程

依无坐标系下不可压缩流体动量微分方程

$$\frac{\partial}{\partial t}(\varphi \boldsymbol{V}_s+\varphi_1\boldsymbol{V})+\varphi^{\frac{2}{3}}\nabla\cdot\boldsymbol{V}_s\boldsymbol{V}_s+\varphi_2\nabla\cdot\boldsymbol{V}\boldsymbol{V}=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \qquad (11\text{-}4\text{-}3)$$

现将 φ 作为参变量处理，则

$$\frac{\partial \boldsymbol{V}}{\partial t}+\varphi\frac{\partial}{\partial t}(\boldsymbol{V}_s-\boldsymbol{V})+\varphi_2\boldsymbol{V}\cdot\nabla(\boldsymbol{V}-\boldsymbol{V}_s)=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \qquad (11\text{-}4\text{-}4)$$

将式(11-4-4)变成柱坐标系对应方程

$$\frac{\partial \boldsymbol{V}}{\partial t}-\frac{\nu}{2}\varphi\left[\frac{\partial}{\partial t}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\boldsymbol{e}_r+\frac{\partial}{\partial t}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\boldsymbol{e}_\varphi+\frac{\partial}{\partial t}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\boldsymbol{e}_z\right]+\varphi_2\Bigg(u_r\frac{\partial}{\partial r}$$
$$+v_\varphi\frac{\partial}{r\partial\varphi}+w_z\frac{\partial}{\partial z}\Bigg)\cdot\left\{\frac{\nu}{2}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\boldsymbol{e}_r+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\boldsymbol{e}_\varphi+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\boldsymbol{e}_z\right]\right\}$$
$$=\boldsymbol{f}+\frac{1}{\rho}\left[\frac{\partial}{\partial r}(p_{rr}\boldsymbol{e}_r+p_{r\varphi}\boldsymbol{e}_\varphi+p_{zr}\boldsymbol{e}_z)+\frac{\partial}{r\partial\varphi}(p_{r\varphi}\boldsymbol{e}_r+p_{\varphi\varphi}\boldsymbol{e}_\varphi+p_{z\varphi}\boldsymbol{e}_z)+\frac{\partial}{\partial z}(p_{rz}\boldsymbol{e}_r+p_{\varphi z}\boldsymbol{e}_\varphi+p_{zz}\boldsymbol{e}_z)\right] \qquad (11\text{-}4\text{-}4)'$$

$$\frac{\partial \boldsymbol{V}}{\partial t}-\frac{\varphi\nu}{2}\left[\frac{\partial}{\partial t}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\boldsymbol{e}_r+\frac{\partial}{\partial t}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\boldsymbol{e}_\varphi+\frac{\partial}{\partial t}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\boldsymbol{e}_z\right]+\varphi_2\frac{\nu}{2}$$
$$\Bigg\{u_r\frac{\partial}{\partial r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\boldsymbol{e}_r+v_\varphi\left[\boldsymbol{e}_r\frac{\partial}{r\partial\varphi}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)-\frac{\boldsymbol{e}_\varphi}{r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\right]+w_\varphi\boldsymbol{e}_r\frac{\partial}{\partial z}\frac{1}{u_r}$$
$$\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+u_r\boldsymbol{e}_\varphi\frac{\partial}{\partial r}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+v_\varphi\left[\boldsymbol{e}_\varphi\frac{\partial}{r\partial\varphi}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)-\frac{\boldsymbol{e}_r}{r}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right]$$
$$+w_z\boldsymbol{e}_\varphi\frac{\partial}{\partial z}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+u_r\boldsymbol{e}_z\frac{\partial}{\partial r}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)+v_\varphi\boldsymbol{e}_z\frac{\partial}{r\partial\varphi}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)+w_z\boldsymbol{e}_z$$
$$\frac{\partial}{\partial z}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\Bigg\}=\boldsymbol{f}+\frac{1}{\rho}\Bigg[\left(\boldsymbol{e}_r\frac{\partial p_{rr}}{\partial r}+\boldsymbol{e}_\varphi\frac{\partial+p_{\varphi r}}{\partial r}+\boldsymbol{e}_z\frac{\partial p_{zr}}{\partial r}\right)+\Bigg(\boldsymbol{e}_r\frac{\partial p_{r\varphi}}{r\partial\varphi}+p_{r\varphi}\frac{\boldsymbol{e}_\varphi}{r}$$
$$+\boldsymbol{e}_\varphi\frac{\partial p_{\varphi\varphi}}{r\partial\varphi}-p_{\varphi\varphi}\frac{\boldsymbol{e}_r}{r}+\boldsymbol{e}_z\frac{\partial p_{z\varphi}}{r\partial\varphi}\Bigg)+\left(\boldsymbol{e}_r\frac{\partial p_{rz}}{\partial z}+\boldsymbol{e}_\varphi\frac{\partial p_{\varphi z}}{\partial z}+\boldsymbol{e}_z\frac{\partial p_{zz}}{\partial z}\right)\Bigg] \qquad (11\text{-}4\text{-}4)''$$

$$\frac{\partial \boldsymbol{V}}{\partial t}-\frac{\varphi\nu}{2}\left[\frac{\partial}{\partial t}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\boldsymbol{e}_r+\frac{\partial}{\partial t}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\boldsymbol{e}_\varphi+\frac{\partial}{\partial t}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\boldsymbol{e}_z\right]+\frac{\varphi_2\nu}{2}$$
$$\Bigg\{\boldsymbol{e}_r u_r\frac{\partial}{\partial r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+v_\varphi\left[\boldsymbol{e}_r\frac{\partial}{r\partial\varphi}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)-\frac{\boldsymbol{e}_\varphi}{r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\right]+w_\varphi\Bigg[\boldsymbol{e}_r\frac{\partial}{\partial z}\frac{1}{u_r}$$
$$\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\Bigg]+u_r\left[\boldsymbol{e}_\varphi\frac{\partial}{\partial r}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right]+v_\varphi\left[\boldsymbol{e}_\varphi\frac{\partial}{r\partial\varphi}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)-\frac{\boldsymbol{e}_r}{r}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right]$$
$$+w_z\left[\boldsymbol{e}_\varphi\frac{\partial}{\partial z}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right]+u_r\left[\boldsymbol{e}_z\frac{\partial}{\partial z}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]+v_\varphi\left[\boldsymbol{e}_z\frac{\partial}{r\partial\varphi}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]$$

$$+w_z\left[e_z\frac{\partial}{\partial z}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]\bigg\}$$

$$=f+\frac{1}{\rho}\bigg\{e_r\frac{\partial}{\partial r}\left(-p+2\mu\frac{\partial u_r}{\partial r}\right)+e_\varphi\frac{\partial}{\partial r}(1-\varphi^{\frac{2}{3}})\mu\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial v_\varphi}{\partial r}-\frac{v_\varphi}{r}\right)+\frac{\partial}{\partial r}\varphi^{\frac{2}{3}}\rho vt'\left(\frac{\partial u_r}{r\partial\varphi}\right)^2$$
$$+e_z\left[\frac{\partial}{\partial r}(1-\varphi^{\frac{2}{3}})\mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial w_z}{\partial r}\right)+\frac{\partial}{\partial r}\varphi^{\frac{2}{3}}\rho vt'\left(\frac{\partial u_r}{\partial z}\right)^2\right]+e_r\left[\frac{\partial}{r\partial\varphi}(1-\varphi^{\frac{2}{3}})\mu\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial u_r}{\partial r}-\frac{v_\varphi}{r}\right)\right.$$
$$\left.+\frac{\partial}{r\partial\varphi}\varphi^{\frac{2}{3}}\rho vt'\left(\frac{\partial v_\varphi}{\partial r}\right)^2\right]+\frac{e_\varphi}{r}\left[(1-\varphi^{\frac{2}{3}})\mu\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial u_r}{r\partial\varphi}-\frac{v_\varphi}{r}\right)+\varphi^{\frac{2}{3}}\rho vt'\left(\frac{\partial v_\varphi}{\partial r}\right)^2\right]+e_\varphi\frac{\partial}{r\partial\varphi}$$
$$\left[-p+2\mu\left(\frac{\partial v_\varphi}{r\partial\varphi}+\frac{u_r}{r}\right)\right]-e_r\frac{1}{r}\left[-p+2\mu\left(\frac{\partial v_\varphi}{r\partial\varphi}+\frac{u_r}{r}\right)\right]+e_z\left[\frac{\partial}{r\partial\varphi}(1-\varphi^{\frac{2}{3}})\mu\left(\frac{\partial v_\varphi}{\partial z}+\frac{\partial w_z}{r\partial\varphi}\right)\right.$$
$$\left.+\frac{\partial}{r\partial\varphi}\varphi^{\frac{2}{3}}\rho vt'\left(\frac{\partial v_\varphi}{\partial z}\right)^2\right]+e_\varphi\left[\frac{\partial}{\partial z}(1-\varphi^{\frac{2}{3}})\mu\left(\frac{\partial w_z}{r\partial\varphi}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{\partial}{\partial z}\rho vt'\left(\frac{\partial w_z}{r\partial\varphi}\right)^2\right]$$
$$+e_z\frac{\partial}{\partial z}\left(-p+2\mu\frac{\partial w_z}{\partial z}\right)\bigg\}\tag{11-4-4}'''$$

将式(11-4-4)‴中φ，$\varphi^{\frac{2}{3}}$作为参变量，并按e_r，e_φ，e_z三个方向列出

$$\frac{\partial u_r}{\partial t}+\frac{\nu\varphi}{2}\frac{\partial}{\partial t}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{\nu}{2}\varphi_2\bigg\{u_r\left[\frac{1}{u_r}\left(\frac{\partial^2 u_r}{r\partial\varphi\partial r}+\frac{\partial^2 u_r}{\partial zr\partial\varphi}\right)-\frac{1}{u_r^2}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\frac{\partial u_r}{\partial r}\right]$$
$$+v_\varphi\left[\frac{1}{u_r}\left(\frac{\partial^2 u_r}{r^2\partial\varphi^2}+\frac{\partial u_r}{\partial zr\partial\varphi}\right)-\frac{1}{u_r^2}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\frac{\partial u_r}{r\partial\varphi}-\frac{1}{r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\right]$$
$$+w_\varphi\left[\frac{1}{u_r}\left(\frac{\partial^2 u_r}{r\partial\varphi\partial z}+\frac{\partial^2 u_r}{\partial z^2}\right)-\frac{1}{u_r^2}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\frac{\partial u_r}{\partial z}\right]\bigg\}$$
$$=f_r-\frac{\partial p}{\rho\partial r}+\nu\left[2\left(\frac{\partial^2 u_r}{\partial r^2}-\frac{u_r}{r^2}\right)+\frac{\partial u_r}{r^2\partial\varphi}+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 u_r}{r\partial\varphi\partial r}+\frac{\partial^2 u_r}{\partial z\partial r}+\frac{\partial^2 u_r}{r^2\partial\varphi^2}+\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial u_r}{r^2\partial\varphi}\right)\right.$$
$$\left.+2\varphi^{\frac{2}{3}}t'\left(\frac{\partial u_r}{r\partial\varphi}\frac{\partial^2 u_r}{r\partial\varphi\partial r}+\frac{\partial u_r}{\partial z}\frac{\partial^2 u_r}{\partial z\partial r}\right)\right]\tag{11-4-5}$$

$$\frac{\partial v_\varphi}{\partial t}-\frac{\nu}{2}\varphi\frac{\partial}{\partial t}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{\nu}{2}\varphi_2\bigg\{u_r\left[\frac{1}{v_\varphi}\left(\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{\partial^2 v_\varphi}{\partial z\partial r}\right)-\frac{1}{v_\varphi^2}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\frac{\partial v_\varphi}{\partial r}\right]$$
$$+v_\varphi\left[\frac{1}{v_\varphi}\left(\frac{\partial^2 v_\varphi}{\partial rr\partial\varphi}+\frac{\partial^2 v_\varphi}{\partial zr\partial\varphi}\right)-\frac{1}{v_\varphi^2}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\frac{\partial v_\varphi}{r\partial\varphi}-\frac{1}{r}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right]$$
$$+w_z\left[\frac{1}{v_\varphi}\left(\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}\right)-\frac{1}{v_\varphi^2}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\frac{\partial v_\varphi}{\partial z}\right]\bigg\}$$
$$=f_\varphi-\frac{\partial p}{\rho r\partial\varphi}+\nu\bigg\{2\left(\frac{\partial^2 v_\varphi}{r^2\partial\varphi^2}-\frac{\partial v_\varphi}{r^2\partial\varphi}\right)+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2 v_\varphi}{\partial z^2}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}+\frac{\partial v_\varphi}{r^2\partial\varphi}\right)$$
$$+\varphi^{\frac{2}{3}}t'\left[2\left(\frac{\partial v_\varphi}{\partial r}\frac{\partial^2 v_\varphi}{\partial rr\partial\varphi}+\frac{\partial v_\varphi}{\partial z}\frac{\partial^2 v_\varphi}{\partial zr\partial\varphi}\right)+\frac{1}{r}\left(\frac{\partial v_\varphi}{\partial r}\right)^2\right]\bigg\}\tag{11-4-6}$$

$$\frac{\partial w_z}{\partial t}-\frac{\nu}{2}\varphi\frac{\partial}{\partial t}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)+\frac{\nu}{2}\varphi_2\bigg\{u_r\left[\frac{1}{w_z}\left(\frac{\partial^2 w_z}{\partial r^2}+\frac{\partial^2 w_z}{r\partial\varphi\partial r}\right)-\frac{1}{w_z^2}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\frac{\partial w_z}{\partial z}\right]$$

$$+ v_\varphi\left[\frac{1}{w_z}\left(\frac{\partial^2 w_z}{\partial r r\partial\varphi}+\frac{\partial w_z}{r^2\partial\varphi^2}\right)-\frac{1}{w_z^2}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\frac{\partial w_z}{r\partial\varphi}\right]+w_z\left[\frac{1}{w_z}\left(\frac{\partial^2 w_z}{\partial r\partial z}+\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)\right.$$

$$\left.\left.-\frac{1}{w_z^2}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\frac{\partial w_z}{\partial z}\right]\right\}=f_z-\frac{\partial p}{\rho\partial z}+\nu\left[2\frac{\partial^2 w_z}{\partial z^2}+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 w_z}{\partial r^2}+\frac{\partial^2 w_z}{r^2\partial\varphi^2}+\frac{\partial^2 w_z}{\partial r\partial z}\right.\right.$$

$$\left.\left.+\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)+2\varphi^{\frac{2}{3}}t'\left(\frac{\partial w_z}{\partial z}\frac{\partial^2 w_z}{\partial z^2}+\frac{\partial w_z}{r\partial\varphi}\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)\right] \qquad (11\text{-}4\text{-}7)$$

以上式(11-4-5)至式(11-4-7)是坐标系不可压缩湍流动量微分方程。

11.4.3 能量微分方程

依无坐标系下不可压缩流能量微分方程式(3-4-4)

$$\frac{\partial}{\partial t}\left(\varphi\frac{V_s^2}{2}+\varphi_1\frac{V^2}{2}\right)+\nabla\cdot\left(\varphi^{\frac{2}{3}}\frac{V_s^2}{2}\boldsymbol{V}_s+\varphi_2\frac{V^2}{2}\boldsymbol{V}\right)$$

$$=\frac{1}{\rho}\nabla(\lambda\nabla T)+q_R+\boldsymbol{f}\cdot(\varphi\boldsymbol{V}_s+\varphi_1\boldsymbol{V})+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{V}_s+\varphi_2\boldsymbol{V})$$

可以写成

$$\frac{\partial}{\partial t}\left(\varphi\frac{V_s^2}{2}+\varphi_1\frac{V^2}{2}\right)+\varphi^{\frac{2}{3}}\frac{V_s^2}{2}\nabla\cdot\boldsymbol{V}_s+\varphi_2\frac{V^2}{2}\nabla\cdot\boldsymbol{V}$$

$$=\frac{1}{\rho}\nabla\cdot(\lambda\nabla T)+q_R+\boldsymbol{f}\cdot(\varphi\boldsymbol{V}_s+\varphi_1\boldsymbol{V})+\frac{1}{\rho}(\varphi^{\frac{2}{3}}\boldsymbol{V}_s+\varphi\boldsymbol{V})\cdot\nabla\cdot\boldsymbol{P} \qquad (11\text{-}4\text{-}8)$$

将式(11-3-1)代入式(11-4-8)，写成

$$\frac{\partial}{\partial t}\left(\varphi\frac{V_s^2}{2}+\varphi_1\frac{V^2}{2}\right)+\varphi_2\boldsymbol{V}\cdot\nabla\left(\frac{V^2}{2}-\frac{V_s^2}{2}\right)$$

$$=\frac{1}{\rho}\nabla\cdot(\lambda\nabla T)+q_R+\boldsymbol{f}\cdot[\varphi\boldsymbol{V}_s+(1-\varphi)\boldsymbol{V}]+\frac{1}{\rho}[\varphi^{\frac{2}{3}}\boldsymbol{V}_s+(1-\varphi^{\frac{2}{3}})\boldsymbol{V}]\cdot\nabla\cdot\boldsymbol{P} \qquad (11\text{-}4\text{-}9)$$

进一步展开

$$\frac{\partial}{\partial t}\left(\frac{V^2}{2}\right)+\varphi\frac{\partial}{\partial t}\left(\frac{V_s^2}{2}-\frac{V^2}{2}\right)+\boldsymbol{V}\cdot\nabla\left(\frac{V^2}{2}-\frac{V_s^2}{2}\right)+\varphi^{\frac{2}{3}}\boldsymbol{V}\cdot\nabla\left(\frac{V_s^2}{2}-\frac{V^2}{2}\right)$$

$$=\frac{1}{\rho}\nabla\cdot(\nabla T)+q_R+\boldsymbol{f}\cdot\boldsymbol{V}+\boldsymbol{f}\cdot\varphi(\boldsymbol{V}_s-\boldsymbol{V})+\frac{1}{\rho}[\boldsymbol{V}\cdot\nabla\cdot\boldsymbol{P}+\varphi^{\frac{2}{3}}(\boldsymbol{V}_s-\boldsymbol{V})\cdot\nabla\cdot\boldsymbol{P}] \qquad (11\text{-}4\text{-}10)$$

现简化推导，将式(11-4-10)难以推导的项目分别进行推演。

由式(4-3-49)至式(4-3-51)可以得

$$u_{sr}^2\approx u_r^2-\frac{\nu}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right) \qquad (a)$$

$$v_{s\varphi}^2\approx v_\varphi^2-\frac{\nu}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right) \qquad (b)$$

$$w_{sz}^2\approx w_z^2-\frac{\nu}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right) \qquad (c)$$

将式(a)，式(b)，式(c)代入式(11-3-10)有关各项。

(1)项：

$$\frac{\partial}{\partial t}\left(\frac{V_s^2}{2}-\frac{V^2}{2}\right)=-\frac{\nu}{2}\frac{\partial}{\partial t}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]$$
$$=-\frac{\nu}{2}\left[\frac{1}{u_r}\left(\frac{\partial^2 u_r}{r\partial\varphi\partial t}+\frac{\partial^2 u_r}{\partial z\partial t}\right)-\frac{1}{u_r^2}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\frac{\partial u_r}{\partial t}+\frac{1}{v_\varphi}\left(\frac{\partial^2 v_\varphi}{\partial r\partial t}+\frac{\partial^2 v_\varphi}{\partial z\partial t}\right)\right.$$
$$\left.-\frac{1}{v_\varphi^2}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\frac{\partial v_\varphi}{\partial t}+\frac{1}{w_z}\left(\frac{\partial^2 w_z}{\partial r\partial t}+\frac{\partial^2 w_z}{r\partial\varphi\partial t}\right)-\frac{1}{w_z^2}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\frac{\partial w_z}{\partial t}\right]\tag{1)'$$

合并(2)，(3)项为

$$(2)+(3)=\mathbf{V}\cdot\nabla\left[\left(\frac{V^2}{2}-\frac{V_s^2}{2}\right)-\varphi^{\frac{2}{3}}\left(\frac{V^2}{2}-\frac{V_s^2}{2}\right)\right]$$
$$=(1-\varphi^{\frac{2}{3}})\frac{\nu}{2}\left\{u_r\frac{\partial}{\partial r}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]\right.$$
$$+v_\varphi\frac{\partial}{r\partial\varphi}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]$$
$$\left.+w_z\frac{\partial}{\partial z}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]\right\}\tag{2)'+(3)'$$

(4)项推导：

$$\boldsymbol{f}\cdot\varphi(\mathbf{V}_s-\mathbf{V})=-\frac{\nu}{2}\varphi\left[\frac{f_r}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{f_\varphi}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{f_z}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]\tag{4)'$$

(5)项推导：

$$\mathbf{V}\cdot\nabla\cdot\boldsymbol{P}=\mathbf{V}\left\{\left(\frac{\partial}{H_1\partial q_1}\boldsymbol{e}_1+\frac{\partial}{H_2\partial q_2}\boldsymbol{e}_2+\frac{\partial}{H_3\partial q_3}\boldsymbol{e}_3\right)\cdot[\boldsymbol{e}_1(p_{11}\boldsymbol{e}_1+p_{21}\boldsymbol{e}_2+p_{31}\boldsymbol{e}_3)\right.$$
$$\left.+\boldsymbol{e}_2(p_{12}\boldsymbol{e}_1+p_{22}\boldsymbol{e}_2+p_{32}\boldsymbol{e}_3)+\boldsymbol{e}_3(p_{13}\boldsymbol{e}_1+p_{23}\boldsymbol{e}_2+p_{33}\boldsymbol{e}_3)]\right\}$$
$$=\mathbf{V}\cdot\left[\frac{\partial}{H_1\partial q_1}(p_{11}\boldsymbol{e}_1+p_{21}\boldsymbol{e}_2+p_{31}\boldsymbol{e}_3)+\frac{\partial}{H_2\partial q_2}(p_{12}\boldsymbol{e}_1+p_{22}\boldsymbol{e}_2+p_{32}\boldsymbol{e}_3)\right.$$
$$\left.+\frac{\partial}{H_3\partial q_3}(p_{13}\boldsymbol{e}_1+p_{23}\boldsymbol{e}_2+p_{33}\boldsymbol{e}_3)\right]$$
$$=\mathbf{V}\cdot\left[\frac{\partial}{H_1\partial q_1}(p_{11}\boldsymbol{e}_1)+\frac{\partial}{H_2\partial q_2}(p_{12}\boldsymbol{e}_1)+\frac{\partial}{H_3\partial q_3}(p_{13}\boldsymbol{e}_1)+\frac{\partial}{H_1\partial q_1}(p_{21}\boldsymbol{e}_2)+\frac{\partial}{H_2\partial q_2}\right.$$
$$\left.(p_{22}\boldsymbol{e}_2)+\frac{\partial}{H_3\partial q_3}(p_{23}\boldsymbol{e}_2)+\frac{\partial}{H_1\partial q_1}(p_{31}\boldsymbol{e}_3)+\frac{\partial}{H_2\partial q_2}(p_{32}\boldsymbol{e}_3)+\frac{\partial}{H_3\partial q_3}(p_{33}\boldsymbol{e}_3)\right]$$
$$=\mathbf{V}\cdot\left[\frac{\boldsymbol{e}_1}{H_1}\frac{\partial p_{11}}{\partial q_1}+\frac{p_{11}}{H_1}\frac{\partial\boldsymbol{e}_1}{\partial q_1}+\frac{\boldsymbol{e}_1}{H_2}\frac{\partial p_{12}}{\partial q_2}+\frac{p_{12}}{H_2}\frac{\partial\boldsymbol{e}_1}{\partial q_2}+\frac{\boldsymbol{e}_1}{H_3}\frac{\partial p_{13}}{\partial q_3}+\frac{p_{13}}{H_3}\frac{\partial\boldsymbol{e}_1}{\partial q_3}\right.$$
$$+\frac{\boldsymbol{e}_2}{H_1}\frac{\partial p_{21}}{\partial q_1}+\frac{p_{21}}{H_1}\frac{\partial\boldsymbol{e}_2}{\partial q_1}+\frac{\boldsymbol{e}_2}{H_2}\frac{\partial p_{22}}{\partial q_2}+\frac{p_{22}}{H_2}\frac{\partial\boldsymbol{e}_2}{\partial q_2}+\frac{\boldsymbol{e}_2}{H_3}\frac{\partial p_{23}}{\partial q_3}+\frac{p_{23}}{H_3}\frac{\partial\boldsymbol{e}_2}{\partial q_3}$$
$$\left.+\frac{\boldsymbol{e}_3}{H_1}\frac{\partial p_{31}}{\partial q_1}+\frac{p_{31}}{H_1}\frac{\partial\boldsymbol{e}_3}{\partial q_1}+\frac{\boldsymbol{e}_3}{H_2}\frac{\partial p_{32}}{\partial q_2}+\frac{p_{32}}{H_2}\frac{\partial\boldsymbol{e}_3}{\partial q_2}+\frac{\boldsymbol{e}_3}{H_3}\frac{\partial p_{33}}{\partial q_3}+\frac{p_{33}}{H_3}\frac{\partial\boldsymbol{e}_3}{\partial q_3}\right]$$

$$= \boldsymbol{V}\cdot\left[\frac{\boldsymbol{e}_1}{H_1}\frac{\partial \boldsymbol{p}_{11}}{\partial q_1}+\frac{\boldsymbol{p}_{11}}{H_1}\left(-\frac{\boldsymbol{e}_2}{H_2}\frac{\partial H_1}{\partial q_2}-\frac{\boldsymbol{e}_3}{H_3}\frac{\partial H_1}{\partial q_3}\right)+\frac{\boldsymbol{e}_1}{H_2}\frac{\partial \boldsymbol{p}_{12}}{\partial q_2}+\frac{\boldsymbol{p}_{12}}{H_2}\left(\frac{\boldsymbol{e}_2}{H_1}\frac{\partial H_2}{\partial q_1}\right)+\frac{\boldsymbol{e}_1}{H_3}\frac{\partial \boldsymbol{p}_{13}}{\partial q_3}\right.$$
$$+\frac{\boldsymbol{p}_{13}}{H_3}\left(\frac{\boldsymbol{e}_3}{H_1}\frac{\partial H_3}{\partial q_1}\right)+\frac{\boldsymbol{e}_2}{H_1}\frac{\partial \boldsymbol{p}_{21}}{\partial q_1}+\frac{\boldsymbol{p}_{21}}{H_1}\left(\frac{\boldsymbol{e}_1}{H_2}\frac{\partial H_1}{\partial q_2}\right)+\frac{\boldsymbol{p}_{22}}{H_2}\left(-\frac{\boldsymbol{e}_3}{H_3}\frac{\partial H_2}{\partial q_3}-\frac{\boldsymbol{e}_1}{H_1}\frac{\partial H_2}{\partial q_1}\right)$$
$$+\frac{\boldsymbol{e}_2}{H_2}\frac{\partial \boldsymbol{p}_{22}}{\partial q_2}+\frac{\boldsymbol{e}_2}{H_3}\frac{\partial \boldsymbol{p}_{23}}{\partial q_3}+\frac{\boldsymbol{p}_{23}}{H_3}\left(\frac{\boldsymbol{e}_3}{H_2}\frac{\partial H_3}{\partial q_2}\right)+\frac{\boldsymbol{e}_3}{H_1}\frac{\partial \boldsymbol{p}_{31}}{\partial q_1}+\frac{\boldsymbol{p}_{31}}{H_1}\left(\frac{\boldsymbol{e}_1}{H_3}\frac{\partial H_1}{\partial q_3}\right)$$
$$\left.+\frac{\boldsymbol{e}_3}{H_2}\frac{\partial \boldsymbol{p}_{32}}{\partial q_2}+\frac{\boldsymbol{p}_{32}}{H_2}\left(\frac{\boldsymbol{e}_2}{H_3}\frac{\partial H_2}{\partial q_3}\right)+\frac{\boldsymbol{e}_3}{H_3}\frac{\partial \boldsymbol{p}_{33}}{\partial q_3}+\frac{\boldsymbol{p}_{33}}{H_3}\left(-\frac{\boldsymbol{e}_1}{H_1}\frac{\partial H_3}{\partial q_1}-\frac{\boldsymbol{e}_2}{H_2}\frac{\partial H_3}{\partial q_2}\right)\right] \tag{5)'}$$

(5)′是曲线坐标公式计算 $\boldsymbol{V}\cdot\nabla\cdot\boldsymbol{P}$ 的公式。以下开始将其转变为只适合柱坐标系计算 $\boldsymbol{V}\cdot\nabla\cdot\boldsymbol{P}$ 的公式。

$$\boldsymbol{V}\cdot\nabla\cdot\boldsymbol{P}=\boldsymbol{V}\cdot\left[\frac{\boldsymbol{e}_r}{1}\frac{\partial \boldsymbol{p}_{rr}}{\partial r}+\frac{\boldsymbol{p}_{rr}}{1}\left(-\frac{\boldsymbol{e}_\varphi}{r}\frac{\partial 1}{\partial\varphi}-\frac{\boldsymbol{e}_z}{1}\frac{\partial 1}{\partial z}\right)+\frac{\boldsymbol{e}_r}{r}\frac{\partial \boldsymbol{p}_{r\varphi}}{\partial\varphi}+\frac{\boldsymbol{p}_{r\varphi}}{1}\left(\frac{\boldsymbol{e}_\varphi}{1}\frac{\partial r}{\partial r}\right)+\frac{\boldsymbol{e}_r}{1}\frac{\partial \boldsymbol{p}_{rz}}{\partial z}+\frac{\boldsymbol{p}_{rz}}{1}\left(\frac{\boldsymbol{e}_z}{1}\frac{\partial 1}{\partial r}\right)\right.$$
$$+\frac{\boldsymbol{e}_\varphi}{1}\frac{\partial \boldsymbol{p}_{\varphi r}}{\partial r}+\frac{\boldsymbol{p}_{\varphi r}}{1}\left(\frac{\boldsymbol{e}_r}{r}\frac{\partial 1}{\partial\varphi}\right)+\frac{\boldsymbol{p}_{\varphi\varphi}}{r}\left(-\frac{\boldsymbol{e}_z}{1}\frac{\partial r}{\partial\varphi}-\frac{\boldsymbol{e}_r}{1}\frac{\partial r}{\partial r}\right)+\frac{\boldsymbol{e}_\varphi}{r}\frac{\partial \boldsymbol{p}_{\varphi\varphi}}{\partial\varphi}+\frac{\boldsymbol{e}_\varphi}{1}\frac{\partial \boldsymbol{p}_{23}}{\partial z}+\frac{\boldsymbol{p}_{23}}{1}\left(\frac{\boldsymbol{e}_z}{r}\frac{\partial 1}{\partial z}\right)$$
$$\left.+\frac{\boldsymbol{e}_z}{1}\frac{\partial \boldsymbol{p}_{31}}{\partial r}+\frac{\boldsymbol{p}_{zr}}{1}\left(\frac{\boldsymbol{e}_r}{1}\frac{\partial 1}{\partial z}\right)+\frac{\boldsymbol{e}_z}{r}\frac{\partial \boldsymbol{p}_{z\varphi}}{\partial\varphi}+\frac{\boldsymbol{p}_{z\varphi}}{r}\left(\frac{\boldsymbol{e}_\varphi}{1}\frac{\partial r}{\partial z}\right)+\frac{\boldsymbol{e}_z}{1}\frac{\partial \boldsymbol{p}_{zz}}{\partial z}+\frac{\boldsymbol{p}_{zz}}{1}\left(-\frac{\boldsymbol{e}_r}{r}\frac{\partial 1}{\partial r}-\frac{\boldsymbol{e}_\varphi}{r}\frac{\partial 1}{\partial\varphi}\right)\right]$$
$$=(u_r\boldsymbol{e}_r+v_\varphi\boldsymbol{e}_\varphi+w_z\boldsymbol{e}_z)\cdot\left[\boldsymbol{e}_r\left(\frac{\partial p_{rr}}{\partial r}+\frac{\partial p_{r\varphi}}{r\partial\varphi}-\frac{p_{\varphi\varphi}}{r}+\frac{\partial p_{rz}}{\partial z}\right)+\boldsymbol{e}_\varphi\left(\frac{\partial p_{\varphi r}}{\partial r}+\frac{\partial p_{\varphi\varphi}}{r\partial\varphi}+\frac{\partial p_{\varphi z}}{\partial z}\right.\right.$$
$$\left.\left.+\frac{p_{r\varphi}}{r}\right)+\boldsymbol{e}_z\left(\frac{\partial p_{zr}}{\partial r}+\frac{\partial p_{z\varphi}}{r\partial\varphi}+\frac{\partial p_{zz}}{\partial z}\right)\right]$$
$$=u_r\left(\frac{\partial p_{rr}}{\partial r}+\frac{\partial p_{r\varphi}}{r\partial\varphi}+\frac{\partial p_{rz}}{\partial z}-\frac{p_{\varphi\varphi}}{r}\right)+v_\varphi\left(\frac{\partial p_{\varphi r}}{\partial r}+\frac{\partial p_{\varphi\varphi}}{r\partial\varphi}+\frac{\partial p_{\varphi z}}{\partial z}+\frac{p_{r\varphi}}{r}\right)$$
$$+w_z\left(\frac{\partial p_{zr}}{\partial r}+\frac{\partial p_{z\varphi}}{r\partial\varphi}+\frac{\partial p_{zz}}{\partial z}\right)$$
$$=u_r\left\{\frac{\partial}{\partial r}\left(-p+2\mu\frac{\partial u_r}{\partial r}\right)+\frac{\partial}{r\partial\varphi}\mu\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial u_r}{r\partial\varphi}-\frac{v_\varphi}{r}\right)+\frac{\partial}{\partial z}\mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial w_z}{\partial r}\right)\right.$$
$$\left.-\frac{1}{r}\left[-p+2\mu\left(\frac{\partial v_\varphi}{r\partial\varphi}+\frac{u_r}{r}\right)\right]\right\}+v_\varphi\left\{\frac{\partial}{\partial r}\mu\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial u_r}{r\partial\varphi}-\frac{v_\varphi}{r}\right)\right.$$
$$\left.+\frac{\partial}{r\partial\varphi}\left[-p+2\mu\left(\frac{\partial v_\varphi}{r\partial\varphi}+\frac{u_r}{r}\right)\right]+\frac{\partial}{\partial z}\mu\left(\frac{\partial w_z}{r\partial\varphi}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{\mu}{r}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial u_r}{r\partial\varphi}+\frac{v_\varphi}{r}\right)\right\}$$
$$+w_z\left[\frac{\partial}{\partial r}\mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial w_z}{\partial r}\right)+\frac{\partial}{r\partial\varphi}\mu\left(\frac{\partial v_\varphi}{\partial z}+\frac{\partial w_z}{r\partial\varphi}\right)+\frac{\partial}{\partial z}\left(-p+2\mu\frac{\partial w_z}{\partial z}\right)\right] \tag{5)''}$$

(6)项推导:

$$\varphi^{\frac{2}{3}}(\boldsymbol{V}_s-\boldsymbol{V})\cdot\nabla\cdot\boldsymbol{P}=-\frac{\nu}{2}\varphi^{\frac{2}{3}}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\boldsymbol{e}_r+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\boldsymbol{e}_\varphi+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\boldsymbol{e}_z\right]$$
$$\cdot\left\{\boldsymbol{e}_r\left[(1-\varphi^{\frac{2}{3}})\left(\frac{\partial p_{r\varphi}}{r\partial\varphi}+\frac{\partial p_{rz}}{\partial z}\right)+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{r\varphi}}{r\partial\varphi}+\frac{\partial p'_{rz}}{\partial z}\right)\right]+\boldsymbol{e}_\varphi\left[(1-\varphi^{\frac{2}{3}})\right.\right.$$
$$\left.\left(\frac{\partial p_{\varphi r}}{\partial r}+\frac{\partial p_{\varphi z}}{\partial z}+\frac{p_{r\varphi}}{r}\right)+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{\varphi r}}{\partial r}+\frac{\partial p'_{\varphi z}}{\partial z}+\frac{p'_{r\varphi}}{r}\right)\right]+\boldsymbol{e}_z\left[(1-\varphi^{\frac{2}{3}})\right.$$
$$\left.\left.\left(\frac{\partial p_{zr}}{\partial r}+\frac{\partial p_{z\varphi}}{r\partial\varphi}\right)+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{zr}}{\partial r}+\frac{\partial p'_{z\varphi}}{r\partial\varphi}\right)\right]\right\}$$

$$
\begin{aligned}
&= -\frac{\nu}{2}\varphi^{\frac{2}{3}}\left\{\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\left[(1-\varphi^{\frac{2}{3}})\left(\frac{\partial p_{r\varphi}}{r\partial\varphi}+\frac{\partial p_{rz}}{\partial z}\right)+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{r\varphi}}{r\partial\varphi}+\frac{\partial p'_{rz}}{\partial z}\right)\right]\right.\\
&\quad+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\left[(1-\varphi^{\frac{2}{3}})\left(\frac{\partial p_{\varphi r}}{\partial r}+\frac{\partial p_{\varphi z}}{\partial z}+\frac{p_{r\varphi}}{r}\right)+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{\varphi r}}{\partial r}+\frac{\partial p'_{\varphi z}}{\partial z}+\frac{p'_{r\varphi}}{r}\right)\right]\\
&\quad\left.+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\left[(1-\varphi^{\frac{2}{3}})\left(\frac{\partial p_{zr}}{\partial r}+\frac{\partial p_{z\varphi}}{r\partial\varphi}\right)+\varphi^{\frac{2}{3}}\left(\frac{\partial p'_{zr}}{\partial r}+\frac{\partial p'_{z\varphi}}{r\partial\varphi}\right)\right]\right\}\\
&= -\frac{\nu}{2}\varphi^{\frac{2}{3}}\left\{\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\left\{(1-\varphi^{\frac{2}{3}})\left[\frac{\partial}{r\partial\varphi}\mu\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial u_r}{r\partial\varphi}-\frac{v_\varphi}{r}\right)\right.\right.\right.\\
&\quad\left.\left.+\frac{\partial}{\partial z}\mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial w_z}{\partial r}\right)\right]+\varphi^{\frac{2}{3}}\left[\frac{\partial}{r\partial\varphi}\rho\nu t'\left(\frac{\partial v_\varphi}{\partial r}\right)^2+\frac{\partial}{\partial z}\rho\nu t'\left(\frac{\partial w_z}{\partial r}\right)^2\right]\right\}\\
&\quad+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\left\{(1-\varphi^{\frac{2}{3}})\left[\frac{\partial}{\partial r}\mu\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial u_r}{r\partial\varphi}-\frac{v_\varphi}{r}\right)+\frac{\partial}{\partial z}\mu\left(\frac{\partial w_z}{r\partial\varphi}+\frac{\partial v_\varphi}{\partial z}\right)\right.\right.\\
&\quad\left.+\frac{\mu}{r}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial u_r}{r\partial\varphi}-\frac{v_\varphi}{r}\right)\right]+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial r}\rho\nu t'\left(\frac{\partial u_r}{r\partial\varphi}\right)^2+\frac{\partial}{\partial z}\rho\nu t'\left(\frac{\partial w_z}{r\partial\varphi}\right)^2\right.\\
&\quad\left.\left.+\frac{1}{r}\rho\nu t'\left(\frac{\partial v_\varphi}{\partial r}\right)^2\right]\right\}+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\left\{(1-\varphi^{\frac{2}{3}})\left[\frac{\partial}{\partial r}\mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial w_z}{\partial r}\right)\right.\right.\\
&\quad\left.\left.\left.+\frac{\partial}{r\partial\varphi}\mu\left(\frac{\partial w_z}{r\partial\varphi}+\frac{\partial v_\varphi}{\partial z}\right)\right]+\varphi^{\frac{2}{3}}\left[\frac{\partial}{\partial r}\rho\nu t'\left(\frac{\partial u_r}{\partial z}\right)^2+\frac{\partial}{r\partial\varphi}\rho\nu t'\left(\frac{\partial v_\varphi}{\partial z}\right)^2\right]\right\}\right\}
\end{aligned}\tag{6}'
$$

将式(1)′～式(6)′各项代回式(11-4-10)，得

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{u_r^2+v_\varphi^2+w_z^2}{2}\right)-\frac{1}{2}\varphi\nu\left[\frac{1}{u_r}\left(\frac{\partial^2 u_r}{r\partial\varphi\partial t}+\frac{\partial^2 u}{\partial z\partial t}\right)-\frac{1}{u_r^2}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\frac{\partial u_r}{\partial t}+\frac{1}{v_\varphi}\left(\frac{\partial^2 v_\varphi}{\partial r\partial t}+\frac{\partial^2 v_\varphi}{\partial z\partial t}\right)\right.\\
&\left.-\frac{1}{v_\varphi^2}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\frac{\partial v_\varphi}{\partial t}+\frac{1}{w_z}\left(\frac{\partial^2 w_z}{\partial r\partial t}+\frac{\partial^2 w_z}{r\partial\varphi\partial t}\right)-\frac{1}{w_\varphi^2}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\frac{\partial w_z}{\partial t}\right]\\
&+(1-\varphi^{\frac{2}{3}})\frac{\nu}{2}\left\{u_r\frac{\partial}{\partial r}\left[\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]\right.\\
&+v_\varphi\frac{\partial}{r\partial\varphi}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]\\
&\left.+w_z\frac{\partial}{\partial z}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\rho\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]\right\}\\
=&\frac{\lambda}{\rho}\left(\frac{\partial^2 T}{\partial r^2}+\frac{\partial^2 T}{r^2\partial\varphi^2}+\frac{\partial^2 T}{\partial z^2}\right)+q_R+f_r u_r+f_\varphi v_\varphi+f_z w_z-\frac{1}{2}\varphi\nu\left[\frac{f_r}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{f_\varphi}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right.\\
&\left.+\frac{f_z}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]+u_r\left[-\frac{\partial p}{\rho\partial r}+2\nu\frac{\partial^2 u_r}{\partial r^2}+\nu\left(\frac{\partial^w v_\varphi}{r\partial\varphi\partial r}-\frac{\partial^2 u_r}{r^2\partial\varphi^2}-\frac{\partial v_\varphi}{r^2\partial\varphi}\right)+\nu\left(\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial^2 w_z}{\partial r\partial_z}\right)\right.\\
&\left.+\frac{p}{r\rho}+\frac{2\nu}{r}\left(\frac{\partial v_\varphi}{r\partial\varphi}+\frac{u_r}{r}\right)\right]+v_\varphi\left[\nu\left(\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{\partial^2 u_r}{r\partial\varphi\partial r}-\frac{\partial v_\varphi}{r\partial r}+\frac{v_\varphi}{r^2}\right)-\frac{\partial p}{\rho r\partial\varphi}+2\nu\left(\frac{\partial v_\varphi}{r^2\partial\varphi^2}+\frac{1}{r}\frac{\partial u_r}{r\partial\varphi}\right)\right.\\
&+\nu\left(\frac{\partial^2 w_z}{r\partial\varphi\partial z}+\frac{\partial^2 w_z}{\partial z^2}\right)+\frac{\nu}{r}\left(\frac{\partial v_\rho}{\partial r}+\frac{\partial u_r}{r\partial\varphi}+\frac{v_\varphi}{r}\right)\Bigg]+w_z\left[\nu\left(\frac{\partial^2 u_r}{\partial z\partial r}+\frac{\partial^2 w_z}{\partial r^2}\right)+\nu\left(\frac{\partial^2 v_\varphi}{\partial zr\partial\varphi}+\frac{\partial^w_z}{r^2\partial\varphi^2}\right)\right.
\end{aligned}
$$

$$-\frac{\partial p}{\rho\partial z}+2D\frac{\partial^2 w_z}{\partial z^2}\bigg]-\frac{1}{2}\varphi^{\frac{2}{3}}\nu\left\{\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\left[(1-\varphi^{\frac{2}{3}})\nu\left(\frac{\partial^2 v_\varphi}{\partial r r\partial\varphi}+\frac{\partial^2 u_r}{r^2\partial\varphi^2}-\frac{\partial v_\varphi}{r^2\partial\varphi}+\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial w_z}{\partial r\partial z}\right)\right.\right.$$

$$\left.\left.+2\varphi^{\frac{2}{3}}\nu t'\left(\frac{\partial v_\varphi}{\partial r}\frac{\partial^2 v_\varphi}{\partial r r\partial\varphi}+\frac{\partial w_z}{\partial r}\frac{\partial^2 w_z}{\partial r r\partial\varphi}\right)\right]\right\}+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\left\{(1-\varphi^{\frac{2}{3}})\nu\left(\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{\partial u_r}{r\partial\varphi\partial r}\right.\right.$$

$$\left.-\frac{\partial v_\varphi}{r\partial\varphi}+\frac{v_\varphi}{r^2}+\frac{\partial w_z}{r\partial\varphi\partial z}+\frac{\partial^2 v_\varphi}{\partial z^2}+\frac{\partial v_\varphi}{r\partial r}+\frac{\partial u_r}{r^2\partial\varphi}-\frac{v_\varphi}{r^2}\right)+2\varphi^{\frac{2}{3}}\nu t'\left[\frac{\partial^2 u_r}{r\partial\varphi}\frac{\partial u_r}{r\partial\varphi\partial r}+\frac{\partial w_z}{r\partial\varphi}\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right.$$

$$\left.\left.+\frac{1}{r}\left(\frac{\partial v_\varphi}{\partial r}\right)^2\right]\right\}+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\left[(1-\varphi^{\frac{2}{3}})\nu\left(\frac{\partial^2 u_r}{\partial z\partial r}+\frac{\partial^2 w_z}{\partial r^2}+\frac{\partial^2 w_z}{r^2\partial\varphi^2}+\frac{\partial v_\varphi}{\partial z r\partial\varphi}\right)\right.$$

$$\left.+2\varphi^{\frac{2}{3}}\nu t'\left(\frac{\partial u_r}{\partial z}\frac{\partial^2 u_r}{\partial z\partial r}+\frac{\partial v_\varphi}{\partial z}\frac{\partial^2 v_\rho}{\partial z r\partial\varphi}\right)\right] \tag{11-4-11}$$

式 (11-4-11)是柱坐标系下不可压缩湍流能量微分方程，当定常流去掉含$\frac{\partial}{\partial t}$项。

11.4.4 动量矩微分方程

依无坐标系下不可压缩湍流动量矩微分方程式(3-4-3)：

$$\frac{\partial}{\partial t}[\boldsymbol{r}\times\rho(\varphi\boldsymbol{V}_s+\varphi\boldsymbol{V})]+\nabla\cdot\boldsymbol{V}_s(\boldsymbol{r}\times\varphi^{\frac{2}{3}}\rho\boldsymbol{V}_s)\ +\nabla\cdot\boldsymbol{V}(\boldsymbol{r}\times\varphi_2\rho\boldsymbol{V})$$

$$=(\boldsymbol{r}\times\boldsymbol{f})\rho+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P})$$

将式(11-4-1)代入上式得

$$\frac{\partial}{\partial t}[\boldsymbol{r}\times\rho(\varphi\boldsymbol{V}_s+\varphi_1\boldsymbol{V})]+\nabla\cdot\boldsymbol{V}(\boldsymbol{r}\times\varphi_2\rho\boldsymbol{V}-\boldsymbol{V}_s)=(\boldsymbol{r}\times\boldsymbol{f})\rho+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{11-4-12}$$

将 φ，φ_2，$\varphi^{\frac{2}{3}}$作为参变量，$\boldsymbol{r}$ 原是参变量，ρ 为常量，则式(11-4-12)为

$$\boldsymbol{r}\times\frac{\partial\boldsymbol{V}}{\partial t}+\varphi\boldsymbol{r}\times\frac{\partial}{\partial t}(\boldsymbol{V}_s-\boldsymbol{V})+\varphi_2\boldsymbol{r}\times(\boldsymbol{V}\cdot\nabla)(\boldsymbol{V}-\boldsymbol{V}_s)=\boldsymbol{r}\times\boldsymbol{f}+\frac{1}{\rho}\boldsymbol{r}\times\nabla\cdot\boldsymbol{P} \tag{11-4-13}$$

转变为柱坐标系下对应方程

$$\boldsymbol{r}\times\frac{\partial\boldsymbol{V}}{\partial t}+\varphi\boldsymbol{r}\times\frac{\partial}{\partial t}(\boldsymbol{V}_s-\boldsymbol{V})+\varphi_2\frac{\nu}{2}\boldsymbol{r}\times\left\{u_r\left[\boldsymbol{e}_r\frac{\partial}{\partial r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\frac{\partial\boldsymbol{e}_r}{\partial r}\right]\right.$$

$$+v_\varphi\left[\boldsymbol{e}_r\frac{\partial}{r\partial\varphi}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\frac{\partial\boldsymbol{e}_r}{r\partial\varphi}\right]+w_z\left[\boldsymbol{e}_r\frac{\partial}{\partial z}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}\right.\right.$$

$$\left.\left.+\frac{\partial u_r}{\partial z}\right)\frac{\partial\boldsymbol{e}_r}{\partial z}\right]+u_r\left[\boldsymbol{e}_\varphi\frac{\partial}{\partial r}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\frac{\partial\boldsymbol{e}_\varphi}{\partial r}\right]+v_\varphi\left[\boldsymbol{e}_\varphi\frac{\partial}{r\partial\varphi}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right.$$

$$\left.+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\frac{\partial\boldsymbol{e}_\varphi}{r\partial\varphi}\right]+w_z\left[\boldsymbol{e}_\varphi\frac{\partial}{\partial z}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\frac{\partial\boldsymbol{e}_\varphi}{\partial z}\right]$$

$$+u_r\left[\boldsymbol{e}_z\frac{\partial}{\partial r}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\frac{\partial\boldsymbol{e}_z}{\partial r}\right]+v_\varphi\left[\boldsymbol{e}_z\frac{\partial}{r\partial\varphi}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right.$$

$$\left.\left.+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\frac{\partial\boldsymbol{e}_z}{\partial r}\right]+w_z\left[\boldsymbol{e}_z\frac{\partial}{\partial z}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\frac{\partial\boldsymbol{e}_z}{\partial z}\right]\right\}$$

$$= \boldsymbol{r} \times \boldsymbol{f} + \frac{1}{\rho}\boldsymbol{r} \times \nabla \cdot \boldsymbol{P} \tag{11-4-13)'$$

$$\boldsymbol{r} \times \frac{\partial \boldsymbol{V}}{\partial t} + \varphi \boldsymbol{r} \times \frac{\partial}{\partial t}(\boldsymbol{V}_s - \boldsymbol{V}) + \varphi_2 \frac{\nu}{2}(r_r \boldsymbol{e}_r + r_\varphi \boldsymbol{e}_\varphi + r_z \boldsymbol{e}_z) \times \left\{\left\{ u_r\left[\frac{\partial}{\partial r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial \varphi} + \frac{\partial u_r}{\partial z}\right)\right]\right.\right.$$
$$+ v_\varphi\left[\frac{\partial}{r\partial \varphi}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi} + \frac{\partial u_r}{\partial z}\right) + \frac{1}{r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi} + \frac{\partial u_r}{\partial z}\right)\right] + w_z\left[\frac{\partial}{\partial z}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi} + \frac{\partial u_r}{\partial z}\right)\right]\Big\}\boldsymbol{e}_r$$
$$+ \left\{u_r\left[\frac{\partial}{\partial r}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r} + \frac{\partial v_\varphi}{\partial z}\right)\right] + v_\varphi\left[\frac{\partial}{r\partial\varphi}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r} + \frac{\partial v_\varphi}{\partial z}\right) - \frac{1}{r}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r} + \frac{\partial v_\varphi}{\partial z}\right)\right]\right.$$
$$+ w_z\left[\frac{\partial}{\partial z}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r} + \frac{\partial v_\varphi}{\partial z}\right)\right]\Big\}\boldsymbol{e}_\varphi + \left\{u_r\left[\frac{\partial}{\partial r}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_z}{r\partial\varphi}\right)\right] + v_\varphi\left[\frac{\partial}{r\partial\varphi}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_z}{r\partial\varphi}\right)\right]\right.$$
$$\left.\left. + w_z\left[\frac{\partial}{\partial z}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_z}{r\partial\varphi}\right)\right]\right\}\boldsymbol{e}_z\right\} = \boldsymbol{r} \times \boldsymbol{f} + \frac{1}{\rho}\boldsymbol{r} \times \nabla \cdot \boldsymbol{P} \tag{11-4-13)''$$

利用柱坐标系下动量微分方程式(11-4-5)至式(11-4-7)，将式(11-4-13)″按三个方向 $\boldsymbol{e}_r$，$\boldsymbol{e}_\varphi$，$\boldsymbol{e}_z$ 分别列出

$$r_\varphi \frac{\partial w_z}{\partial t} - r_z \frac{\partial v_\varphi}{\partial t} - \frac{\nu}{2}\varphi\left\{r_\varphi\left[\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_z}{r\partial\varphi}\right)\right] - r_z\left[\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r} + \frac{\partial v_\varphi}{\partial z}\right)\right]\right\} + \varphi_2\frac{\nu}{2}\left\{r_\varphi\left\{u_r\left[\frac{\partial}{\partial r}\frac{1}{w_z}\right.\right.\right.$$
$$\left.\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_z}{r\partial\varphi}\right)\right] + v_\varphi\left[\frac{\partial}{r\partial\varphi}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_z}{r\partial\varphi}\right)\right] + w_z\left[\frac{\partial}{\partial z}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_z}{r\partial\varphi}\right)\right]\Big\} - r_z\Big\{u_r\left[\frac{\partial}{\partial r}\frac{1}{v_\varphi}\right.$$
$$\left.\left(\frac{\partial v_\varphi}{\partial r} + \frac{\partial v_\varphi}{\partial z}\right)\right] + v_\varphi\left[\frac{\partial}{r\partial\varphi}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r} + \frac{\partial v_\varphi}{\partial z}\right) - \frac{1}{r}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r} + \frac{\partial v_\varphi}{\partial z}\right)\right] + w_z\left[\frac{\partial}{\partial z}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r} + \frac{\partial v_\varphi}{\partial z}\right)\right]\Big\}\Big\}$$
$$= r_\varphi f_z - r_z f_\varphi + \frac{1}{\rho}\left(r_z\frac{\partial p}{r\partial\varphi} - r_\varphi\frac{\partial p}{\partial z}\right) + \nu\left\{r_\varphi\left[2\frac{\partial^2 w_z}{\partial z^2} + (1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 w_z}{\partial r^2} + \frac{\partial^2 w_z}{r^2\partial\varphi^2} + \frac{\partial^2 w_z}{\partial r\partial z} + \frac{\partial w_z}{r\partial\varphi\partial z}\right)\right.\right.$$
$$\left. + 2\varphi^{\frac{2}{3}}t\left(\frac{\partial w_z}{\partial z}\frac{\partial^2 w_z}{\partial z^2} + \frac{\partial w_z}{r\partial\varphi}\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)\right] - r_z\left[2\left(\frac{\partial^2 v_\varphi}{r^2\partial\varphi^2} - \frac{\partial v_\varphi}{r^2\partial\varphi}\right) + (1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{\partial^2 v_\varphi}{\partial z^2}\right.\right.$$
$$\left.\left.\left. + \frac{\partial^2 v_\varphi}{r\partial\varphi\partial r} + \frac{\partial^2 v_\varphi}{r\partial\varphi\partial z} - \frac{\partial v_\varphi}{r^2\partial\varphi}\right) + \varphi^{\frac{2}{3}}t'\left(\frac{\partial v_\varphi}{\partial r}\frac{\partial^2 v_\varphi}{\partial r r\partial\varphi} + \frac{\partial v_\varphi}{\partial z}\frac{\partial^2 v_\varphi}{\partial z r\partial\varphi}\right) + \varphi^{\frac{2}{3}}t'\frac{1}{r}\left(\frac{\partial v_\varphi}{\partial r}\right)^2\right]\right\} \tag{11-4-14}$$

$$r_z\frac{\partial u_r}{\partial t} - r_r\frac{\partial w_z}{\partial t} - \frac{\nu}{2}\varphi\left\{r_z\left[\frac{1}{u_r}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_r}{r\partial\varphi}\right)\right] - r_r\left[\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_z}{r\partial\varphi}\right)\right]\right\} + \frac{\nu}{2}\varphi_2\left\{r_z\left\{u_r\left[\frac{\partial}{\partial r}\frac{1}{u_r}\right.\right.\right.$$
$$\left.\left(\frac{\partial u_r}{r\partial\varphi} + \frac{\partial u_r}{\partial z}\right)\right] + v_\varphi\left[\frac{\partial}{r\partial\varphi}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi} + \frac{\partial u_r}{\partial z}\right) + \frac{1}{r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi} + \frac{\partial u_r}{\partial z}\right)\right] + w_z\left[\frac{\partial}{\partial z}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi} + \frac{\partial u_r}{\partial z}\right)\right]\Big\}$$
$$- r_r\left\{u_r\left[\frac{\partial}{\partial r}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_z}{r\partial\varphi}\right)\right] + v_\varphi\left[\frac{\partial}{r\partial\varphi}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_z}{r\partial\varphi}\right)\right] + w_z\left[\frac{\partial}{\partial z}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_z}{r\partial\varphi}\right)\right]\right\}\Big\}$$
$$= r_z f_r - r_r f_z + \frac{1}{\rho}\left(r_r\frac{\partial p}{\partial z} - r_z\frac{\partial p}{\partial r}\right) + \nu\left\{r_z\left[2\left(\frac{\partial^2 u_r}{\partial r^2} - \frac{u_r}{r^2}\right) + \frac{\partial u_r}{r^2\partial\varphi} + (1-\varphi^{\frac{2}{3}})\left(\frac{\partial u_r}{r\partial\varphi\partial r}\right.\right.\right.$$
$$\left.\left. + \frac{\partial^2 u_r}{\partial z\partial r} + \frac{\partial^2 u_r}{\partial z^2} + \frac{\partial^2 u_r}{r^2\partial\varphi^2} + \frac{\partial u_r}{r^2\partial\varphi}\right) + 2\varphi^{\frac{2}{3}}t'\left(\frac{\partial u_r}{r\partial\varphi}\frac{\partial^2 u_r}{r\partial\varphi\partial r} + \frac{\partial u_r}{\partial z}\frac{\partial^2 u_r}{\partial z\partial r}\right)\right] - r_r\left[2\frac{\partial^2 w_z}{\partial z^2} + (1-\varphi^{\frac{2}{3}})\right.$$

$$\left(\frac{\partial^2 w_z}{\partial r^2}+\frac{\partial^2 w_z}{r^2\partial\varphi^2}+\frac{\partial^2 w_z}{\partial r\partial z}+\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)-2\varphi^{\frac{2}{3}}t'\left(\frac{\partial w_z}{\partial z}\frac{\partial^2 w_z}{\partial z^2}+\frac{\partial w_z}{r\partial\varphi}\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)\Big]\Big\} \tag{11-4-15}$$

$$r_r\frac{\partial v_\varphi}{\partial t}-r_\varphi\frac{\partial u_r}{\partial t}-\frac{\nu}{2}\varphi\left\{r_r\left[\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right]-r_\varphi\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\right]\right\}+\frac{\nu}{2}\varphi_2\left\{r_r\left\{u_r\left[\frac{\partial}{\partial r}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}\right.\right.\right.\right.$$
$$\left.\left.+\frac{\partial v_\varphi}{\partial z}\right)\right]+v_\varphi\left[\frac{\partial}{r\partial\varphi}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)-\frac{1}{r}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right]+w_z\left[\frac{\partial}{\partial z}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right]\Big\}-r_\varphi\Big\{u_r\Big[\frac{\partial}{\partial r}$$
$$\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\Big]+v_\varphi\left[\frac{\partial}{r\partial\varphi}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+\frac{1}{r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\right]+w_z\left[\frac{\partial}{\partial z}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\right]\Big\}\Big\}$$
$$=r_rf_\varphi-r_\varphi f_r+\frac{1}{\rho}\left(r_\varphi\frac{\partial p}{\partial r}-r_r\frac{\partial p}{r\partial\varphi}\right)+\nu\Big\{r_r\Big[2\left(\frac{\partial^2 v_\varphi}{r^2\partial\varphi^2}-\frac{\partial v_\varphi}{r^2\partial\varphi}\right)+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}\right.$$
$$\left.+\frac{\partial^2 v_\varphi}{\partial z^2}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}-\frac{\partial v_\varphi}{r^2\partial\varphi}\right)+2\varphi^{\frac{2}{3}}t'\left(\frac{\partial v_\varphi}{\partial r}\frac{\partial^2 v_\varphi}{\partial r r\partial\varphi}+\frac{\partial v_\varphi}{\partial z}\frac{\partial^2 v_\varphi}{\partial z\partial\varphi}\right)+\varphi^{\frac{2}{3}}t'\frac{1}{r}\left(\frac{\partial v_\varphi}{\partial r}\right)^2\Big]-r_\varphi\Big[2\left(\frac{\partial^2 u_r}{\partial r^2}-\frac{u_r}{r^2}\right)$$
$$+\frac{\partial u_r}{r^2\partial\varphi}+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 u_r}{r\partial\varphi\partial r}+\frac{\partial^2 u_r}{\partial z\partial r}+\frac{\partial^2 u_r}{r^2\partial\varphi^2}+\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial u_r}{r^2\partial\varphi}\right)+2\varphi^{\frac{2}{3}}t'\left(\frac{\partial u_r}{r\partial\varphi}\frac{\partial^2 u_r}{r\partial z\partial\varphi\partial r}+\frac{\partial u_r}{\partial z}\frac{\partial^2 u_r}{\partial z\partial r}\right)\Big]\Big\} \tag{11-4-16}$$

式(11-4-14)至式(11-4-16)，是柱坐标系下不可压缩湍流动量矩微分方程。

11.5 可压缩湍流微分方程组

本节特点：流体在运动过程中，密度变化，涡旋的密度 ρ_s 在平移运动微团的流场运动，所到之处，其压力自动调节到与所在地流体压力一样，但其密度 ρ_s，温度 T_s 与该地的流体不同。

以下推导，要应用可压缩流体的法向应力表达式，由流体力学可知

$$p_{rr}=\left(-p-\frac{2}{3}\mu\nabla\cdot\mathbf{V}\right)+2\mu\frac{\partial u_r}{\partial r} \tag{11-5-1}$$

$$p_{\varphi\varphi}=\left(-p-\frac{2}{3}\mu\nabla\cdot\mathbf{V}\right)+2\mu\left(\frac{\partial u_\varphi}{r\partial\varphi}+\frac{u_r}{r}\right) \tag{11-5-2}$$

$$p_{zz}=\left(-p-\frac{2}{3}\mu\nabla\cdot\mathbf{V}\right)+2\mu\frac{\partial u_z}{\partial z} \tag{11-5-3}$$

11.5.1 连续性微分方程

将无坐标系下可压缩湍流式(3-7-1)，变成柱坐标系下对应方程：

$$\varphi\frac{\mathrm{D}\rho_s}{\mathrm{D}t}+(1-\varphi)\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho_s\nabla\cdot\mathbf{V}_s+\rho\nabla\cdot\mathbf{V}=0$$

利用式(4-5-1)将式(3-7-1)改为

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}\rho\left(\frac{T}{T_s}-1\right)+\rho\left(\frac{T}{T_s}\nabla\cdot\mathbf{V}_s+\nabla\cdot\mathbf{V}\right)=0 \tag{11-5-4}$$

引入式(4-5-2)与式(4-5-7)，则为

$$[\varphi(\alpha-1)+1]\frac{D\rho}{Dt}+\rho(\alpha\nabla\cdot\boldsymbol{V}_s+\nabla\cdot\boldsymbol{V})=0 \tag{11-5-4}'$$

将式(4-3-49)至式(4-3-51)变成对应柱坐标系下公式，代入式(11-5-4)′，全式除以 ρ，对应 1kg 流体而言

$$[\varphi(\alpha-1)+1]\frac{D\rho}{\rho Dt}+\frac{\partial}{\partial r}\left[u_r(\alpha+1)-\frac{\alpha\nu}{2u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\right]+\frac{\partial}{r\partial\varphi}\left[v_\varphi(\alpha+1)-\frac{\alpha\nu}{2v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right]$$
$$+\frac{\partial}{\partial z}\left[w_z(\alpha+1)-\frac{\alpha\nu}{2w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]=0 \tag{11-5-5}$$

式(11-5-5)就是柱坐标系下可压缩湍流连续性微分方程。一般情况下，应用无坐标系下连续性微分方程式(3-7-1)′，将它们代入动量、能量、动量矩的无坐标系下的方程，以得到简化公式。

11.5.2 动量微分方程

为推导柱坐标系下可压缩湍流动量微分方程，选列出式(3-7-2)：

$$\varphi\frac{D}{Dt}(\rho_s\boldsymbol{V}_s)+\varphi_1\frac{D}{Dt}(\rho\boldsymbol{V})+\rho_s\boldsymbol{V}_s(\nabla\cdot\boldsymbol{V}_s)+\rho\boldsymbol{V}(\nabla\cdot\boldsymbol{V})=\boldsymbol{f}[\varphi\rho_s+(1-\varphi)\rho]+\nabla\cdot\boldsymbol{P}$$

将式(3-7-1)代入式(3-7-2)′

$$\varphi\frac{D(\rho_s\boldsymbol{V}_s)}{Dt}+\varphi_1\frac{D}{Dt}(\rho\boldsymbol{V})-\boldsymbol{V}_s\left(\varphi\frac{D\rho_s}{Dt}+\varphi_1\frac{D\rho}{Dt}+\rho\nabla\cdot\boldsymbol{V}\right)+\rho\boldsymbol{V}\nabla\cdot\boldsymbol{V}$$
$$=\boldsymbol{f}[\varphi\rho_s+(1-\varphi)\rho]+\nabla\cdot\boldsymbol{P} \tag{3-7-2}'$$

$$\varphi\left(\frac{D}{Dt}\rho_s\boldsymbol{V}_s-\boldsymbol{V}_s\frac{D\rho_s}{Dt}\right)+(1-\varphi)\left(\frac{D}{Dt}\rho\boldsymbol{V}-\boldsymbol{V}_s\frac{D\rho}{Dt}\right)+\rho(\boldsymbol{V}-\boldsymbol{V}_s)\nabla\cdot\boldsymbol{V}$$
$$=\boldsymbol{f}[\varphi\rho_s+(1-\varphi)\rho]+\nabla\cdot\boldsymbol{P} \tag{3-7-2}''$$

$$\varphi\left(\rho_s\frac{D\boldsymbol{V}_s}{Dt}+\boldsymbol{V}_s\frac{D\rho_s}{Dt}-\boldsymbol{V}_s\frac{D\rho_s}{Dt}\right)+(1-\varphi)\left(\rho\frac{D\boldsymbol{V}}{Dt}+\boldsymbol{V}\frac{D\rho}{Dt}-\boldsymbol{V}_s\frac{D\rho}{Dt}\right)+\rho(\boldsymbol{V}-\boldsymbol{V}_s)\nabla\cdot\boldsymbol{V}$$
$$=\boldsymbol{f}[\varphi\rho_s+(1-\varphi)\rho]+\nabla\cdot\boldsymbol{P} \tag{3-7-2}'''$$

$$\varphi\rho_s\frac{D\boldsymbol{V}_s}{Dt}+(1-\varphi)\rho\frac{D\boldsymbol{V}}{Dt}+(1-\varphi)(\boldsymbol{V}-\boldsymbol{V}_s)\frac{D\rho}{Dt}+\rho\ (\boldsymbol{V}-\boldsymbol{V}_s)\nabla\cdot\boldsymbol{V}$$
$$=\boldsymbol{f}[\varphi\rho_s+(1-\varphi)\rho]+\nabla\cdot\boldsymbol{P} \tag{3-7-2}''''$$

将式(4-5-1)代入式(3-7-2)″″，除以 ρ

$$\varphi\frac{T}{T_s}\frac{D\boldsymbol{V}_s}{Dt}+(1-\varphi)\rho\frac{D\boldsymbol{V}}{Dt}+(1-\varphi)(\boldsymbol{V}-\boldsymbol{V}_s)\frac{1}{\rho}\frac{D\rho}{Dt}+\boldsymbol{V}\cdot\nabla\ (\boldsymbol{V}-\boldsymbol{V}_s)$$
$$=\boldsymbol{f}\left[\varphi\frac{T}{T_s}+(1-\varphi)\right]+\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{11-5-6}$$

将式(4-5-2)与式(4-5-7)代入式(11-5-6)

$$\frac{D\boldsymbol{V}}{Dt}+\varphi\left(\alpha\frac{D\boldsymbol{V}_s}{Dt}-\frac{D\boldsymbol{V}}{Dt}\right)+(1-\varphi)(\boldsymbol{V}-\boldsymbol{V}_s)\frac{1}{\rho}\frac{D\rho}{Dt}+\boldsymbol{V}\cdot\nabla\ (\boldsymbol{V}-\boldsymbol{V}_s)$$
$$=\boldsymbol{f}[\varphi\alpha+(1-\varphi)]+\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{11-5-6}'$$

式中，$\alpha=\dfrac{1}{\eta}$，将变成柱坐标系下对应方程。为此，利用式(4-3-49)至式(4-3-51)，将涡旋

微团速度变成连续相速度。按三个坐标轴分别列出方程。

$$
\begin{aligned}
&\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+v_\varphi\frac{\partial u_r}{r\partial\varphi}+w_z\frac{\partial u_r}{\partial z}-\frac{v_\varphi^2}{r}+\varphi\left[(\alpha-1)\frac{\partial u_r}{\partial t}-\frac{2\nu}{2u_r}\left(\frac{\partial^2 u_r}{r\partial\varphi\partial t}+\frac{\partial^2 u_r}{\partial z\partial t}\right)\right]+(1-\varphi)\frac{\nu}{2u_r}\\
&\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\frac{\nu}{2}\left[u_r\frac{\partial}{\partial r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+v_\varphi\frac{\partial}{r\partial\varphi}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+w_z\frac{\partial}{\partial z}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\right]\\
&=[(\alpha-1)\varphi+1]f_r-\frac{\partial p}{\rho\partial r}+\nu\left[\frac{4}{3}\left(\frac{\partial^2 u_r}{\partial r^2}-\frac{\partial v_\varphi}{r\partial\varphi}\right)+\frac{2}{3}\left(\frac{\partial u_r}{\partial r}+\frac{\partial w_z}{\partial z}-\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}-\frac{\partial^2 w_z}{\partial z\partial r}\right)\right.\\
&\quad\left.+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 u_r}{r^2\partial\varphi^2}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2 w_z}{\partial r\partial z}+\frac{\partial u_r}{\partial z^2}-\frac{\partial v_\varphi}{r^2\partial\varphi}\right)+\varphi^{\frac{2}{3}}2t'\left(\frac{\partial v_\varphi}{\partial r}\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}+\frac{\partial w_z}{\partial r}\frac{\partial^2 w_z}{\partial r\partial z}\right)\right]
\end{aligned}
\tag{11-5-7}
$$

$$
\begin{aligned}
&\frac{\partial v_\varphi}{\partial t}+u_r\frac{\partial v_\varphi}{\partial r}+v_\varphi\frac{\partial v_\varphi}{r\partial\varphi}+w_z\frac{\partial v_\varphi}{\partial z}+\frac{v_\varphi u_r}{r}+\varphi\left[(\alpha-1)\frac{\partial v_\varphi}{\partial t}-\frac{\alpha\nu}{2v_\varphi}\left(\frac{\partial^2 v_\varphi}{\partial r\partial t}+\frac{\partial^2 v_\varphi}{\partial z\partial t}\right)\right]+(1-\varphi)\frac{\nu}{2v_\varphi}\\
&\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\frac{\nu}{2}\left[u_r\frac{\partial}{\partial r}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+v_\varphi\frac{\partial}{r\partial\varphi}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+w_z\frac{\partial}{\partial z}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right]\\
&=[(\alpha-1)\varphi+1]f_v-\frac{\partial p}{\rho r\partial\varphi}+\nu\left\{\frac{4}{3}\frac{\partial^2 v_\varphi}{r^2\partial\varphi^2}-\frac{2}{3}\frac{\partial^2 u_r}{r\partial\varphi\partial r}-\frac{2}{3}\frac{\partial^2 w_z}{r\partial\varphi\partial z}+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{\partial^2 u_r}{r\partial\varphi\partial r}\right.\right.\\
&\quad\left.\left.+\frac{\partial^2 w_z}{r\partial\varphi\partial z}+\frac{\partial^2 v_\varphi}{\partial z^2}+\frac{1}{r}\frac{\partial v_\varphi}{\partial r}+\frac{\partial u_r}{r^2\partial\varphi}-\frac{v_\varphi}{r^2}\right)+\varphi^{\frac{2}{3}}\left[2t'\frac{\partial u_r}{r\partial\varphi}\frac{\partial^2 u_r}{r\partial\varphi\partial r}+2t'\frac{\partial w_z}{r\partial\varphi}\frac{\partial^2 w_z}{r\partial\varphi\partial z}+t'\left(\frac{\partial v_\varphi}{\partial r}\right)^2\right]\right\}
\end{aligned}
\tag{11-5-8}
$$

$$
\begin{aligned}
&\frac{\partial w_z}{\partial t}+u_r\frac{\partial w_z}{\partial r}+v_\varphi\frac{\partial w_z}{r\partial\varphi}+w_z\frac{\partial w_z}{\partial z}+\varphi\left[(\alpha-1)\frac{\partial w_z}{\partial t}-\frac{\alpha\nu}{2w_z}\left(\frac{\partial^2 w_z}{\partial r\partial t}+\frac{\partial^2 w_z}{r\partial\varphi\partial t}\right)\right]+(1-\varphi)\frac{\nu}{2w_z}\left(\frac{\partial w_z}{\partial r}\right.\\
&\left.+\frac{\partial w_z}{r\partial\varphi}\right)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\frac{\nu}{2}\left[u_r\frac{\partial}{\partial r}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)+v_\varphi\frac{\partial}{r\partial\varphi}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)+w_z\frac{\partial}{\partial z}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]\\
&=[(\alpha-1)\varphi+1]f_z-\frac{\partial p}{\rho\partial z}+\nu\left[\frac{4}{3}\frac{\partial^2 w_z}{\partial z^2}-\frac{2}{3}\left(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}\right)+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial^2 w_z}{\partial r\partial z}\right.\right.\\
&\quad\left.\left.+\frac{\partial^2 v_\varphi}{r\partial z\partial\varphi}+\frac{\partial^2 w_z}{r^2\partial\varphi^2}\right)+\varphi^{\frac{2}{3}}2t'\left(\frac{\partial u_r}{\partial z}\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial v_\varphi}{\partial z}\frac{\partial^2 v_\varphi}{r\partial z\partial\varphi}\right)\right]
\end{aligned}
\tag{11-5-9}
$$

将式(11-5-7)～式(11-5-9)按流体力学受力规律进行组合为

$$
\begin{aligned}
&\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+v_\varphi\frac{\partial u_r}{r\partial\varphi}+w_z\frac{\partial u_r}{\partial z}+\frac{u_r u_\varphi}{r}+\varphi\left[(\alpha-1)\frac{\partial u_r}{\partial t}-\frac{2\nu}{2u_r}\left(\frac{\partial^2 u_r}{r\partial\varphi\partial t}+\frac{\partial^2 u_r}{\partial z\partial t}\right)\right]+(1-\varphi)\frac{\nu}{2u_r}\\
&\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\frac{\nu}{2}\left[u_r\frac{\partial}{\partial r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+v_\varphi\frac{\partial}{r\partial\varphi}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)+w_z\frac{\partial}{\partial z}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\right]\\
&=[(\alpha-1)\varphi+1]f_r-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{4}{3}\frac{\partial^2 u_r}{\partial r^2}+\frac{2}{3}\left(\frac{\partial u_r}{\partial r}-\frac{\partial^2 u_r}{r\partial\varphi\partial r}-\frac{\partial^2 u_r}{\partial r\partial z}\right)+(1-\varphi^{\frac{2}{3}})\right.\\
&\quad\left.\left(\frac{\partial^2 u_r}{r^2\partial\varphi^2}+\frac{\partial^2 u_r}{r\partial\varphi\partial r}+\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial u_r}{r^2\partial\varphi}\right)+\varphi^{\frac{2}{3}}2t'\left(\frac{\partial u_r}{r\partial\varphi}\frac{\partial^2 u_r}{r\partial\varphi\partial r}+\frac{\partial u_r}{\partial z}\frac{\partial^2 u_r}{\partial z^2}\right)\right]
\end{aligned}
\tag{11-5-10}
$$

$$
\frac{\partial v_\varphi}{\partial t}+u_r\frac{\partial v_\varphi}{\partial r}+v_\varphi\frac{\partial v_\varphi}{r\partial\varphi}+w_z\frac{\partial v_\varphi}{\partial z}-\frac{v_\varphi^2}{r}+\varphi\left[(\alpha-1)\frac{\partial v_\varphi}{\partial t}-\frac{\alpha\nu}{2v_\varphi}\left(\frac{\partial^2 v_\varphi}{\partial r\partial t}+\frac{\partial^2 v_\varphi}{\partial z\partial t}\right)\right]+(1-\varphi)\frac{\nu}{2v_\varphi}
$$

$$\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\frac{\nu}{2}\left[u_r\frac{\partial}{\partial r}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+v_\varphi\frac{\partial}{r\partial\varphi}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)+w_z\frac{\partial}{\partial z}\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right]$$

$$=[(\alpha-1)\varphi+1]f_v-\frac{\partial p}{\rho r\partial\varphi}+\nu\left[\frac{4}{3}\left(\frac{\partial^2 v_\varphi}{r^2\partial\varphi^2}-\frac{\partial v_\varphi}{r^2\partial\varphi}\right)-\frac{2}{3}\left(\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}\right)+(1-\varphi^{\frac{2}{3}})\right.$$

$$\left.\left(\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}-\frac{\partial v_\varphi}{r^2\partial\varphi}\right)+\varphi^{\frac{2}{3}}2t'\left(\frac{\partial v_\varphi}{\partial r}\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}+\frac{\partial v_\varphi}{\partial z}\frac{\partial^2 v_\varphi}{\partial z\partial r}\right)+\varphi^{\frac{2}{3}}t'\left(\frac{\partial^2 v_\varphi}{\partial r}\right)^2\right]$$

$$(11\text{-}5\text{-}11)$$

$$\frac{\partial w_z}{\partial t}+u_r\frac{\partial w_z}{\partial r}+v_\varphi\frac{\partial w_z}{r\partial\varphi}+w_z\frac{\partial w_z}{\partial z}+\varphi\left[(\alpha-1)\frac{\partial w_z}{\partial t}-\frac{\alpha\nu}{2w_z}\left(\frac{\partial^2 w_z}{\partial r\partial t}+\frac{\partial^2 w_z}{r\partial\varphi\partial t}\right)\right]+(1-\varphi)\frac{\nu}{2w_z}\left(\frac{\partial w_z}{\partial r}\right.$$

$$\left.+\frac{\partial w_z}{r\partial\varphi}\right)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\frac{\nu}{2}\left[u_r\frac{\partial}{\partial r}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)+v_\varphi\frac{\partial}{r\partial\varphi}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)+w_z\frac{\partial}{\partial z}\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]$$

$$=[(\alpha-1)\varphi+1]f_z-\frac{\partial p}{\rho\partial z}+\nu\left[\frac{4}{3}\frac{\partial^2 w_z}{\partial z^2}+\frac{2}{3}\left(\frac{\partial w_z}{\partial z}-\frac{\partial^2 w_z}{\partial z\partial r}-\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)+(1-\varphi^{\frac{2}{3}})\right.$$

$$\left.\left(\frac{\partial^2 w_z}{\partial r\partial z}+\frac{\partial^2 w_z}{r^2\partial\varphi^2}+\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)+\varphi^{\frac{2}{3}}2t'\left(\frac{\partial w_z}{\partial r}\frac{\partial^2 w_z}{\partial r\partial z}-\frac{\partial w_z}{r\partial\varphi}\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)\right]\qquad(11\text{-}5\text{-}12)$$

式(11-5-10)至式(11-5-12)三式，就是柱坐标系下可压缩湍流动量微分方程，定常流时去掉含$\frac{\partial}{\partial t}$项。

11.5.3 能量微分方程

依式(3-7-4)：

$$\varphi\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho_s\left(e_s+\frac{V_s^2}{2}\right)\right]+\rho_s\left(e_s+\frac{V_s^2}{2}\right)\nabla\cdot\boldsymbol{V}_s+\varphi_1\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho\left(e+\frac{V^2}{2}\right)\right]+\rho\left(e+\frac{V^2}{2}\right)\nabla\cdot\boldsymbol{V}$$

$$=\nabla\cdot(\lambda\nabla T)+\rho q_{\mathrm{R}}+\boldsymbol{f}(\varphi\rho_s\boldsymbol{V}_s+\varphi_1\rho\boldsymbol{V})+\nabla\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{V}_s+\boldsymbol{V})$$

将式(3-7-1)代入上式，化简后为

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho\left(e+\frac{V^2}{2}\right)\right]+\left[(e-e_s)+\left(\frac{V^2}{2}-\frac{V_s^2}{2}\right)\right]\rho\nabla\cdot\boldsymbol{V}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho_s\left(e_s+\frac{V_s^2}{2}\right)-\rho\left(e+\frac{V^2}{2}\right)\right]+$$

$$\varphi\left(e_s+\frac{V_s^2}{2}\right)\frac{\mathrm{D}}{\mathrm{D}t}(\rho-\rho_s)-\left(e+\frac{V_s^2}{2}\right)\frac{\mathrm{D}\rho}{\mathrm{D}t}$$

$$=\nabla\cdot(\lambda\nabla T)+\rho q_{\mathrm{R}}+\boldsymbol{f}[\varphi\rho_s\boldsymbol{V}_s+(1-\varphi)\rho\boldsymbol{V}]+\nabla\cdot\boldsymbol{P}\cdot[\varphi^{\frac{2}{3}}\boldsymbol{V}_s+(1-\varphi^{\frac{2}{3}})\boldsymbol{V}]\qquad(11\text{-}5\text{-}13)$$

利用式(4-5-1)与式(4-5-2)，为近似处理，取 $T_{s0}\approx 0$，当 $T_{s0}>T$，式(4-5-2)′取(+)号，当 $T_{s0}<T$，取(−)号。

$$T_s=T(1\pm e^{-BF_i})\qquad(4\text{-}5\text{-}2)'$$

令

$$\eta=1\pm e^{-BF_i}\qquad(11\text{-}5\text{-}14)$$

这样，$\frac{T}{T_s}=\frac{1}{n}=\alpha$，$T_s=nT$，$e=C_pT$，$e_s=C_pT_s$ 将它们代入式(11-5-13)

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho\left(C_pT+\frac{V^2}{2}\right)\right]+\rho\left[C_pT(1-n)+\frac{1}{2}(V^2-V_s^2)\right]\nabla\cdot\boldsymbol{V}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}\rho\left(C_pT+\alpha\frac{V_s^2}{2}-C_pT-\frac{V^2}{2}\right)$$
$$+\varphi\left(C_pnT+\frac{V_s^2}{2}\right)\frac{\mathrm{D}}{\mathrm{D}t}C_pT(1-n)-\left(C_pnT\frac{V_s^2}{2}\right)\frac{\mathrm{D}\rho}{\mathrm{D}t}$$
$$=\nabla\cdot(\lambda\nabla T)+\rho q_{\mathrm{R}}+\boldsymbol{f}\cdot[\rho\boldsymbol{V}+\varphi(\rho_s\boldsymbol{V}_s-\rho\boldsymbol{V})]+\nabla\cdot\boldsymbol{P}\cdot[\varphi^{\frac{2}{3}}\boldsymbol{V}_s+(1-\varphi^{\frac{2}{3}})\boldsymbol{V}] \tag{11-5-13$'$}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho\left(C_pT+\frac{V^2}{2}\right)\right]+\rho\left[C_pT(1-n)+\frac{1}{2}(V^2-V_s^2)\right]\nabla\cdot\boldsymbol{V}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}\rho\left(\alpha\frac{V_s^2}{2}-\frac{V^2}{2}\right)$$
$$+\left(C_pnT+\frac{V_s^2}{2}\right)\left[\varphi C_p(1-n)\frac{\mathrm{D}T}{\mathrm{D}t}-\frac{\mathrm{D}\rho}{\mathrm{D}t}\right]$$
$$=\nabla\cdot(\lambda\nabla T)+\rho q_{\mathrm{R}}+\boldsymbol{f}\rho[\boldsymbol{V}+\varphi(\alpha\boldsymbol{V}_s-\boldsymbol{V})]+\nabla\cdot\boldsymbol{P}\cdot[\boldsymbol{V}+\varphi^{\frac{2}{3}}(\boldsymbol{V}_s-\boldsymbol{V})] \tag{11-5-13$''$}$$

全式除以 ρ，并按流体力学有意义原则整理为

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(C_pT+\frac{V^2}{2}\right)+\left(C_pT+\frac{V^2}{2}\right)\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\boldsymbol{V}\cdot\nabla\left[C_pT(1-n)+\frac{1}{2}(V^2-V_s^2)\right]$$
$$+\varphi\left[\frac{\mathrm{D}}{\mathrm{D}t}\left(\alpha\frac{V_s^2}{2}\frac{V^2}{2}\right)+\left(\alpha\frac{V_s^2}{2}-\frac{V^2}{2}\right)\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}\right]+\left(C_pnT+\frac{V_s^2}{2}\right)\left[\varphi C_p(1-n)\frac{\mathrm{D}T}{\mathrm{D}t}-\frac{\mathrm{D}\rho}{\mathrm{D}t}\right]$$
$$=\frac{1}{\rho}\nabla\cdot(\lambda\nabla T)+q_{\mathrm{R}}+\boldsymbol{f}\cdot[\boldsymbol{V}+\varphi(\alpha\boldsymbol{V}_s-\boldsymbol{V})]+\frac{1}{\rho}[\boldsymbol{V}+\varphi^{\frac{2}{3}}(\boldsymbol{V}_s-\boldsymbol{V})]\cdot\nabla\cdot\boldsymbol{P} \tag{11-5-13$'''$}$$

转化柱坐标系，只含连续相变量的微分方程

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(C_pT+\frac{V^2}{2}\right)+\left(C_pT+\frac{V^2}{2}\right)\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\left(u_r\frac{\partial}{\partial r}+v_\varphi\frac{\partial}{r\partial\varphi}+w_z\frac{\partial}{\partial z}\right)\left[C_pT(1-n)\right.$$
$$\left.+\nu\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}+\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}+\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]+\varphi\left\{\frac{\mathrm{D}}{\mathrm{D}t}\left[(\alpha-1)\frac{V^2}{2}-\frac{\alpha\nu}{2}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}+\frac{\partial v_\varphi}{\partial r}\right.\right.\right.$$
$$\left.\left.+\frac{\partial v_\varphi}{\partial z}+\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]+\left[(\alpha-1)\frac{V^2}{2}-\frac{\alpha\nu}{2}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}+\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}+\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}$$
$$\left.+\left[C_pnT+\frac{V^2}{2}-\frac{\nu}{2}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}+\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}+\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]\left[\varphi C_p(1-n)\frac{\mathrm{D}T}{\mathrm{D}t}-\frac{\mathrm{D}\rho}{\mathrm{D}t}\right]\right\}$$
$$=\frac{\lambda}{\rho}\left(\frac{\partial^2T}{\partial r^2}+\frac{\partial^2T}{r\partial\varphi}+\frac{\partial^2T}{\partial z^2}\right)+q_{\mathrm{R}}+f_ru_r+f_\varphi v_\varphi+f_zw_z-\frac{\varphi\nu\alpha}{2}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)f_r+\frac{1}{v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}\right.\right.$$
$$\left.\left.+\frac{\partial v_\varphi}{\partial z}\right)+\frac{1}{w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]+u_r\left\{\frac{p}{\rho r}-\frac{\partial p}{\rho\partial r}+\nu\left[\frac{4}{3}\left(\frac{\partial^2u_r}{\partial r^2}-\frac{\partial v_\varphi}{r^2\partial\varphi}\right)-\frac{2}{3}\left(\frac{\partial^2v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2w_z}{\partial z\partial r}\right)\right.\right.$$
$$\left.\left.+\frac{2}{3}\left(\frac{\partial u_r}{r\partial r}+\frac{\partial w_z}{r\partial\varphi\partial z}\right)\right]\right\}+v_\varphi\left\{-\frac{\partial p}{\rho r\partial\varphi}+\nu\left[\frac{4}{3}\frac{\partial^2v_\varphi}{r^2\partial\varphi^2}-\frac{2}{3}\left(\frac{\partial^2u_r}{r\partial\varphi\partial r}+\frac{\partial^2w_z}{r\partial r\partial\varphi}\right)+\frac{\partial u_r}{r^2\partial\varphi}\right]\right\}$$
$$+w_z\left\{-\frac{\partial p}{\rho\partial z}+\nu\left[\frac{4}{3}\frac{\partial^2w_z}{\partial z^2}-\frac{2}{3}\left(\frac{\partial^2u_r}{\partial r\partial z}+\frac{\partial^2v_\varphi}{r\partial\varphi\partial z}\right)\right]\right\}+\varphi^{\frac{2}{3}}\frac{\nu}{2u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\left[(1-\varphi^{\frac{2}{3}})\nu\left(\frac{\partial^2u_r}{r^2\partial\varphi^2}\right.\right.$$
$$\left.\left.+\frac{\partial^2u_r}{\partial z^2}+\frac{\partial^2u_r}{r\partial\varphi\partial r}+\frac{\partial^2u_r}{\partial r\partial z}+\frac{\partial u_r}{r^2\partial\varphi}\right)+2\varphi^{\frac{2}{3}}\nu t'\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial^2u_r}{r\partial\varphi\partial r}\frac{\partial u_r}{\partial z}\frac{\partial u_r}{\partial z\partial r}\right)\right]+\varphi^{\frac{2}{3}}\frac{\nu}{2v_\varphi}$$

$$\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\left\{(1-\varphi^{\frac{2}{3}})\nu\left(\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{\partial^2 v_\varphi}{\partial z^2}+\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}+\frac{\partial^2 v_\varphi}{r\partial z\partial\varphi}+\frac{\partial v_\varphi}{r^2\partial\varphi^2}\right)+\varphi^{\frac{2}{3}}\nu t'\left[2\left(\frac{\partial v_\varphi}{\partial z}\right.\right.\right.$$

$$\left.\left.\left.\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}+\frac{\partial v_\varphi}{\partial z}\frac{\partial^2 v_\varphi}{r\partial z\partial\varphi}\right)-\frac{1}{r}\left(\frac{\partial v_\varphi}{\partial r}\right)^2\right]\right\}+\varphi^{\frac{2}{3}}\frac{\nu}{2w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{\partial z}\right)\left[(1-\varphi^{\frac{2}{3}})\nu\left(\frac{\partial^2 w_z}{\partial r^2}\right.\right.$$

$$\left.\left.+\frac{\partial^2 w_z}{\partial r\partial z}+\frac{\partial^2 w_z}{r\partial\varphi\partial r}\right)+2\varphi^{\frac{2}{3}}\nu t'\left(\frac{\partial w_z}{\partial r}+\frac{\partial^2 w_z}{r\partial\varphi\partial r}\frac{\partial w_z}{r\partial\varphi}\frac{\partial^2 w_z}{r\partial\varphi\partial z}\right)\right] \tag{11-5-15}$$

式(11-5-15)就是可压缩湍流在柱坐标系下能量微分方程，当定常流去掉含$\frac{\partial}{\partial t}$项。

11.5.4　动量矩微分方程

将式(3-7-1)′代入式(3-7-3)

$$\varphi\frac{\mathrm{D}(\boldsymbol{r}\times\rho_s\boldsymbol{V}_s)}{\mathrm{D}t}-(\boldsymbol{r}\times\rho_s\boldsymbol{V}_s)\frac{1}{\rho_s}\left[\varphi\frac{\mathrm{D}\rho_s}{\mathrm{D}t}+(1-\varphi)\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\nabla\cdot\boldsymbol{V}\right]+(1-\varphi)\frac{\mathrm{D}(\boldsymbol{r}\times\rho\boldsymbol{V})}{\mathrm{D}t}$$

$$+(\boldsymbol{r}\times\rho\boldsymbol{V})\nabla\cdot\boldsymbol{V}=\boldsymbol{r}\times[\varphi\rho_s+(1-\varphi)\rho]\boldsymbol{f}+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{3-7-3$'$}$$

整理为

$$\frac{\mathrm{D}(\boldsymbol{r}\times\rho\boldsymbol{V})}{\mathrm{D}t}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}[\boldsymbol{r}\times(\rho_s\boldsymbol{V}_s-\rho\boldsymbol{V})]+\boldsymbol{V}\cdot\nabla[\boldsymbol{r}\times\rho(\boldsymbol{V}-\boldsymbol{V}_s)]-(\boldsymbol{r}\times\boldsymbol{V}_s)$$

$$\left[\varphi\frac{\mathrm{D}}{\mathrm{D}t}+(\rho_s-\rho)-\frac{\mathrm{D}\rho}{\mathrm{D}t}\right]=\boldsymbol{r}\times[\varphi\rho_s+(1-\varphi)\rho]\boldsymbol{f}+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{3-7-3$''$}$$

将 $\rho_s=\frac{\rho T}{T_s}$代入式(3-7-3)″，利用式(4-5-2)，将式(4-5-7)简化为

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{r}\times\rho\boldsymbol{V})+\varphi\frac{\mathrm{D}}{\mathrm{D}t}[\boldsymbol{r}\times\rho(\alpha\boldsymbol{V}_s-\boldsymbol{V})]+\boldsymbol{V}\cdot\nabla[\boldsymbol{r}\times\rho(\boldsymbol{V}-\boldsymbol{V}_s)]-(\boldsymbol{r}\times\boldsymbol{V}_s)$$

$$[\varphi(\alpha-1)-1]\frac{\mathrm{D}\rho}{\mathrm{D}t}=[1+\varphi(\alpha-1)]\rho(\boldsymbol{r}\times\boldsymbol{f})+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{11-5-16}$$

对式(11-5-16)各项分别进行推演

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{r}\times\rho\boldsymbol{V})=\frac{\mathrm{D}}{\mathrm{D}t}\rho[(r_2V_3-r_3V_2)\boldsymbol{e}_1+(r_3V_1-r_1V_3)\boldsymbol{e}_2+(r_1V_2-r_2V_1)\boldsymbol{e}_3]$$

$$=[(r_2V_3-r_3V_2)\boldsymbol{e}_1+(r_3V_1-r_1V_3)\boldsymbol{e}_2+(r_1V_2-r_2V_1)\boldsymbol{e}_3]\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\frac{\partial}{\partial t}[(r_2V_3-$$

$$r_2V_2)\boldsymbol{e}_1+(r_3V_1-r_1V_3)\boldsymbol{e}_2+(r_1V_2-r_2V_1)\boldsymbol{e}_3]+V_1\frac{\partial}{H_1\partial q_1}[(r_2V_3-r_2V_2)\boldsymbol{e}_1$$

$$+(r_3V_1-r_1V_3)\boldsymbol{e}_2+(r_1V_2-r_2V_1)\boldsymbol{e}_3]+V_2\frac{\partial}{H_2\partial q_2}[(r_2V_3-r_2V_2)\boldsymbol{e}_1+(r_3V_1$$

$$-r_1V_3)\boldsymbol{e}_2+(r_1V_2-r_2V_1)\boldsymbol{e}_3]+V_3\frac{\partial}{H_3\partial q_3}[(r_2V_3-r_2V_2)\boldsymbol{e}_1+(r_3V_1-r_1V_3)$$

$$\boldsymbol{e}_2+(r_1V_2-r_2V_1)\boldsymbol{e}_3]$$

$$=\boldsymbol{e}_r(r_\varphi w_z-r_zv_\varphi)\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\boldsymbol{e}_r\left\{\frac{\partial}{\partial t}(r_2V_3-r_3V_2)+u_r\frac{\partial}{\partial r}(r_\varphi w_z-r_zv_\varphi)\right.$$

$$\left.+v_\varphi\left[\frac{\partial}{r\partial\varphi}(r_\varphi w_z-r_zv_\varphi)+\frac{1}{r}(r_\varphi w_z-r_zv_\varphi)\right]+w_z\frac{\partial}{\partial z}(r_\varphi w_z-r_zv_\varphi)\right\}$$

$$+e_\varphi(r_z u_r - r_r w_z)\frac{D\rho}{Dt} + e_\varphi\rho\left\{\frac{\partial}{\partial t}(r_3 V_1 - r_1 V_3) + u_r\frac{\partial}{\partial r}(r_z u_r - r_r w_z)\right.$$
$$+v_\varphi\left[\frac{\partial}{r\partial\varphi}(r_z u_r - r_r w_z) - \frac{1}{r}(r_z u_r - r_r w_z)\right] + w_z\frac{\partial}{\partial z}(r_z u_r - r_r w_z)\Big\}$$
$$+e_z(r_r v_\varphi - r_\varphi u_r)\frac{D\rho}{Dt} + \rho e_z\left[\frac{\partial}{\partial t}(r_1 V_2 - r_2 V_1) + u_r\frac{\partial}{\partial r}(r_r v_\varphi - r_\varphi u_r)\right.$$
$$\left.+v_\varphi\frac{\partial}{r\partial\varphi}(r_r v_\varphi - r_\varphi u_r) + w_z\frac{\partial}{\partial z}(r_r v_\varphi - r_\varphi u_r)\right] \tag{a}$$

式(11-5-16)中第二项推导

$$\varphi\frac{D}{Dt}[\boldsymbol{r}\times\rho(\alpha\boldsymbol{V}_s - \boldsymbol{V})] = \varphi\frac{D}{Dt}[\rho\boldsymbol{r}\times(\alpha\boldsymbol{V}_s - \boldsymbol{V})]$$
$$=\varphi[\boldsymbol{r}\times(\alpha\boldsymbol{V}_s - \boldsymbol{V})]\frac{D\rho}{Dt} + \varphi\rho\frac{D}{Dt}[\boldsymbol{r}\times(\alpha\boldsymbol{V}_s - \boldsymbol{V})] \tag{b}$$

令

$$R_2^3 = r_2\left[(\alpha-1)V_3 - \frac{\alpha\nu}{2V_3}\left(\frac{\partial V_3}{H_1\partial q_1} + \frac{\partial V_3}{H_2\partial q_2}\right)\right] - r_3\left[(\alpha-1)V_2 - \frac{\alpha\nu}{2V_2}\left(\frac{\partial V_2}{H_1\partial q_1} + \frac{\partial V_2}{H_3\partial q_3}\right)\right] \tag{11-5-17}$$

$$R_3^1 = r_3\left[(\alpha-1)V_1 - \frac{\alpha\nu}{2V_1}\left(\frac{\partial V_1}{H_2\partial q_2} + \frac{\partial V_1}{H_3\partial q_3}\right)\right] - r_1\left[(\alpha-1)V_3 - \frac{\alpha\nu}{2V_3}\left(\frac{\partial V_3}{H_1\partial q_1} + \frac{\partial V_3}{H_2\partial q_2}\right)\right] \tag{11-5-18}$$

$$R_1^2 = r_1\left[(\alpha-1)V_2 - \frac{\alpha\nu}{2V_2}\left(\frac{\partial V_2}{H_1\partial q_1} + \frac{\partial V_2}{H_3\partial q_3}\right)\right] - r_2\left[(\alpha-1)V_1 - \frac{\alpha\nu}{2V_1}\left(\frac{\partial V_1}{H_2\partial q_2} + \frac{\partial V_1}{H_3\partial q_3}\right)\right] \tag{11-5-19}$$

(b)项推导，除以 ρ 可简化为

$$\varphi(R_2^3 e_1 + R_3^1 e_2 + R_1^2 e_3)\frac{D\rho}{\rho Dt} + \varphi\left[\frac{\partial}{\partial t}(R_2^3 e_1 + R_3^1 e_2 + R_1^2 e_3) + V_1\frac{\partial}{H_1\partial q_1}(R_2^3 e_1 + R_3^1 e_2 + R_1^2 e_3)\right.$$
$$\left.+V_2\frac{\partial}{H_2\partial q_2}(R_2^3 e_1 + R_3^1 e_2 + R_1^2 e_3) + V_3\frac{\partial}{H_3\partial q_3}(R_2^3 e_1 + R_3^1 e_2 + R_1^2 e_3)\right]$$

$$=\varphi(R_2^3 e_1 + R_3^1 e_2 + R_1^2 e_3)\frac{D\rho}{Dt} + \varphi\left\{\frac{\partial}{\partial t}(R_2^3 e_1 + R_3^1 e_2 + R_1^2 e_3) + V_1\left[\left(\frac{e_1}{H_1}\frac{\partial R_2^3}{\partial q_1} + \frac{e_2}{H_1}\frac{\partial R_3^1}{\partial q_1} + \frac{e_3}{H_1}\frac{\partial R_1^2}{\partial q_1}\right)\right.\right.$$
$$+\left(\frac{R_2^3}{H_1}\frac{\partial e_1}{\partial q_1} + \frac{R_3^1}{H_1}\frac{\partial e_2}{\partial q_1} + \frac{R_1^2}{H_1}\frac{\partial e_3}{\partial q_1}\right)\Bigg] + V_2\left[\left(\frac{e_1}{H_2}\frac{\partial R_2^3}{\partial q_2} + \frac{e_2}{H_2}\frac{\partial R_3^1}{\partial q_2} + \frac{e_3}{H_2}\frac{\partial R_1^2}{\partial q_2}\right) + \left(\frac{R_2^3}{H_2}\frac{\partial e_1}{\partial q_2}\right.\right.$$
$$\left.\left.+\frac{R_3^1}{H_2}\frac{\partial e_2}{\partial q_2} + \frac{R_1^2}{H_2}\frac{\partial e_3}{\partial q_2}\right)\right] + V_3\left[\left(\frac{e_1}{H_3}\frac{\partial R_2^3}{\partial q_3} + \frac{e_2}{H_3}\frac{\partial R_3^1}{\partial q_3} + \frac{e_3}{H_3}\frac{\partial R_1^2}{\partial q_3}\right) + \left(\frac{R_2^3}{H_3}\frac{\partial e_1}{\partial q_3} + \frac{R_3^1}{H_3}\frac{\partial e_2}{\partial q_3} + \frac{R_1^2}{H_3}\frac{\partial e_3}{\partial q_3}\right)\right]\Bigg\}$$

$$=\varphi(R_2^3 e_1 + R_3^1 e_2 + R_1^2 e_3)\frac{D\rho}{Dt} + \varphi\left\{\frac{\partial}{\partial t}(R_2^3 e_1 + R_3^1 e_2 + R_1^2 e_3) + V_1\left[\frac{e_1}{H_1}\frac{\partial R_2^3}{\partial q_1} + \frac{e_2}{H_1}\frac{\partial R_3^1}{\partial q_1} + \frac{e_3}{H_1}\frac{\partial R_1^2}{\partial q_1}\right.\right.$$
$$\left.+\frac{R_2^3}{H_1}\left(-\frac{e_2}{H_2}\frac{\partial H_1}{\partial q_2} - \frac{e_3}{H_3}\frac{\partial H_1}{\partial q_3}\right) + \frac{R_3^1}{H_1}\frac{e_1}{H_2}\frac{\partial H_1}{\partial q_2} + \frac{R_1^2}{H_1}\frac{e_1}{H_3}\frac{\partial H_1}{\partial q_3}\right] + V_2\left[\frac{e_1}{H_2}\frac{\partial R_2^3}{\partial q_2} + \frac{e_2}{H_2}\frac{\partial R_3^1}{\partial q_2}\right.$$

$$+\frac{e_3}{H_2}\frac{\partial R_1^2}{\partial q_2}+\frac{R_2^3}{H_2H_1}\frac{e_2}{}\frac{\partial H_2}{\partial q_1}+\frac{R_3^1}{H_2}\left(-\frac{e_3}{H_3}\frac{\partial H_2}{\partial q_3}-\frac{e_1}{H_1}\frac{\partial H_2}{\partial q_1}\right)+\frac{R_1^2}{H_2H_3}\frac{e_2}{}\frac{\partial H_2}{\partial q_3}\Bigg]+V_3\Bigg[\frac{e_1}{H_3}\frac{\partial R_2^3}{\partial q_3}$$

$$+\frac{e_2}{H_3}\frac{\partial R_3^1}{\partial q_3}+\frac{e_3}{H_3}\frac{\partial R_1^2}{\partial q_3}+\frac{R_2^3}{H_3H_1}\frac{e_3}{}\frac{\partial H_1}{\partial q_1}+\frac{R_3^1}{H_3H_2}\frac{e_3}{}\frac{\partial H_3}{\partial q_2}+\frac{R_1^2}{H_3}\left(-\frac{e_1}{H_1}\frac{\partial H_3}{\partial q_1}-\frac{e_2}{H_2}\frac{\partial H_3}{\partial q_2}\right)\Bigg]\Bigg\}$$

$$=\varphi(R_2^3e_1+R_3^1e_2+R_1^2e_3)\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\varphi\Bigg[\frac{\partial}{\partial t}(R_2^3e_1+R_3^1e_2+R_1^2e_3)+V_1\left(e_r\frac{\partial}{\partial r}R_2^3+e_\varphi\frac{\partial}{\partial r}R_3^1\right.$$

$$\left.+e_z\frac{\partial}{\partial r}R_1^2\right)+V_2\left(e_r\frac{\partial}{r\partial\varphi}R_2^3+e_\varphi\frac{\partial}{r\partial\varphi}R_3^1+e_z\frac{\partial}{r\partial\varphi}R_1^2+\frac{e_2}{r}R_2^3-e_r\frac{1}{r}R_3^1\right)+V_3\left(e_r\frac{\partial}{\partial z}R_2^3\right.$$

$$\left.+e_\varphi\frac{\partial}{\partial z}R_3^1+e_z\frac{\partial}{\partial z}R_1^2\right)\Bigg] \tag{b}'$$

推导式(11-5-16)中第三项

$$\boldsymbol{V}\cdot\nabla[\rho\boldsymbol{r}\times(\boldsymbol{V}-\boldsymbol{V}_s)]=[\boldsymbol{r}\times(\boldsymbol{V}-\boldsymbol{V}_s)](\boldsymbol{V}\cdot\nabla)\rho+\rho(\boldsymbol{V}\cdot\nabla)[\boldsymbol{P}\times(\boldsymbol{V}-\boldsymbol{V}_s)] \tag{c}$$

首先将式(c)除以 ρ，令

$$R_{23}=\frac{r_2}{V_3}\left(\frac{\partial V_3}{H_1\partial q_1}+\frac{\partial V_3}{H_2\partial q_2}\right)-\frac{r_3}{V_2}\left(\frac{\partial V_2}{H_1\partial q_1}+\frac{\partial V_2}{H_3\partial q_3}\right) \tag{11-5-20}$$

$$R_{31}=\frac{r_3}{V_1}\left(\frac{\partial V_2}{H_2\partial q_2}+\frac{\partial V_1}{H_3\partial q_3}\right)-\frac{r_2}{V_3}\left(\frac{\partial V_3}{H_1\partial q_1}+\frac{\partial V_3}{H_2\partial q_2}\right) \tag{11-5-21}$$

$$R_{12}=\frac{r_1}{V_2}\left(\frac{\partial V_2}{H_1\partial q_1}+\frac{\partial V_2}{H_3\partial q_3}\right)-\frac{r_2}{V_1}\left(\frac{\partial V_1}{H_2\partial q_2}+\frac{\partial V_1}{H_3\partial q_3}\right) \tag{11-5-22}$$

将式(11-5-20)～式(11-5-22)代入式(c)

$$\boldsymbol{V}\cdot\nabla[\rho\boldsymbol{r}\times(\boldsymbol{V}-\boldsymbol{V}_s)]=\frac{\nu}{2}\Bigg[\frac{1}{\rho}(R_{23}e_1+R_{31}e_2+R_{12}e_3)(\boldsymbol{V}\cdot\nabla)\rho+e_1(\boldsymbol{V}\cdot\nabla)R_{23}+R_{23}(\boldsymbol{V}\cdot\nabla)e_1$$

$$+e_2(\boldsymbol{V}\cdot\nabla)R_{31}+R_{31}(\boldsymbol{V}\cdot\nabla)e_2+e_3(\boldsymbol{V}\cdot\nabla)R_{12}+R_{12}(\boldsymbol{V}\cdot\nabla)e_3\Bigg]$$

$$=\frac{\nu}{2}\Bigg[\frac{1}{\rho}(R_{23}e_1+R_{31}e_2+R_{12}e_3)(\boldsymbol{V}\cdot\nabla)\rho+e_1(\boldsymbol{V}\cdot\nabla)R_{23}+e_2(\boldsymbol{V}\cdot\nabla)R_{31}$$

$$+e_3(\boldsymbol{V}\cdot\nabla)R_{12}+R_{23}\left(V_1\frac{\partial e_1}{H_1\partial q_1}+V_2\frac{\partial e_1}{H_2\partial q_2}+V_3\frac{\partial e_1}{H_3\partial q_3}\right)$$

$$+R_{31}\left(V_1\frac{\partial e_2}{H_1\partial q_1}+V_2\frac{\partial e_2}{H_2\partial q_2}+V_3\frac{\partial e_2}{H_3\partial q_3}\right)+R_{12}\left(V_1\frac{\partial e_3}{H_1\partial q_1}\right.$$

$$\left.+V_2\frac{\partial e_3}{H_2\partial q_2}+V_3\frac{\partial e_3}{H_3\partial q_3}\right)\Bigg]$$

$$=\frac{\nu}{2}\Bigg[\frac{1}{\rho}(R_{23}e_1+R_{31}e_2+R_{12}e_3)(\boldsymbol{V}\cdot\nabla)\rho+e_1(\boldsymbol{V}\cdot\nabla)R_{23}$$

$$+e_2(\boldsymbol{V}\cdot\nabla)R_{31}+e_3(\boldsymbol{V}\cdot\nabla)R_{12}+R_{23}V_2\frac{e_2}{H_2}-R_{31}V_2\frac{e_1}{H_2}\Bigg] \tag{c}'$$

推导式(11-5-16)中第四项，除以 ρ，则

$$(\boldsymbol{r}\times\boldsymbol{V}_s)[\varphi(\alpha-1)-1]\frac{\mathrm{D}\rho}{\mathrm{D}t}$$

$$=[\varphi(\alpha-1)-1](\boldsymbol{r}\times\boldsymbol{V}_s)\frac{\mathrm{D}\rho}{\mathrm{D}t}$$

$$=\frac{1}{\rho}[\varphi(\alpha-1)-1]\left\{\left\{r_2\left[V_3-\frac{\nu}{2V_3}\left(\frac{\partial V_3}{H_1\partial q_1}+\frac{\partial V_3}{H_2\partial q_2}\right)\right]-r_3\left[V_2-\frac{\nu}{2V_2}\left(\frac{\partial V_2}{H_1\partial q_1}+\frac{\partial V_2}{H_3\partial q_3}\right)\right]\right\}\boldsymbol{e}_1\right.$$

$$+\left\{r_3\left[V_1-\frac{\nu}{2V_1}\left(\frac{\partial V_1}{H_2\partial q_2}+\frac{\partial V_1}{H_3\partial q_3}\right)\right]-r_1\left[V_3-\frac{\nu}{2V_3}\left(\frac{\partial V_3}{H_1\partial q_1}+\frac{\partial V_3}{H_2\partial q_2}\right)\right]\right\}\boldsymbol{e}_2$$

$$\left.+\left\{r_1\left[V_2-\frac{\nu}{2V_2}\left(\frac{\partial V_2}{H_1\partial q_1}+\frac{\partial V_2}{H_3\partial q_3}\right)\right]-r_2\left[V_1-\frac{\nu}{2V_1}\left(\frac{\partial V_1}{H_2\partial q_2}+\frac{\partial V_1}{H_3\partial q_3}\right)\right]\right\}\boldsymbol{e}_3\right\}\qquad(\mathrm{d})$$

推导式(11-5-16)右边第一项，除以 ρ

$$[1+\varphi(\alpha-1)](\boldsymbol{r}\times\boldsymbol{f})$$

$$=[1+\varphi(\alpha-1)][(r_2f_3-r_3f_2)\boldsymbol{e}_1+(r_3f_1-r_1f_3)\boldsymbol{e}_2+(r_1f_2-r_2f_1)\boldsymbol{e}_3]\qquad(\mathrm{e})$$

推导式(11-5-16)右边第二项

$$\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P})=\nabla\cdot[\boldsymbol{r}\times(\boldsymbol{p}_1\boldsymbol{e}_1+\boldsymbol{p}_2\boldsymbol{e}_2+\boldsymbol{p}_3\boldsymbol{e}_3)]$$

$$=\nabla\cdot\{\boldsymbol{r}\times[(p_{11}\boldsymbol{e}_1+p_{12}\boldsymbol{e}_2+p_{13}\boldsymbol{e}_3)\boldsymbol{e}_1+(p_{21}\boldsymbol{e}_1+p_{22}\boldsymbol{e}_2+p_{23}\boldsymbol{e}_3)\boldsymbol{e}_2+(p_{31}\boldsymbol{e}_1+p_{32}\boldsymbol{e}_2+p_{33}\boldsymbol{e}_3)\boldsymbol{e}_3]\}$$

$$=\nabla\cdot\{\boldsymbol{r}\times[\boldsymbol{e}_1(p_{11}\boldsymbol{e}_1+p_{21}\boldsymbol{e}_2+p_{31}\boldsymbol{e}_3)+\boldsymbol{e}_2(p_{12}\boldsymbol{e}_1+p_{22}\boldsymbol{e}_2+p_{32}\boldsymbol{e}_3)+\boldsymbol{e}_3(p_{13}\boldsymbol{e}_1+p_{23}\boldsymbol{e}_2+p_{33}\boldsymbol{e}_3)]\}$$

$$=\nabla\cdot\{[r_2(p_{13}\boldsymbol{e}_1+p_{23}\boldsymbol{e}_2+p_{33}\boldsymbol{e}_3)-r_3(p_{12}\boldsymbol{e}_1+p_{22}\boldsymbol{e}_2+p_{32}\boldsymbol{e}_3)]\boldsymbol{e}_1+[r_3(p_{11}\boldsymbol{e}_1+p_{21}\boldsymbol{e}_2+p_{31}\boldsymbol{e}_3)-r_1(p_{13}\boldsymbol{e}_1+p_{23}\boldsymbol{e}_2+p_{33}\boldsymbol{e}_3)\boldsymbol{e}_3]\}$$

$$=r_2\left[\frac{\boldsymbol{e}_1}{H_1}\frac{\partial p_{13}}{\partial q_1}+\frac{p_{13}}{H_1}\left(-\frac{\boldsymbol{e}_2}{H_2}\frac{\partial H_1}{\partial q_2}-\frac{\boldsymbol{e}_3}{H_3}\frac{\partial H_1}{\partial q_3}\right)+\frac{\boldsymbol{e}_2}{H_1}\frac{\partial p_{23}}{\partial q_1}+\frac{p_{23}}{H_1}\frac{\boldsymbol{e}_1}{H_2}\frac{\partial H_1}{\partial q_2}+\frac{\boldsymbol{e}_3}{H_1}\frac{\partial p_{33}}{\partial q_1}\right.$$

$$\left.+\frac{p_{33}}{H_1}\frac{\boldsymbol{e}_1}{H_3}\frac{\partial H_1}{\partial q_3}\right]-r_3\left[\frac{\boldsymbol{e}_1}{H_1}\frac{\partial p_{12}}{\partial q_1}+\frac{p_{12}}{H_1}\left(-\frac{\boldsymbol{e}_2}{H_2}\frac{\partial H_1}{\partial q_2}-\frac{\boldsymbol{e}_3}{H_3}\frac{\partial H_1}{\partial q_3}\right)+\frac{\boldsymbol{e}_2}{H_1}\frac{\partial p_{22}}{\partial q_1}\right.$$

$$\left.+\frac{p_{22}}{H_1}\frac{\boldsymbol{e}_1}{H_2}\frac{\partial H_1}{\partial q_2}+\frac{\boldsymbol{e}_3}{H_1}\frac{\partial p_{32}}{\partial q_1}+\frac{p_{32}}{H_1}\frac{\boldsymbol{e}_1}{H_3}\frac{\partial H_1}{\partial q_3}\right]+r_3\left[\frac{\boldsymbol{e}_1}{H_2}\frac{\partial p_{11}}{\partial q_2}+\frac{p_{11}}{H_2}\frac{\boldsymbol{e}_2\partial H_2}{\partial q_2}+\frac{\boldsymbol{e}_2}{H_2}\frac{\partial p_{21}}{\partial q_2}\right.$$

$$\left.+\frac{p_{21}}{H_2}\left(-\frac{\boldsymbol{e}_3}{H_3}\frac{\partial H_2}{\partial q_3}-\frac{\boldsymbol{e}_1}{H_1}\frac{\partial H_2}{\partial q_1}\right)+\frac{\boldsymbol{e}_3}{H_2}\frac{\partial p_{31}}{\partial q_2}+\frac{p_{31}}{H_2}\frac{\boldsymbol{e}_2}{H_3}\frac{\partial H_2}{\partial q_3}\right]-r_1\left[\frac{\boldsymbol{e}_1}{H_2}\frac{\partial p_{13}}{\partial q_2}\right.$$

$$\left.+\frac{p_{13}}{H_2}\frac{\boldsymbol{e}_2}{H_1}\frac{\partial H_2}{\partial q_1}+\frac{\boldsymbol{e}_2}{H_2}\frac{\partial p_{23}}{\partial q_2}+\frac{p_{23}}{H_2}\left(-\frac{\boldsymbol{e}_3}{H_3}\frac{\partial H_2}{\partial q_3}-\frac{\boldsymbol{e}_1}{H_1}\frac{\partial H_2}{\partial q_1}\right)+\frac{\boldsymbol{e}_3}{H_2}\frac{\partial p_{33}}{\partial q_2}+\frac{p_{33}}{H_2}\frac{\boldsymbol{e}_2}{H_3}\frac{\partial H_2}{\partial q_3}\right]$$

$$+r_1\left[\frac{\boldsymbol{e}_1}{H_3}\frac{\partial p_{12}}{\partial q_3}+\frac{p_{12}}{H_3}\frac{\boldsymbol{e}_3}{H_1}\frac{\partial H_3}{\partial q_1}+\frac{\boldsymbol{e}_2}{H_3}\frac{\partial p_{22}}{\partial q_3}+\frac{p_{22}}{H_3}\frac{\boldsymbol{e}_3}{H_2}\frac{\partial H_3}{\partial q_2}+\frac{\boldsymbol{e}_3}{H_3}\frac{\partial p_{32}}{\partial q_3}\right.$$

$$\left.+\frac{p_{32}}{H_3}\left(-\frac{\boldsymbol{e}_1}{H_1}\frac{\partial H_3}{\partial q_1}-\frac{\boldsymbol{e}_2}{H_2}\frac{\partial H_3}{\partial q_2}\right)\right]-r_2\left[\frac{\boldsymbol{e}_1}{H_3}\frac{\partial p_{11}}{\partial q_3}+\frac{p_{11}}{H_3}\frac{\boldsymbol{e}_3}{H_1}\frac{\partial H_3}{\partial q_1}+\frac{\boldsymbol{e}_2}{H_3}\frac{\partial p_{21}}{\partial q_3}\right.$$

$$\left.+\frac{p_{21}}{H_3}\frac{\boldsymbol{e}_3}{H_2}\frac{\partial H_3}{\partial q_2}+\frac{\boldsymbol{e}_3}{H_3}\frac{\partial p_{31}}{\partial q_3}+\frac{p_{31}}{H_3}\left(-\frac{\boldsymbol{e}_1}{H_1}\frac{\partial H_3}{\partial q_1}-\frac{\boldsymbol{e}_2}{H_2}\frac{\partial H_3}{\partial q_2}\right)\right]\qquad(\mathrm{f})$$

将(f)项变成坐标系对应的公式

$$r_\varphi\left\{\boldsymbol{e}_r\left[\mu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 w_z}{\partial r^2}+\frac{\partial^2 u_r}{\partial z\partial r}\right)+2\varphi^{\frac{2}{3}}\rho\nu t'\frac{\partial w_z}{\partial r}\frac{\partial^2 w_z}{\partial r^2}\right]+\boldsymbol{e}_\varphi\left[\mu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 w_z}{r\partial\varphi\partial r}+\frac{\partial^2 v_\varphi}{\partial z\partial r}\right)\right.\right.$$

$$+2\varphi^{\frac{2}{3}}\rho\nu t'\frac{\partial w_z}{r\partial\varphi}\frac{\partial^2 w_z}{r\partial\varphi\partial r}\Big]+e_z\Big[-\frac{\partial p}{\partial r}+\mu\Big(\frac{4}{3}\frac{\partial^2 w_z}{\partial z\partial r}-\frac{2}{3}\frac{\partial^2 u_r}{\partial r^2}-\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}\Big)\Big]\Big\}-r_z\Big\{e_r\Big[\mu(1-\varphi^{\frac{2}{3}})\Big(\frac{\partial^2 v_\varphi}{\partial r^2}$$

$$+\frac{\partial^2 u_r}{r\partial\varphi\partial r}-\frac{1}{r}\frac{\partial v_\varphi}{\partial r}+\frac{v_\varphi}{r^2}\Big)+2\varphi^{\frac{2}{3}}\rho\nu t'\frac{\partial v_\varphi}{\partial r}\frac{\partial^2 v_\varphi}{\partial r^2}\Big]+e_\varphi\Big[-\frac{\partial p}{\partial r}+\mu\Big(\frac{4}{3}\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}-\frac{2}{3}\frac{\partial^2 u_r}{\partial r^2}-\frac{\partial^2 w_z}{\partial z\partial r}$$

$$+2\frac{\partial u_r}{r\partial r}-2\frac{u_r}{r^2}\Big)\Big]+e_z\Big[\mu(1-\varphi^{\frac{2}{3}})\Big(\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{\partial^2 w_z}{r\partial\varphi\partial r}\Big)+2\varphi^{\frac{2}{3}}\rho\nu t'\frac{\partial v_\varphi}{\partial z}\frac{\partial^2 v_\varphi}{\partial z\partial r}\Big]\Big\}+r_z\Big\{e_r\Big[-\frac{\partial p}{r\partial\varphi}$$

$$+\mu\Big(\frac{4}{3}\frac{\partial^2 u_r}{r\partial\varphi\partial r}-\frac{2}{3}\frac{\partial^2 v_\varphi}{r^2\partial\varphi^2}+\frac{\partial^2 w_z}{r\partial z\partial\varphi}\Big)-\frac{1}{r}\mu(1-\varphi^{\frac{2}{3}})\Big(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial v_\varphi}{\partial r}+\frac{v_\varphi}{r}\Big)+\frac{1}{r}\varphi^{\frac{2}{3}}\rho\nu t'\Big(\frac{\partial u_r}{r\partial\varphi}\Big)^2\Big]$$

$$+e_\varphi\frac{1}{r}\Big[-p+\mu\Big(\frac{4}{3}\frac{\partial u_r}{\partial r}-\frac{2}{3}\frac{\partial v_\varphi}{r\partial\varphi}-\frac{2}{3}\frac{\partial w_z}{\partial z}\Big)\Big]+e_z\Big[\mu(1-\varphi^{\frac{2}{3}})\Big(\frac{\partial^2 u_r}{r\partial z\partial\varphi}+\frac{\partial^2 w_z}{r\partial\varphi\partial r}\Big)+2\varphi^{\frac{2}{3}}\rho$$

$$\nu t'\frac{\partial u_r}{\partial z}\frac{\partial^2 u_r}{r\partial z\partial\varphi}\Big]\Big\}-r_r\Big\{e_r\Big[\mu(1-\varphi^{\frac{2}{3}})\Big(\frac{\partial^2 w_z}{r\partial\varphi\partial r}+\frac{\partial^2 u_r}{r\partial z\partial\varphi}-\frac{1}{r}\frac{\partial w_z}{r\partial\varphi}-\frac{1}{r}\frac{\partial v_\varphi}{\partial z}\Big)+\varphi^{\frac{2}{3}}\rho\nu t'\Big(\frac{\partial w_z}{r\partial\varphi}\Big)^2$$

$$+2\varphi^{\frac{2}{3}}\rho\nu t'\Big(\frac{\partial w_z}{\partial r}\frac{\partial^2 w_z}{r\partial\varphi\partial r}\Big)\Big]+e_\varphi\Big[\mu(1-\varphi^{\frac{2}{3}})\Big(\frac{\partial^2 w_z}{r^2\partial\varphi^2}+\frac{\partial^2 v_\varphi}{r\partial z\partial\varphi}+\frac{1}{r}\frac{\partial w_z}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial z}\Big)+2\varphi^{\frac{2}{3}}\rho\nu t'$$

$$\frac{\partial w_z}{r\partial\varphi}\frac{\partial^2 w_z}{r^2\partial\varphi^2}+\varphi^{\frac{2}{3}}\rho\nu t'\Big(\frac{\partial w_z}{\partial r}\Big)^2\Big]+e_z\Big[-\frac{\partial p}{r\partial\varphi}+\mu\Big(\frac{4}{3}\frac{\partial^2 w_z}{r\partial z\partial\varphi}-\frac{2}{3}\frac{\partial^2 u_r}{r\partial\varphi\partial r}-\frac{2}{3}\frac{\partial v_\varphi}{r^2\partial\varphi^2}\Big)\Big]\Big\}$$

$$+r_r\Big\{e_r\Big[\mu(1-\varphi^{\frac{2}{3}})\Big(\frac{\partial^2 v_\varphi}{\partial r\partial z}+\frac{\partial^2 u_r}{r\partial\varphi\partial z}-\frac{1}{r}\frac{\partial v_\varphi}{r\partial z}\Big)+2\varphi^{\frac{2}{3}}\rho\nu t'\frac{\partial v_\varphi}{\partial r}\frac{\partial^2 v_\varphi}{\partial r\partial z}\Big]+e_\varphi\Big[-\frac{\partial p}{\partial z}+\mu\Big(\frac{4}{3}\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}$$

$$-\frac{2}{3}\frac{\partial^2 u_r}{\partial r\partial z}-\frac{2}{3}\frac{\partial^2 w_z}{\partial z^2}+\frac{1}{r}\frac{\partial u_r}{\partial z}\Big)\Big]+e_z\Big[\mu(1-\varphi^{\frac{2}{3}})\Big(\frac{\partial^2 v_\varphi}{\partial z^2}+\frac{\partial^2 w_z}{r\partial\varphi\partial z}\Big)+2\varphi^{\frac{2}{3}}\rho\nu t'\frac{\partial v_\varphi}{\partial z}\frac{\partial^2 v_\varphi}{+\partial z^2}\Big]\Big\}$$

$$-r_\varphi\Big\{e_r\Big[-\frac{\partial p}{\partial z}+\mu\Big(\frac{4}{3}\frac{\partial^2 u_r}{\partial r\partial z}-\frac{2}{3}\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}-\frac{2}{3}\frac{\partial^2 w_z}{\partial z^2}\Big)\Big]+e_\varphi\Big[\mu(1-\varphi^{\frac{2}{3}})\Big(\frac{\partial^2 u_r}{r\partial\varphi\partial z}+\frac{\partial^2 v_\varphi}{\partial r\partial z}$$

$$-\frac{1}{r}\frac{\partial v_\varphi}{\partial z}\Big)+2\varphi^{\frac{2}{3}}\rho\nu t'\frac{\partial u_r}{r\partial\varphi}\frac{\partial^2 u_r}{r\partial\varphi\partial z}\Big]+e_z\Big[\mu(1-\varphi^{\frac{2}{3}})\Big(\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial^2 w_z}{\partial r\partial z}\Big)+2\varphi^{\frac{2}{3}}\rho\nu t'\frac{\partial u_r}{\partial z}\frac{\partial^2 u_r}{\partial z^2}\Big]\Big\} \tag{f$'$}$$

将式(a)，式(f)′除以 ρ 后，与式(b)′，式(c)′，式(d)，式(e)一起代入式(11-5-16)，按力矩定义确定转轴，分别列出

$$(r_\varphi w_z-r_zv_\varphi)\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\frac{\partial}{\partial t}(r_\varphi w_z-r_zv_\varphi)+u_r\frac{\partial}{\partial r}(r_\varphi w_z-r_zv_\varphi)+v_\varphi\Big[\frac{\partial}{r\partial\varphi}(r_\varphi w_z-r_zv_\varphi)$$

$$+\frac{1}{r}(r_\varphi w_z-r_zv_\varphi)\Big]+w_z\frac{\partial}{\partial z}(r_\varphi w_z-r_zv_\varphi)+\varphi R_2^3\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\varphi\Big[\frac{\partial}{\partial t}R_2^3+u_r\frac{\partial}{\partial r}R_2^3+v_\varphi\Big(\frac{\partial}{r\partial\varphi}R_2^3$$

$$+\frac{1}{r}R_2^3\Big)+w_z\frac{\partial}{\partial z}R_2^3\Big]+\frac{\nu}{2}\Big[\frac{1}{\rho}R_{23}(\mathbf{V}\cdot\nabla)\rho+(\mathbf{V}\cdot\nabla)R_{23}+v_\varphi\frac{R_{23}}{r}\Big]+\frac{1}{\rho}[\varphi(\alpha-1)-1]$$

$$\Big\{r_\varphi\Big[w_z-\frac{\nu}{2w_z}\Big(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\Big)\Big]-r_z\Big[v_\varphi-\frac{\nu}{2v_\varphi}\Big(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\Big)\Big]\Big\}\frac{\mathrm{D}p}{\mathrm{D}t}$$

$$=[1+\varphi(\alpha-1)-1](r_\varphi f_z-r_zf_\varphi)+\frac{1}{\rho}\Big(r_\varphi\frac{\partial p}{\partial z}-r_z\frac{\partial p}{r\partial\varphi}\Big)+\nu\Big\{r_\varphi\Big(\frac{4}{3}\frac{\partial^2 w_z}{\partial z\partial r}+\frac{2}{3}\frac{\partial^2 w_z}{\partial z\partial r}\Big)-r_z$$

$$\left(\frac{4}{3}\frac{\partial^2 v_\varphi}{r\partial\varphi\partial r}+\frac{2}{3}\frac{\partial^2 v_\varphi}{r^2\partial\varphi^2}\right)+(1-\varphi^{\frac{2}{3}})\left[r_\varphi\left(\frac{\partial^2 w_z}{\partial r^2}+\frac{\partial^2 w_z}{r\partial\varphi\partial r}-\frac{\partial^2 w_z}{\partial r\partial z}\right)-r_z\left(\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{\partial^2 v_\varphi}{r^2\partial\varphi}-\frac{\partial^2 v_\varphi}{r\partial r\partial\varphi}\right)\right]$$

$$+2\varphi^{\frac{2}{3}}t'\left[r_\varphi\left(\frac{\partial w_z}{\partial r}\frac{\partial^2 w_z}{\partial r^2}+\frac{\partial w_z}{r\partial\varphi}\frac{\partial^2 w_z}{r\partial\varphi\partial r}\right)-r_z\left(\frac{\partial v_\varphi}{\partial r}\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{\partial v_\varphi}{\partial z}\frac{\partial^2 v_\varphi}{\partial z\partial r}\right)\right]\Bigg\} \quad (11\text{-}5\text{-}23)$$

$$(r_z u_r-r_r w_z)\frac{D\rho}{\rho Dt}+\frac{\partial}{\partial t}(r_z u_r-r_r w_z)+u_r\frac{\partial}{\partial r}(r_z u_r-r_r w_z)+v_\varphi\left[\frac{\partial}{r\partial\varphi}(r_z u_r-r_r w_z)\right.$$

$$\left.-\frac{1}{r}(r_z u_r-r_r w_z)\right]+w_z\frac{\partial}{\partial z}(r_z u_r-r_r w_z)+\varphi R_3^1\frac{D\rho}{\rho Dt}+\varphi\left[\frac{\partial}{\partial t}R_3^1+u_r\frac{\partial}{\partial r}R_3^1+v_\varphi\left(\frac{\partial}{r\partial\varphi}R_3^1\right.\right.$$

$$\left.\left.-\frac{1}{r}R_3^1\right)+w_z\frac{\partial}{\partial z}R_3^1\right]+\frac{\nu}{2}\left[\frac{1}{\rho}R_{31}(\boldsymbol{V}\cdot\nabla)\rho+(\boldsymbol{V}\cdot\nabla)R_{31}-R_{31}\frac{v_\varphi}{r}\right]+\frac{1}{\rho}[\varphi(\alpha-1)-1]$$

$$\left\{r_z\left[u_r-\frac{\nu}{2u_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\right]-r_r\left[w_z-\frac{\nu}{2w_z}\left(\frac{\partial w_z}{\partial r}+\frac{\partial w_z}{r\partial\varphi}\right)\right]\right\}\frac{D\rho}{Dt}$$

$$=[1+\varphi(\alpha-1)](r_z f_r-r_r f_z)+\frac{1}{\rho}\left(r_z\frac{\partial p}{\partial r}-r_r\frac{\partial p}{\partial z}\right)+\nu\left\{r_z\left(\frac{4}{3}\frac{\partial^2 u_r}{r\partial r\partial\varphi}+\frac{4}{3}\frac{\partial u_r}{r\partial r}\right.\right.$$

$$\left.+\frac{2}{3}\frac{\partial^2 u_r}{\partial r^2}+2\frac{u_r}{r^2}-2\frac{\partial u_r}{r\partial r}\right)-r_r\left(\frac{4}{3}\frac{\partial^2 w_z}{r\partial\varphi\partial z}+\frac{2}{3}\frac{\partial^2 w_z}{\partial z^2}\right)+(1-\varphi^{\frac{2}{3}})\left[r_z\left(\frac{\partial^2 u_r}{r^2\partial\varphi^2}+\frac{\partial^2 u_r}{r\partial\varphi\partial z}\right.\right.$$

$$\left.\left.-\frac{\partial^2 u_r}{r\partial\varphi\partial r}-\frac{1}{r}\frac{\partial u_r}{r\partial\varphi}\right)-r_r\left(\frac{\partial^2 w_z}{r\partial\varphi\partial r}+\frac{\partial^2 w_z}{r^2\partial\varphi^2}+\frac{1}{r}\frac{\partial w_z}{\partial r}-\frac{\partial^2 w_z}{r\partial\varphi\partial z}-\frac{1}{r}\frac{\partial w_z}{r\partial\varphi}\right)\right]+\varphi^{\frac{2}{3}}t'\left[\frac{1}{r}\left(\frac{\partial u_r}{r\partial\varphi}\right)^2\right.$$

$$\left.+2\frac{\partial u_r}{\partial z}\frac{\partial^2 u_r}{rz\partial\varphi}\right]-r_r\left[2\left(\frac{\partial w_z}{\partial r}\frac{\partial^2 w_z}{r\partial\varphi\partial r}+\frac{\partial w_z}{r\partial\varphi}\frac{\partial^2 w_z}{r^2\partial\varphi^2}\right)+\frac{1}{r}\left(\frac{\partial w_z}{\partial r}\right)^2-\frac{1}{r}\left(\frac{\partial w_z}{r\partial\varphi}\right)^2\right]\Bigg\} \quad (11\text{-}5\text{-}24)$$

$$(r_r v_\varphi-r_\varphi u_r)\frac{D\rho}{\rho Dt}+\frac{\partial}{\partial t}(r_r v_\varphi-r_\varphi u_r)+u_r\frac{\partial}{\partial r}(r_r v_\varphi-r_\varphi u_r)+v_\varphi\frac{\partial}{r\partial\varphi}(r_r v_\varphi-r_\varphi u_r)+w_z\frac{\partial}{\partial z}$$

$$(r_r v_\varphi-r_\varphi u_r)+\varphi R_1^2\frac{D\rho}{\rho Dt}+\varphi\left[\frac{\partial}{\partial t}R_1^2+u_r\frac{\partial}{\partial r}R_1^2+v_\varphi\frac{\partial}{r\partial\varphi}R_1^2+w_z\frac{\partial}{\partial z}R_1^2\right]+\frac{\nu}{2}\left[\frac{1}{\rho}R_{12}(\boldsymbol{V}\cdot\nabla)\rho\right.$$

$$\left.+(\boldsymbol{V}\cdot\nabla)R_{12}\right]+\frac{1}{\rho}[\varphi(\alpha-1)-1]\left\{r_r\left[v_\varphi-\frac{\nu}{2v_\varphi}\left(\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_\varphi}{\partial z}\right)\right]-r_\varphi\left[u_r-\frac{\nu}{uv_r}\left(\frac{\partial u_r}{r\partial\varphi}+\frac{\partial u_r}{\partial z}\right)\right]\right\}\frac{D\rho}{Dt}$$

$$=[1+\varphi(\alpha-1)](r_r f_\varphi-r_\varphi f_r)+\frac{1}{\rho}\left(r_r\frac{\partial p}{r\partial\varphi}-r_\varphi\frac{\partial p}{\partial r}\right)+\nu\left\{r_r\left(\frac{4}{3}\frac{\partial^2 v_\varphi}{r\partial\varphi\partial z}+\frac{2}{3}\frac{\partial^2 v_\varphi}{r^2\partial\varphi^2}\right)-r_\varphi\right.$$

$$\left(\frac{4}{3}\frac{\partial^2 u_r}{\partial r\partial z}+\frac{2}{3}\frac{\partial^2 u_r}{\partial r^2}\right)-(1-\varphi^{\frac{2}{3}})\left[r_r\left(\frac{\partial^2 v_\varphi}{\partial r\partial z}+\frac{\partial^2 v_\varphi}{\partial z^2}-\frac{\partial^2 v_\varphi}{r\partial z\partial\varphi}-\frac{1}{r}\frac{\partial v_\varphi}{\partial z}\right)-r_\varphi\left(\frac{\partial^2 u_r}{r\partial\varphi\partial z}+\frac{\partial^2 u_r}{\partial z^2}\right.\right.$$

$$\left.\left.\left.-\frac{\partial^2 u_r}{\partial z\partial r}\right)\right]+2\varphi^{\frac{2}{3}}t'\left[r_r\left(\frac{\partial v_\varphi}{\partial r}\frac{\partial^2 v_\varphi}{\partial r\partial z}+\frac{\partial v_\varphi}{\partial z}\frac{\partial^2 v_\varphi}{\partial z^2}\right)-r_\varphi\left(\frac{\partial u_r}{r\partial\varphi}\frac{\partial^2 u_r}{r\partial\varphi\partial z}+\frac{\partial u_r}{\partial z}\frac{\partial^2 u_r}{\partial z^2}\right)\right]\right\}$$

$$(11\text{-}5\text{-}25)$$

式(11-5-23)至式(1-5-25)就是柱坐标系下可压缩湍流动量矩微分方程，当定常流时去掉含$\frac{\partial}{\partial t}$项。

第 12 章　球坐标系流体运动微分方程

本章讨论不可压缩与可压缩两种情况下的湍流运动。球坐标系选定如图 12-1 所示。由流体力学可知

$$p_{rr} = -p + 2\mu\frac{\partial u_r}{\partial r} \tag{a}$$

$$p_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{u_r}{r}\right) \tag{b}$$

$$p_{\varphi\varphi} = -p + 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial w_\varphi}{\partial\varphi} + \frac{u_r}{r} + \cot\theta\,\frac{v_\theta}{r}\right) \tag{c}$$

$$p_{r\theta} = p_{\theta r} = \mu\left(\frac{\partial u_r}{r\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) \tag{d}$$

$$p_{\theta\varphi} = p_{\varphi\theta} = \mu\left(\frac{\partial w_\varphi}{r\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial r} - \cot\theta\,\frac{w_\varphi}{r}\right) \tag{e}$$

$$p_{13} = p_{31} = \mu\left(\frac{\partial w_\varphi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{w_\varphi}{r}\right) \tag{f}$$

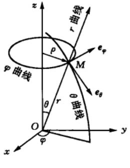


图 12-1　球坐标系

球坐标系下湍流附加剪应力为

$$p'_{r\theta} = \rho\nu t'\left(\frac{\partial v_\theta}{\partial r}\right)^2 \tag{g}$$

$$p'_{\theta r} = \rho\nu t'\left(\frac{\partial u_r}{r\partial\theta}\right)^2 \tag{h}$$

$$p'_{\theta\varphi} = \rho\nu t'\left(\frac{\partial w_\varphi}{r\partial\theta}\right)^2 \tag{i}$$

$$p'_{\varphi\theta} = \rho\nu t'\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)^2 \tag{j}$$

$$p'_{\varphi r} = \rho\nu t\left(\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)^2 \tag{k}$$

$$p'_{13} = \rho\nu t\left(\frac{\partial w_\varphi}{\partial r}\right)^2 \tag{l}$$

12.1　不可压缩湍流微分方程组

所谓方程组，是指连续性、动量、能量、动量矩四个方程。

12.1.1　连续性微分方程

依不可压缩无坐标系下连续性微分方程式(3-4-1)，可得

$$
2\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+\cot\theta\,\frac{v}{r}+\frac{\partial v_\theta}{r\partial\theta}+\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}-\varphi^{\frac{2}{3}}\frac{\nu}{2}\left\{\frac{2}{ru_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\frac{\partial}{\partial r}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]\right.
$$

$$
\left.+\frac{\cot\theta}{r}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+\frac{\partial}{r\partial\theta}\left[\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right]+\frac{\partial}{r\sin\theta\partial\varphi}\left[\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]\right\}
$$

(12-1-1)

12.1.2 动量微分方程

将式(3-4-1)

$$
\varphi^{\frac{2}{3}}\nabla\cdot\mathbf{V}_s=-\varphi_2\nabla\cdot\mathbf{V}
$$

代入式(3-4-2)，$\rho_s=\rho=$常数，全式除以 ρ，展开，合并，整理为

$$
\frac{\partial}{\partial t}\mathbf{V}+\varphi\frac{\partial}{\partial t}(\mathbf{V}_s-\mathbf{V})+\varphi^{\frac{2}{3}}\mathbf{V}\cdot\nabla(\mathbf{V}-\mathbf{V}_s)=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot\mathbf{P}\tag{12-1-2}
$$

$$
\frac{\partial}{\partial t}\mathbf{V}-\varphi\frac{\nu}{2}\left[\frac{\partial}{\partial t}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\mathbf{e}_r+\frac{\partial}{\partial t}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\mathbf{e}_\theta+\frac{\partial}{\partial t}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\mathbf{e}_\varphi\right]
$$

$$
+\varphi^{\frac{2}{3}}\frac{\nu}{2}\left\{u_r\frac{\partial}{\partial r}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\mathbf{e}_r+\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\mathbf{e}_\theta+\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\varphi}\right)\mathbf{e}_\varphi\right]\right.
$$

$$
+v_\theta\frac{\partial}{r\partial\theta}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\mathbf{e}_r+\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\mathbf{e}_\theta+\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\varphi}\right)\mathbf{e}_\varphi\right]
$$

$$
\left.+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\mathbf{e}_r+\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\mathbf{e}_\theta+\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\mathbf{e}_\varphi\right]\right\}
$$

$$
=\boldsymbol{f}+\frac{1}{\rho}\left[\frac{\partial}{\partial r}(p_{rr}\mathbf{e}_r+p_{\theta r}\mathbf{e}_\theta+p_{\varphi r}\mathbf{e}_\varphi)+\frac{\partial}{r\partial\theta}(p_{r\theta}\mathbf{e}_r+p_{\theta\theta}\mathbf{e}_\theta+p_{\varphi\theta}\mathbf{e}_\varphi)+\frac{\partial}{r\sin\theta\partial\varphi}(p_{r\varphi}\mathbf{e}_r+p_{\theta\varphi}\mathbf{e}_\theta+p_{\varphi\varphi}\mathbf{e}_\varphi)\right]
$$

(12-1-2)′

$$
\frac{\partial}{\partial t}\mathbf{V}-\varphi\frac{\nu}{2}\left[\frac{\partial}{\partial t}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\mathbf{e}_r+\frac{\partial}{\partial t}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\mathbf{e}_\theta+\frac{\partial}{\partial t}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\varphi}\right)\mathbf{e}_\varphi\right]
$$

$$
+\varphi^{\frac{2}{3}}\frac{\nu}{2}\left\{u_r\left[\mathbf{e}_r\frac{\partial}{\partial r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\frac{\partial\mathbf{e}_r}{\partial r}+\mathbf{e}_\theta\frac{\partial}{\partial r}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right.\right.
$$

$$
\left.+\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\frac{\partial\mathbf{e}_\theta}{\partial r}+\mathbf{e}_\varphi\frac{\partial}{\partial r}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)+\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\frac{\partial\mathbf{e}_\varphi}{\partial r}\right]
$$

$$
+v_\theta\left[\mathbf{e}_r\frac{\partial}{r\partial\theta}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\frac{\partial\mathbf{e}_r}{r\partial\theta}+\mathbf{e}_\theta\frac{\partial}{r\partial\theta}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right.
$$

$$
\left.+\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\frac{\partial\mathbf{e}_\theta}{r\partial\theta}+\mathbf{e}_\varphi\frac{\partial}{r\partial\theta}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\varphi}\right)+\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\varphi}\right)\frac{\partial\mathbf{e}_\varphi}{r\partial\varphi}\right]+w_\varphi
$$

$$
\left[\mathbf{e}_r\frac{\partial}{r\sin\theta\partial\varphi}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\frac{\partial\mathbf{e}_r}{r\sin\theta\partial\varphi}+\mathbf{e}_\theta\frac{\partial}{r\sin\theta\partial\varphi}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right.
$$

$$
\left.\left.+\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\frac{\partial\mathbf{e}_\theta}{r\sin\theta\partial\varphi}+\mathbf{e}_\varphi\frac{\partial}{r\sin\theta\partial\varphi}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\varphi}\right)+\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\frac{\partial\mathbf{e}_\varphi}{r\sin\theta\partial\varphi}\right]\right\}
$$

$$
=\boldsymbol{f}+\frac{1}{\rho}\left[\mathbf{e}_r\frac{\partial p_{rr}}{\partial r}-p_{rr}\left(\frac{\mathbf{e}_\theta\partial 1}{r\partial\theta}-\mathbf{e}_\varphi\frac{\partial 1}{\partial\varphi}\right)+\mathbf{e}_\theta\frac{\partial p_{\theta r}}{\partial r}+\frac{p_{\theta r}}{1}\frac{\mathbf{e}_r\partial 1}{\partial r}+p_{\varphi r}\frac{\mathbf{e}_r\partial 1}{\partial\varphi}+\mathbf{e}_r\frac{\partial p_{r\theta}}{r\partial\theta}+\mathbf{e}_\theta p_{r\theta}\frac{1}{r}\frac{\partial r}{\partial\varphi}\right.
$$

$$+\frac{e_\theta\partial p_{\theta\theta}}{r\,\partial\theta}+\frac{p_{\theta\theta}}{r}\left(-\frac{e_\varphi\partial r}{\partial\varphi}-e_r\frac{\partial r}{\partial r}\right)+\frac{e_\varphi\partial p_{\varphi\theta}}{r\,\partial\theta}+\frac{p_{\varphi\theta}e_\theta\partial r}{r\,\partial\theta}+e_r\frac{\partial p_{r\varphi}}{r\sin\theta\partial\varphi}+\frac{e_\varphi p_{r\varphi}\partial r\sin\theta}{r\sin\theta\partial r}$$

$$+\frac{e_\theta\partial p_{\theta\varphi}}{r\sin\theta\partial\varphi}+\frac{e_\varphi p_{\theta\varphi}\partial r\sin\theta}{r\sin\theta\;\;\partial r}+e_\varphi\frac{\partial p_{\varphi\varphi}}{r\sin\theta\partial\varphi}+\frac{p_{\varphi\varphi}}{r\sin\theta}\left(-\frac{e_r\partial r\sin\theta}{\partial r}-\frac{e_\theta\partial r\sin\theta}{r\partial\theta}\right)\Bigg]\Bigg] \qquad (12\text{-}1\text{-}2)''$$

$$\frac{\partial}{\partial t}(u_re_r+v_\theta e_\theta+w_\varphi e_\varphi)-\frac{\nu\varphi}{2}\left[\frac{\partial}{\partial t}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)e_r+\frac{\partial}{\partial t}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)e_\theta+\frac{\partial}{\partial t}\frac{1}{w_\varphi}\right.$$

$$\left.\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\varphi}\right)e_\varphi\right]+\frac{\varphi^{\frac{2}{3}}\nu}{2}\left\{u_r\left[e_r\frac{\partial}{\partial r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+e_\theta\frac{\partial}{\partial r}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+e_\varphi\frac{\partial}{\partial r}\right.\right.$$

$$\left.\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\varphi}\right)\right]+v_\theta\left[e_r\frac{\partial}{r\partial\theta}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+e_\varphi\frac{1}{r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+e_\theta\frac{\partial}{r\partial\theta}\right.$$

$$\left.\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)-\frac{e_r}{r}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+e_\varphi\frac{\partial}{r\partial\theta}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]+w_\varphi\left[e_r\frac{\partial}{r\sin\theta\partial\varphi}\right.$$

$$\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\frac{e_\varphi}{r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+e_\theta\frac{\partial}{r\sin\theta\partial\varphi}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+\frac{e_\varphi}{r}\cot\theta$$

$$\left.\left.\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+e_\varphi\frac{\partial}{r\sin\theta\partial\varphi}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)-\left(\frac{e_r}{r}+\frac{e_\theta}{r}\cot\theta\right)\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]\right\}$$

$$=f_re_r+f_\theta e_\theta+f_\varphi e_\varphi+\frac{1}{\rho}\left\{e_r\left(-\frac{\partial p}{\partial r}+2\mu\frac{\partial^2u_r}{\partial r^2}\right)+e_\theta\left[\mu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2u_r}{r\partial\theta\partial r}+\frac{\partial^2v_\theta}{\partial r^2}-\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r^2}\right)+\right.\right.$$

$$\left.\varphi^{\frac{2}{3}}2\rho\nu t'\frac{\partial u_r}{r\partial\theta}\frac{\partial^2u_r}{r\partial\theta\partial r}\right]+e_\varphi\left[\mu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2u_r}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2w_\varphi}{\partial r^2}-\frac{1}{r}\frac{\partial w_\varphi}{\partial r}+\frac{w_\varphi}{r^2}\right)+\varphi^{\frac{2}{3}}2\rho\nu t'\frac{\partial u_r}{r\sin\theta\partial\varphi\partial r}\right]+$$

$$e_r\left[\mu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2v_\theta}{r\partial r\partial\theta}+\frac{\partial^2u_r}{r^2\partial\theta^2}-\frac{1}{r}\frac{\partial v_\theta}{r\partial\theta}\right)+\varphi^{\frac{2}{3}}2\rho\nu t'\frac{\partial v_\theta}{\partial r}\frac{\partial^2v_\theta}{r\partial r\partial\theta}\right]+e_\theta\frac{1}{r}\left[\mu(1-\varphi^{\frac{2}{3}})\right.$$

$$\left.\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial u_r}{r\partial\theta}-\frac{v_\theta}{r}\right)+\varphi^{\frac{2}{3}}\rho\nu t'\left(\frac{\partial v_\theta}{\partial r}\right)^2\right]+e_\theta\left[-\frac{\partial p}{r\partial\theta}+2\mu\left(\frac{\partial^2v_\theta}{r^2\partial\theta^2}+\frac{1}{r}\frac{\partial u_r}{r\partial\theta}\right)\right]-e_r\left[-\frac{p}{r}+\right.$$

$$\left.2\mu\left(\frac{\partial v_\theta}{r^2\partial\theta}+\frac{u_r}{r^2}\right)\right]+e_\varphi\left[\mu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2v_\theta}{r\sin\theta\partial\varphi r\partial\theta}+\frac{\partial^2w_\varphi}{r^2\partial\theta^2}-\frac{1}{r}\cot\theta\frac{\partial w_\varphi}{r\partial\theta}+\csc^2\theta\frac{w_\varphi}{r^2}\right)+\right.$$

$$\left.\varphi^{\frac{2}{3}}2\rho\nu t'\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\frac{\partial^2v_\theta}{r\sin\theta\partial\varphi r\partial\theta}\right)\right]+e_r\left[\mu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2w_\varphi}{r\sin\theta r\partial\varphi}+\frac{\partial^2u_r}{r^2\sin^2\theta\partial\varphi^2}-\frac{1}{r}\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\right)+\right.$$

$$\left.\varphi^{\frac{2}{3}}2\rho\nu t'\frac{\partial w_\varphi}{\partial r}\frac{\partial^2w_\varphi}{r\sin\theta r\partial\varphi}\right]+e_\varphi\frac{1}{r}\left[\mu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial u_r}{r\sin\theta\partial\varphi}-\frac{w_\varphi}{r}\right)+\varphi^{\frac{2}{3}}\rho\nu t'\left(\frac{\partial w_\varphi}{\partial r}\right)^2\right]+$$

$$e_\theta\left[\mu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2w_\varphi}{r\sin\theta\partial\theta r\partial\varphi}+\frac{\partial^2v_\theta}{r^2\sin^2\theta\partial\varphi^2}-\frac{1}{r}\cot\theta\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\right)+2\varphi^{\frac{2}{3}}\rho\nu t'\frac{\partial w_\varphi}{r\partial\theta}\frac{\partial^2w_\varphi}{r\partial\theta r\sin\theta\partial\varphi}\right]+$$

$$e_\varphi\csc\theta\cot\theta\frac{1}{r}\left[\mu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial w_\varphi}{r\partial\theta}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\cot\theta\frac{w_\varphi}{r}\right)+\varphi^{\frac{2}{3}}\rho\nu t'\left(\frac{\partial w_\varphi}{r\partial\theta}\right)^2\right]+$$

$$e_\varphi\left[-\frac{\partial p}{r\sin\theta\partial\varphi}+2\mu\left(\frac{\partial^2w_\varphi}{r^2\sin^2\theta\partial\varphi^2}+\frac{1}{r}\frac{\partial^2u_r}{r\sin\theta\partial\varphi}-\frac{1}{r}\cot\theta\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right]-(e_r+e_\theta\cot\theta)\frac{1}{r}$$

$$\left.\left[-p+2\mu\left(\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}+\frac{u_r}{r}-\cot\theta\frac{v_\theta}{r}\right)\right]\right\} \qquad (12\text{-}1\text{-}2)'''$$

按 e_r，e_θ，e_φ 三个曲线列出方程：

$$\frac{\partial}{\partial t} \quad u_r \quad - \quad \frac{\nu}{2} \quad \varphi \quad \frac{\partial}{\partial t} \quad \frac{1}{u_r} \quad \left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right) \quad + \quad \varphi^{\frac{2}{3}} \quad \frac{\nu}{2}$$

$$\left\{u_r\left[\frac{\partial}{\partial r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]+v_\theta\left[\frac{\partial}{r\partial\theta}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\right.\right.$$

$$\left.\left.\frac{1}{r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]+w_\varphi\left[\frac{\partial}{r\sin\theta\partial\varphi}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\frac{1}{r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]\right\}$$

$$=f_r-\frac{\partial p}{\rho\partial r}+2\nu\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r^2\sin\theta\partial\varphi}-\frac{u_r}{r^2}\right)+(2+\cot\theta)\frac{p}{\rho r}+\nu\left[(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 u_r}{r\partial\theta\partial r}+\right.\right.$$

$$\left.\left.\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 u_r}{r^2\partial\theta^2}+\frac{\partial^2 u_2}{r^2\sin^2\theta\partial\varphi}+\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r^2\sin\theta\partial\varphi}\right)+2\varphi^{\frac{2}{3}}t'\left(\frac{\partial u_r}{r\partial\theta}\frac{\partial^2 u_r}{r\partial\theta\partial r}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}\right)\right]$$

$$(12\text{-}1\text{-}3)$$

$$\frac{\partial}{\partial t}v_\theta-\frac{\nu}{2}\varphi\frac{\partial}{\partial t}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+\varphi^{\frac{2}{3}}\frac{\nu}{2}\left\{u_r\left[\frac{\partial}{\partial r}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+v_\theta\left[\frac{\partial}{r\partial\theta}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)-\right.\right.\right.$$

$$\left.\left.\frac{1}{r}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right]+w_\varphi\left[\frac{\partial}{r\sin\theta\partial\varphi}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+\frac{1}{r}\cot\theta\,\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right]\right\}$$

$$=f_\theta-\frac{\partial p}{\rho r\partial\theta}+2\nu\left[\frac{\partial^2 v_\theta}{r^2\partial\theta^2}+(1+\cot\theta)\frac{v_\theta}{r^2}-\frac{\partial v_\theta}{r^2\partial\theta}-\frac{1}{r}\cot\theta\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right]+\nu\left[(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 v_\theta}{\partial r^2}+\right.\right.$$

$$\left.\left.\frac{\partial^2 v_\theta}{r\sin\theta\partial\varphi r\partial\theta}+\frac{\partial^2 v_\theta}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2 v_\theta}{r\partial r\partial\theta}+\csc\theta\cot\theta\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}\right)\right]+\nu\varphi^{\frac{2}{3}}t'$$

$$\left[2\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\frac{\partial^2 v_\theta}{r\sin\theta\partial\varphi r\partial\theta}+\frac{\partial v_\theta}{\partial r}\frac{\partial^2 v_\theta}{r\partial r\partial\theta}\right)+\left(\frac{\partial v_\theta}{\partial r}\right)^2\right] \qquad (12\text{-}1\text{-}3)'$$

$$\frac{\partial}{\partial t}w_\varphi-\frac{\nu}{2}\varphi\frac{\partial}{\partial t}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)+\varphi^{\frac{2}{3}}\frac{\nu}{2}\left\{u_r\frac{\partial}{\partial r}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)+v_\theta\frac{\partial}{r\partial\theta}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)+\right.$$

$$\left.w_\varphi\left[\frac{\partial}{r\sin\theta\partial\varphi}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)-\frac{1}{r}(1+\cot\theta)\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]\right\}$$

$$=f_\varphi-\frac{\partial p}{\rho r\sin\theta\partial\varphi}+\nu\left\{2\left[\frac{\partial^2 w_\varphi}{r^2\sin^2\theta\partial\varphi^2}-(1+\cot\theta)\frac{1}{r}\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\right]+(1-\varphi^{\frac{2}{3}})\left[\frac{\partial^2 w_\varphi}{\partial r^2}+\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+\right.\right.$$

$$\left.\frac{\partial^2 w_\varphi}{r\sin\theta\partial r\partial\varphi}+\frac{\partial^2 w_\varphi}{r\partial\theta r\sin\theta\partial\varphi}+(\csc\theta-\cot^2\theta)\csc\theta\frac{w_\varphi}{r^2}-\cot\theta(\csc\theta+1)\frac{\partial w_\varphi}{r\partial\theta}-\frac{1}{r}\frac{\partial w_\varphi}{\partial r}-\left(1+\cot\theta\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}\right)\right]$$

$$\left.+\varphi^{\frac{2}{3}}t'\left[2\left(\frac{\partial w_\varphi}{\partial r}\frac{\partial^2 w_\varphi}{\partial r r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{r\partial\theta}\frac{\partial^2 w_\varphi}{r\partial\theta r\sin\theta\partial\varphi}\right)+\left(\frac{\partial w_\varphi}{\partial r}\right)^2+\left(\frac{\partial w_\varphi}{r\partial\theta}\right)^2\right]\right\} \qquad (12\text{-}1\text{-}3)''$$

式(12-1-3)至式(12-1-3)″是球坐标系下，不可压缩湍流动量微分方程。当定常流时，去掉含$\frac{\partial}{\partial t}$项。

12.1.3　能量微分方程

当流体微团在运动过程中，其密度保持不变时，无坐标系，能量微分方程式(3-4-4)

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\varphi\frac{v_s^2}{2}+\varphi_1\frac{v^2}{2}\right)+\nabla\cdot\left(\varphi^{\frac{2}{3}}\frac{v_s^2}{2}\boldsymbol{V}_s+\varphi_2\frac{v^2}{2}\boldsymbol{V}\right)\\
&=\frac{1}{\rho}\nabla\cdot(\nabla T)+q_R+\boldsymbol{f}\cdot(\varphi\boldsymbol{V}_s+\varphi_1\boldsymbol{V})+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{V}_s+\varphi_2\boldsymbol{V})
\end{aligned}
$$

将不可压缩湍流连续性微分方程式(3-4-1)

$$\varphi^{\frac{2}{3}}\nabla\cdot\boldsymbol{V}_s+\varphi_2\nabla\cdot\boldsymbol{V}=0$$

代入式(3-4-4)，然后全式进行推导

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{v^2}{2}\right)+\frac{\partial}{\partial t}\varphi\left(\frac{v_s^2}{2}-\frac{v^2}{2}\right)+\varphi\boldsymbol{V}\cdot\nabla\left(\frac{v^2}{2}-\frac{v_s^2}{2}\right)\\
&=\frac{1}{\rho}\nabla\cdot(\nabla T)+q_R+\boldsymbol{f}\cdot[\boldsymbol{V}+\varphi(\boldsymbol{V}_s-\boldsymbol{V})]+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}[\boldsymbol{V}+\varphi^{\frac{2}{3}}(\boldsymbol{V}_s-\boldsymbol{V})] \qquad (12\text{-}1\text{-}4)
\end{aligned}
$$

将式(12-1-4)化掉分散相涡旋微团速度 v_s 与 $\boldsymbol{V}_s$，全式变成只含连续相速度 v，$\boldsymbol{V}$。然后，以 u_r，v_θ，w_φ 表示的连续相速度出现全式。

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{u_r^2}{2}+\frac{v_\theta^2}{2}+\frac{w_\varphi^2}{2}\right)-\nu\varphi\frac{\partial}{\partial t}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)+\\
&\nu\varphi_2\left(u_r\frac{\partial}{\partial r}+v_\theta\frac{\partial}{r\partial\theta}+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}\right)\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\\
&=\frac{1}{\rho}\nabla\cdot(\nabla T)+q_R+\boldsymbol{f}\cdot[\boldsymbol{V}+\varphi(\boldsymbol{V}_s-\boldsymbol{V})]+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}[\boldsymbol{V}+\varphi^{\frac{2}{3}}(\boldsymbol{V}_s-\boldsymbol{V})] \qquad (12\text{-}1\text{-}4)'
\end{aligned}
$$

$$
\begin{aligned}
&u_r\frac{\partial u_r}{\partial t}+v_\theta\frac{\partial v_\theta}{\partial t}+w_\varphi\frac{\partial w_\varphi}{\partial t}-\nu\varphi\left(\frac{\partial^2u_r}{r\partial\theta\partial t}+\frac{\partial^2u_r}{r\sin\theta\partial\varphi\partial t}+\frac{\partial^2v_\theta}{\partial r\partial t}+\frac{\partial^2v_\theta}{r\sin\theta\partial\varphi\partial t}+\frac{\partial^2w_\varphi}{\partial r\partial t}+\frac{\partial^2w_\varphi}{r\partial\theta\partial t}\right)+\\
&\nu\varphi\left[u_r\left(\frac{\partial^2u_r}{r\partial\theta\partial r}+\frac{\partial^2u_r}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2v_\theta}{\partial r^2}+\frac{\partial^2v_\theta}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2w_\varphi}{\partial r^2}+\frac{\partial^2w_\varphi}{r\partial\theta\partial r}\right)+v_\theta\left(\frac{\partial^2u_r}{r^2\partial\theta^2}+\frac{\partial^2u_r}{r\sin\theta\partial\varphi r\partial\theta}+\right.\right.\\
&\left.\frac{\partial^2v_\theta}{\partial rr\partial\theta}+\frac{\partial^2v_\theta}{r\sin\theta\partial\varphi r\partial\theta}+\frac{\partial^2w_\varphi}{\partial rr\partial\theta}+\frac{\partial^2w_\varphi}{r^2\partial\theta^2}\right)+w_\varphi\left(\frac{\partial^2u_r}{r\partial\theta r\sin\theta\partial\varphi}+\frac{\partial^2u_r}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2v_\theta}{\partial rr\sin\theta\partial\varphi}+\right.\\
&\left.\left.\frac{\partial^2v_\theta}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2w_\varphi}{\partial rr\sin\theta\partial\varphi}+\frac{\partial^2w_\varphi}{r\partial\theta r\sin\theta\partial\varphi}\right)\right]\\
&=\frac{\lambda}{\rho}\left(\frac{\partial^2T}{\partial r^2}+\frac{\partial^2T}{r^2\partial\theta^2}+\frac{\partial^2T}{r^2\sin^2\theta\partial\varphi^2}\right)+q_R+f_ru_r+f_\theta v_\theta+f_\varphi w_\varphi-\frac{\nu\varphi}{2}\left[f_r\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\right.\\
&\left.f_\theta\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+f_\varphi\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]+\left\{2\nu\left[\frac{\partial^2u_r}{\partial r^2}+\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}-\frac{u_r}{r^2}+\right.\right.\\
&\left.(2+\cot\theta)\frac{p}{\rho r}\right]+\nu\left[(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2u_r}{r\partial\theta\partial r}+\frac{\partial^2u_r}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2u_r}{r^2\partial\theta^2}+\frac{\partial^2u_r}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r^2\sin\theta\partial\varphi}\right)+\right.\\
&\left.\left.2\varphi^{\frac{2}{3}}t'\left(\frac{\partial u_r}{r\partial\theta}\frac{\partial^2u_r}{r\partial\theta\partial r}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\frac{\partial^2u_r}{r\sin\theta\partial\varphi\partial r}\right)\right]-\frac{\partial p}{\rho\partial r}\right\}\left[u_r-\varphi^{\frac{2}{3}}\frac{\nu}{2}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]+\\
&\left\{2\nu\left[\frac{\partial^2v_\theta}{r^2\partial\theta^2}+(1+\cot\theta)\frac{v_\theta}{r^2}-\frac{\partial v_\theta}{r^2\partial\theta}-\frac{1}{r}\cot\theta\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right]+\nu\left[(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2v_\theta}{\partial r^2}+\frac{\partial^2v_\theta}{r\sin\theta\partial\varphi r\partial\theta}+\right.\right.\right.\\
&\left.\left.\frac{\partial^2v_\theta}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2v_\theta}{\partial rr\partial\theta}+\csc\theta\cot\theta\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\frac{1}{r^2}+\frac{\partial v_\theta}{\partial\theta}\right)\right]+\nu\varphi^{\frac{2}{3}}t'\left[2\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\frac{\partial^2v_\theta}{r\sin\theta\partial\varphi r\partial\theta}+\right.\right.
\end{aligned}
$$

$$\frac{\partial v_\theta}{\partial r}\frac{\partial^2 v_\theta}{\partial r\partial\theta}\Big)+\left(\frac{\partial v_\theta}{\partial r}\right)^2\Big]-\frac{\partial p}{\rho r\partial\theta}\Big\}\left[v_\theta-\varphi^{\frac{2}{3}}\frac{\nu}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right]+\Big\{2\nu\Big[\frac{\partial^2 w_\varphi}{r^2\sin^2\theta\partial\varphi^2}-$$

$$(1+\cot\theta)\frac{1}{r}\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\Big]+\nu(1-\varphi^{\frac{2}{3}})\Big[\frac{\partial^2 w_\varphi}{\partial r^2}+\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+\frac{\partial^2 w_\varphi}{\partial r r\sin\theta\partial\varphi}+\frac{\partial^2 w_\varphi}{r\partial\theta r\sin\theta\partial\varphi}+(\csc\theta-\cot\theta)$$

$$\csc\theta\frac{w_\varphi}{r^2}-\cot\alpha(\csc\theta+1)\frac{\partial w_\varphi}{r\partial\theta}-\frac{\partial w_\varphi}{r\partial r}-(1+\cot\theta)\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}\Big]+2\varphi^{\frac{2}{3}}t'\Big[2\Big(\frac{\partial w_\varphi}{\partial r}\frac{\partial^2 w_\varphi}{\partial r r\sin\theta\partial\varphi}+$$

$$\frac{\partial w_\varphi}{r\partial\theta}\frac{\partial^2 w_\varphi}{r\partial\theta r\sin\theta\partial\varphi}\Big)+\left(\frac{\partial w_\varphi}{\partial r}\right)^2+\left(\frac{\partial w_\varphi}{r\partial\theta}\right)^2\Big]-\frac{\partial p}{\rho r\sin\theta\partial\varphi}\Big\}\left[w_\varphi-\varphi^{\frac{2}{3}}\frac{\nu}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]$$

(12-1-4)″

式(12-1-4)″就是球坐标系下不可压缩湍流能量微分方程。当定常流时，可令含$\frac{\partial}{\partial t}$项为零。

12.1.4 动量矩微分方程

在无坐标系条件下利用动量矩守恒原则建立起动量矩微分方程式(3-4-3)

$$\frac{\partial}{\partial t}[\boldsymbol r\times\rho(\varphi\boldsymbol V_s+\varphi_1\boldsymbol V)]+\nabla\cdot\boldsymbol V_s(\boldsymbol r\times\varphi^{\frac{2}{3}}\rho\boldsymbol V_s)+\nabla\cdot\boldsymbol V(\boldsymbol r\times\varphi_2\rho\boldsymbol V)$$
$$=\rho(\boldsymbol r\times\boldsymbol f)+\nabla\cdot(\boldsymbol r\times\boldsymbol P)$$

式(3-4-3)是在系统体积不变条件下得到的，现在讨论流体为不可压缩情况下，单位重量下的动量矩微分方程。因此全式除以 ρ，则为

$$\frac{\partial}{\partial t}[\boldsymbol r\times(\varphi\boldsymbol V_s+\varphi_1\boldsymbol V)]+\nabla\cdot\boldsymbol V_s(\boldsymbol r\times\varphi^{\frac{2}{3}}\boldsymbol V_s)+\nabla\cdot\boldsymbol V(\boldsymbol r\times\varphi\boldsymbol V)$$
$$=(\boldsymbol r\times\boldsymbol f)+\frac{1}{\rho}\nabla\cdot(\boldsymbol r\times\boldsymbol P)\qquad(3\text{-}4\text{-}3)'$$

将不可压缩湍流运动时连续性微分方程代入式(3-4-3)′，进行整理后为

$$\frac{\partial}{\partial t}(\boldsymbol r\times\boldsymbol V)+\frac{\partial}{\partial t}[\boldsymbol r\times\varphi(\boldsymbol V_s-\boldsymbol V)]-\varphi\boldsymbol V\cdot\nabla[\boldsymbol r\times(\boldsymbol V-\boldsymbol V_s)]=\boldsymbol r\times\boldsymbol f+\frac{1}{\rho}(\boldsymbol r\times\boldsymbol P)\qquad(3\text{-}4\text{-}3)''$$

为以下推导简便，令

$$[R_\varphi^\theta]=\left[\frac{r_\theta}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)-\frac{r_\varphi}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right]\qquad(12\text{-}1\text{-}5)$$

$$[R_r^\varphi]=\left[\frac{r_\varphi}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)-\frac{r_r}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]\qquad(12\text{-}1\text{-}5)'$$

$$[R_\theta^r]=\left[\frac{r_r}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)-\frac{r_\theta}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]\qquad(12\text{-}1\text{-}5)''$$

$$\frac{\partial}{\partial t}(\boldsymbol r\times\boldsymbol V)-\frac{\nu}{2}\varphi\frac{\partial}{\partial t}\Big\{\left[\frac{r_\theta}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)-\frac{r_\varphi}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right]\boldsymbol e_r+\Big[\frac{r_\varphi}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)-$$

$$\frac{r_r}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\Big]\boldsymbol e_\theta+\left[\frac{r_r}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)-\frac{r_\theta}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]\boldsymbol e_\varphi\Big\}-\frac{\nu}{2}\varphi_2\Big(u_r\frac{\partial}{\partial r}+$$

$$v_\theta \frac{\partial}{r\partial\theta} + w_\varphi \frac{\partial}{r\sin\theta\partial\varphi}\bigg)\bigg\{\bigg[\frac{r_\theta}{w_\varphi}\bigg(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\bigg)-\frac{r_\varphi}{v_\theta}\bigg(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\bigg)\bigg]\boldsymbol{e}_r+\bigg[\frac{r_\varphi}{u_r}\bigg(\frac{\partial u_r}{r\partial\theta}+$$

$$\frac{\partial u_r}{r\sin\theta\partial\varphi}\bigg)-\frac{r_r}{w_\varphi}\bigg(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\bigg)\bigg]\boldsymbol{e}_\theta+\bigg[\frac{r_r}{v_\theta}\bigg(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\bigg)-\frac{r_\theta}{u_r}\bigg(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\bigg)\bigg]\boldsymbol{e}_\varphi\bigg\}$$

$$=\boldsymbol{r}\times\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{12-1-6}$$

$$\frac{\partial}{\partial t}(\boldsymbol{r}\times\boldsymbol{V})-\frac{\nu}{2}\varphi\bigg[\frac{\partial}{\partial t}[R_\varphi^\theta]\boldsymbol{e}_r+\frac{\partial}{\partial t}[R_r^\varphi]\boldsymbol{e}_\theta+\frac{\partial}{\partial t}[R_\theta^r]\boldsymbol{e}_\varphi\bigg]-\frac{\nu}{2}\varphi_2\bigg[u_r\frac{\partial}{\partial r}[R_\varphi^\theta]\boldsymbol{e}_r+u_r[R_\varphi^\theta]\frac{\partial \boldsymbol{e}_r}{\partial r}+$$

$$v_\theta\frac{\partial}{r\partial\theta}[R_\varphi^\theta]\boldsymbol{e}_r+v_\theta[R_\varphi^\theta]\frac{\partial \boldsymbol{e}_r}{r\partial\theta}+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}[R_\varphi^\theta]\boldsymbol{e}_r+w_\varphi[R_\varphi^\theta]\frac{\partial \boldsymbol{e}_r}{r\sin\theta\partial\varphi}+u_r\frac{\partial}{\partial r}[R_r^\varphi]\boldsymbol{e}_\theta+$$

$$u_r[R_r^\varphi]\frac{\partial \boldsymbol{e}_\theta}{\partial r}+v_\theta\frac{\partial}{r\partial\theta}[R_r^\varphi]\boldsymbol{e}_\theta+v_\theta[R_r^\varphi]\frac{\partial \boldsymbol{e}_\theta}{r\partial\theta}+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}[R_r^\varphi]\boldsymbol{e}_\theta+w_\varphi[R_r^\varphi]\frac{\partial \boldsymbol{e}_\theta}{r\sin\theta\partial\varphi}+u_r\frac{\partial}{\partial r}$$

$$[R_\theta^r]\boldsymbol{e}_\varphi+u_r[R_\theta^r]\frac{\partial \boldsymbol{e}_\varphi}{\partial r}+v_\theta\frac{\partial}{r\partial\theta}[R_\theta^r]\boldsymbol{e}_\varphi+v_\theta[R_\theta^r]\frac{\partial \boldsymbol{e}_\varphi}{r\partial\theta}+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}[R_\theta^r]\boldsymbol{e}_\varphi+w_\varphi[R_\theta^r]\frac{\partial \boldsymbol{e}_\varphi}{r\sin\theta\partial\varphi}\bigg]$$

$$=\boldsymbol{r}\times\boldsymbol{f}+\frac{1}{\rho}\boldsymbol{r}\times(\nabla\cdot\boldsymbol{P}) \tag{12-1-6$'$}$$

$$\frac{\partial}{\partial t}(\boldsymbol{r}\times\boldsymbol{V})-\frac{\nu}{2}\varphi\bigg[\frac{\partial}{\partial t}[R_\varphi^\theta]\boldsymbol{e}_r+\frac{\partial}{\partial t}[R_r^\varphi]\boldsymbol{e}_\theta+\frac{\partial}{\partial t}[R_\theta^r]\boldsymbol{e}_\varphi\bigg]-\frac{\nu}{2}\varphi_2\bigg\{\bigg[u_r\frac{\partial}{\partial r}[R_\varphi^\theta]\boldsymbol{e}_r+v_\theta\frac{\partial}{r\partial\theta}$$

$$[R_\varphi^\theta]\boldsymbol{e}_r+v_\theta[R_\varphi^\theta]\frac{1}{r}\boldsymbol{e}_\theta+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}[R_\varphi^\theta]\boldsymbol{e}_r+w_\varphi[R_\varphi^\theta]\frac{1}{r}\boldsymbol{e}_\varphi\bigg]+\bigg[u_r\frac{\partial}{\partial r}[R_r^\varphi]\boldsymbol{e}_\theta+v_\theta\frac{\partial}{r\partial\theta}$$

$$[R_r^\varphi]\boldsymbol{e}_\theta-v_\theta[R_r^\varphi]\frac{1}{r}\boldsymbol{e}_r+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}[R_r^\varphi]\boldsymbol{e}_\theta+w_\varphi[R_r^\varphi]\frac{\cot\theta}{r}\boldsymbol{e}_\varphi\bigg]+\bigg[u_r\frac{\partial}{\partial r}[R_\theta^r]\boldsymbol{e}_\varphi+v_\theta\frac{\partial}{r\partial\theta}$$

$$[R_\theta^r]\boldsymbol{e}_\varphi+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}[R_\theta^r]\boldsymbol{e}_\varphi-\frac{1}{r}(\boldsymbol{e}_r+\boldsymbol{e}_\theta\cot\theta)w_\varphi[R_\theta^r]\bigg]\bigg\}$$

$$=\boldsymbol{r}\times\boldsymbol{f}+\frac{1}{\rho}\boldsymbol{r}\times(\nabla\cdot\boldsymbol{P}) \tag{12-1-6$''$}$$

$$\frac{\partial}{\partial t}(r_\theta w_\varphi-r_\varphi v_\theta)-\frac{\nu}{2}\varphi\frac{\partial}{\partial t}[R_\varphi^\theta]-\frac{\nu}{2}\varphi_2\bigg[u_r[R_\varphi^\theta]'_{e_r}+v_\theta[R_\varphi^\theta]'_{e_\theta}+w_\varphi[R_\varphi^\theta]'_{e_\varphi}+\frac{v_\theta}{r}[R_\varphi^\theta]+$$

$$\frac{w_\varphi}{r}[R_\varphi^\theta]\bigg]=r_\theta f_\varphi-r_\varphi f_\theta+(RP)_\varphi^\theta \tag{12-1-7}$$

$$\frac{\partial}{\partial t}(r_\varphi u_r-r_r w_\varphi)-\frac{\nu}{2}\varphi\frac{\partial}{\partial t}[R_r^\varphi]-\frac{\nu}{2}\varphi_2\bigg[u_r[R_r^\varphi]'_{e_r}+v_\theta[R_r^\varphi]'_{e_\theta}+w_\varphi[R_r^\varphi]'_{e_\varphi}+\frac{w_\varphi}{r}[R_r^\varphi]$$

$$\cot\theta\,\frac{1}{r}-v_\theta[R_r^\varphi]\frac{1}{r}\bigg]=r_\varphi f_r-r_r f_\varphi+(RP)_r^\varphi \tag{12-1-7$'$}$$

$$\frac{\partial}{\partial t}(r_r v_\theta-r_\theta u_r)-\frac{\nu}{2}\varphi\frac{\partial}{\partial t}[R_\theta^r]-\frac{\nu}{2}\varphi_2\bigg[u_r[R_\theta^r]'_{e_r}+v_\theta[R_\theta^r]'_{e_\theta}+w_\varphi[R_\theta^r]'_{e_\varphi}-\frac{1}{r}(1+\cot\theta)$$

$$w_\varphi[R_\theta^r]\bigg]=r_r f_\theta-r_\theta f_r+(RP)_\theta^r \tag{12-1-7$''$}$$

式(12-1-7)～式(12-1-7)″中

$$[R_\varphi^\theta]'_{e_r}=\frac{\partial}{\partial r}[R_\varphi^\theta]=r_\theta\bigg[\frac{1}{w_\varphi}\bigg(\frac{\partial^2 w_\varphi}{\partial r^2}+\frac{\partial^2 w_\varphi}{r\partial\theta\partial r}\bigg)-\frac{1}{w_\varphi^2}\bigg(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\bigg)\frac{\partial w_\varphi}{\partial r}\bigg]-$$

$$r_\varphi\left[\frac{1}{v_\theta}\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{\partial^2 v_\theta}{r\sin\theta\partial\varphi\partial r}\right)-\frac{1}{v_\theta^2}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\frac{\partial v_\theta}{\partial r}\right] \tag{12-1-8}$$

$$[R_\varphi^\theta]'_{e_\theta}=\frac{\partial}{r\partial\theta}[R_\varphi^\theta]=r_\theta\left[\frac{1}{w_\varphi}\left(\frac{\partial^2 w_\varphi}{\partial r r\partial\theta}+\frac{\partial^2 w_\varphi}{r^2\partial\theta}\right)-\frac{1}{w_\varphi^2}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\frac{\partial w_\varphi}{r\partial\theta}\right]-$$
$$r_\varphi\left[\frac{1}{v_\theta}\left(\frac{\partial^2 v_\theta}{\partial r r\partial\theta}+\frac{\partial^2 v_\theta}{r\sin\theta\partial\varphi r\partial\theta}\right)-\frac{1}{v_\theta^2}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\frac{\partial v_\theta}{r\partial\theta}\right] \tag{12-1-9}$$

$$[R_\varphi^\theta]'_{e_\varphi}=\frac{\partial}{r\sin\theta\partial\varphi}[R_\varphi^\theta]=r_\theta\left[\frac{1}{w_\varphi}\left(\frac{\partial^2 w_\varphi}{\partial r r\sin\theta\partial\varphi}+\frac{\partial^2 w_\varphi}{r\partial\theta r\sin\theta\partial\varphi}\right)-\frac{1}{w_\varphi^2}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\frac{\partial w_\varphi}{r\partial\sin\theta\partial\varphi}\right]-$$
$$r_\varphi\left[\frac{1}{v_\theta}\left(\frac{\partial^2 v_\theta}{\partial r r\sin\theta\partial\varphi}+\frac{\partial^2 v_\theta}{r^2\sin^2\theta\partial\varphi^2}\right)-\frac{1}{v_\theta^2}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\frac{\partial v_\theta}{r\partial\sin\theta\partial\varphi}\right] \tag{12-1-10}$$

$$[R_r^\varphi]'_{e_r}=\frac{\partial}{\partial r}[R_r^\varphi]=r_\varphi\left[\frac{1}{u_r}\left(\frac{\partial^2 u_r}{r\partial\theta\partial r}+\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}\right)-\frac{1}{u_r^2}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\frac{\partial u_r}{\partial r}\right]-$$
$$r_r\left[\frac{1}{w_\varphi}\left(\frac{\partial^2 w_\varphi}{\partial r^2}+\frac{\partial^2 w_\varphi}{r\partial\theta\partial r}\right)-\frac{1}{w_\varphi^2}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\frac{\partial w_\varphi}{\partial r}\right] \tag{12-1-11}$$

$$[R_r^\varphi]'_{e_\theta}=\frac{\partial}{r\partial\theta}[R_r^\varphi]=r_\varphi\left[\frac{1}{u_r}\left(\frac{\partial^2 u_r}{r^2\partial\theta^2}+\frac{\partial^2 u_r}{r\sin\theta\partial\varphi r\partial\theta}\right)-\frac{1}{u_r^2}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\frac{\partial u_r}{r\partial\theta}\right]-$$
$$r_r\left[\frac{1}{w_\varphi}\left(\frac{\partial^2 w_\varphi}{\partial r r\partial\theta}+\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}\right)-\frac{1}{w_\varphi^2}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\frac{\partial w_\varphi}{r\partial\theta}\right] \tag{12-1-12}$$

$$[R_r^\varphi]'_{e_\varphi}=\frac{\partial}{r\sin\theta\partial\varphi}[R_r^\varphi]=r_\varphi\left[\frac{1}{u_r}\left(\frac{\partial^2 u_r}{r\partial\theta r\sin\theta\partial\varphi}+\frac{\partial^2 u_r}{r^2\sin^2\theta\partial\varphi^2}\right)-\frac{1}{u_r^2}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\frac{\partial u_r}{r\sin\theta\partial\varphi}\right]-$$
$$r_r\left[\frac{1}{w_\varphi}\left(\frac{\partial^2 w_\varphi}{\partial r r\sin\theta\partial\varphi}+\frac{\partial^2 w_\varphi}{r\partial\theta r\sin\partial\varphi}\right)-\frac{1}{w_\varphi^2}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\right] \tag{12-1-13}$$

$$[R_\theta^r]'_{e_r}=\frac{\partial}{\partial r}[R_\varphi^r]=r_r\left[\frac{1}{v_\theta}\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{\partial^2 v_\theta}{r\sin\theta\partial\varphi\partial r}\right)-\frac{1}{v_\theta^2}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\frac{\partial v_\theta}{\partial r}\right]-$$
$$r_\theta\left[\frac{1}{u_r}\left(\frac{\partial^2 u_r}{r\partial\theta\partial r}+\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}\right)-\frac{1}{u_r^2}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\frac{\partial u_r}{\partial r}\right] \tag{12-1-14}$$

$$[R_\theta^r]'_{e_\theta}=\frac{\partial}{r\partial\theta}[R_\theta^r]=r_r\left[\frac{1}{v_\theta}\left(\frac{\partial^2 v_\theta}{\partial r r\partial\theta}+\frac{\partial^2 v_\theta}{r\sin\theta\partial\varphi\partial\theta}\right)-\frac{1}{v_\theta^2}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\frac{\partial v_\theta}{r\partial\theta}\right]-$$
$$r_\theta\left[\frac{1}{u_r}\left(\frac{\partial^2 u_r}{r^2\partial\theta}+\frac{\partial^2 u_r}{r\sin\theta\partial\varphi r\partial\theta}\right)-\frac{1}{u_r^2}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\frac{\partial u_r}{r\partial\theta}\right] \tag{12-1-15}$$

$$[R_\theta^r]'_{e_\varphi}=\frac{\partial}{r\sin\theta\partial\varphi}[R_\theta^r]=r_r\left[\frac{1}{v_\theta}\left(\frac{\partial^2 v_\theta}{\partial r r\sin\theta\partial\varphi}+\frac{\partial^2 v_\theta}{r^2\sin^2\theta\partial\varphi^2}\right)-\frac{1}{v_\theta^2}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right]-$$

$$r_\theta\left[\frac{1}{u_r}\left(\frac{\partial^2 u_r}{r\partial\theta r\sin\theta\partial\varphi}+\frac{\partial^2 u_r}{r^2\sin^2\theta\partial\varphi^2}\right)-\frac{1}{u_r^2}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\frac{\partial u_r}{r\sin\theta\partial\varphi}\right] \tag{12-1-16}$$

$$[RP]_\varphi^\theta=r_\theta\left\{-\frac{\partial p}{\rho r\sin\theta\partial\varphi}+2\nu\left[\frac{\partial^2 w_\varphi}{r^2\sin^2\theta\partial\varphi^2}-(1+\cot\theta)\frac{1}{r}\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\right]+\nu(1-\varphi^{\frac{2}{3}})\left[\frac{\partial^2 w_\varphi}{\partial r^2}+\right.\right.$$
$$\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+\frac{\partial^2 w_\varphi}{\partial rr\sin\theta\partial\varphi}+\frac{\partial^2 w_\varphi}{r\partial\theta r\sin\theta\partial\varphi}+(\csc\theta-\cot^2\theta)\csc\theta\,\frac{w_\varphi}{r^2}-\cot\theta(\csc\theta+1)\frac{\partial w_\varphi}{r\partial\theta}-$$
$$\left.\frac{1}{r}\frac{\partial w_\varphi}{\partial r}-(1+\cot\theta)\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}\right]+\varphi^{\frac{2}{3}}\nu t'\left[2\left(\frac{\partial w_\varphi}{\partial r}\frac{\partial^2 w_\varphi}{\partial rr\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{r\partial\varphi}\frac{\partial^2 w_\varphi}{r\partial\theta r\sin\theta\partial\varphi}\right)+\right.$$
$$\left.\left.\left(\frac{\partial w_\varphi}{\partial r}\right)^2+\left(\frac{\partial w_\varphi}{r\partial\theta}\right)^2\right]\right\}-r_\varphi\left\{-\frac{\partial p}{\rho r\partial\theta}+2\nu\left[\frac{\partial^2 v_\theta}{r^2\partial\theta^2}+(1+\cot\theta)\frac{v_\theta}{r^2}-\frac{\partial v_\theta}{r^2\partial\theta}-\frac{1}{r}\cot\theta\right.\right.$$
$$\left.\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right]+\nu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{\partial^2 v_\theta}{r\sin\theta\partial\varphi r\partial\theta}+\frac{\partial^2 v_\theta}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2 v_\theta}{\partial rr\partial\theta}+\csc\theta\cot\theta\,\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\right.$$
$$\left.\left.\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}\right)+\varphi^{\frac{2}{3}}\nu t'\left[2\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\frac{\partial^2 v_\theta}{r\sin\theta\partial\varphi r\partial\theta}+\frac{\partial v_\theta}{\partial r}\frac{\partial^2 v_\theta}{\partial rr\partial\theta}\right)+\left(\frac{\partial v_\theta}{\partial r}\right)^2\right]\right\} \tag{12-1-17}$$

$$[RP]_r^\varphi=r_\varphi\left\{-\frac{\partial p}{r\partial r}+2\nu\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r\sin\partial\varphi}-\frac{u_r}{r^2}\right)+(2+\cot\theta)\frac{p}{\rho r}+\nu\left[(1-\varphi^{\frac{2}{3}})\right.\right.$$
$$\left(\frac{\partial^2 u_r}{r\partial\theta\partial r}+\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 u_r}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r^2\sin\theta\partial\varphi}\right)+2\varphi^{\frac{2}{3}}t'\left(\frac{\partial u_r}{r\partial\theta}\frac{\partial^2 u_r}{r\partial\theta\partial r}+\right.$$
$$\left.\left.\left.\frac{\partial u_r}{r\sin\theta\partial\varphi}\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}\right)\right]\right\}-r_r\left\{-\frac{\partial p}{\rho r\sin\theta\partial\varphi}+2\nu\left[\frac{\partial^2 w_\varphi}{r^2\sin^2\theta\partial\varphi^2}-(1+\cot\theta)\frac{1}{r}\right.\right.$$
$$\left.\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\right]+\nu(1-\varphi^{\frac{2}{3}})\left[\frac{\partial^2 w_\varphi}{\partial r^2}+\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+\frac{\partial^2 w_\varphi}{\partial rr\sin\theta\partial\varphi}+\frac{\partial^2 w_\varphi}{r\partial\theta r\sin\theta\partial\varphi}+(\csc\theta-\cot^2\theta)\right.$$
$$\left.\csc\theta\,\frac{w_\varphi}{r^2}-\cot\theta(\csc\theta+1)\frac{\partial w_\varphi}{r\partial\theta}-\frac{\partial w_\varphi}{r\partial r}-(1+\cot\theta)-\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}\right]+\varphi^{\frac{2}{3}}\nu t'$$
$$\left.\left[2\left(\frac{\partial w_\varphi}{\partial r}\frac{\partial^2 w_\varphi}{\partial rr\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{r\partial\theta}\frac{\partial^2 w_\varphi}{r\partial\theta r\sin\theta\partial\varphi}\right)+\left(\frac{\partial w_\varphi}{\partial r}\right)^2+\left(\frac{\partial w_\varphi}{r\partial\theta}\right)^2\right]\right\} \tag{12-1-18}$$

$$[RP]_\theta^r=r_r\left\{-\frac{\partial p}{\rho r\partial\theta}+2\nu\left[\frac{\partial^2 v_\theta}{r^2\partial\theta^2}-(1+\cot\theta)\frac{v_\theta}{r^2}-\frac{\partial v_\theta}{r^2\partial\theta}-\frac{1}{r}\cot\theta\,\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right]+\nu(1-\varphi^{\frac{2}{3}})\right.$$
$$\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{\partial^2 v_\theta}{r\sin\theta\partial\varphi r\partial\theta}+\frac{\partial^2 v_\theta}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2 v_\theta}{\partial rr\partial\theta}+\csc\theta\cot\theta\,\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}\right)+\varphi^{\frac{2}{3}}\nu t'$$
$$\left.\left[2\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\frac{\partial^2 v_\theta}{r\sin\theta\partial\varphi r\partial\theta}+\frac{\partial v_\theta}{\partial r}\frac{\partial^2 v_\theta}{\partial rr\partial\theta}\right)+\left(\frac{\partial v_\theta}{\partial r}\right)^2\right]\right\}-r_\theta\left\{-\frac{\partial p}{\rho\partial r}+2\nu\left(\frac{\partial^2 u_r}{\partial r^2}+\right.\right.$$
$$\left.\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}-\frac{u_r}{r^2}\right)+(2+\cot\theta)\frac{p}{\rho r}+\nu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 u_r}{r^2\partial\theta^2}+\right.$$

$$\left.\left.\left.\frac{\partial^2 u_r}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial u_r}{r^2\sin\partial\varphi}\right)+2\varphi^{\frac{2}{3}}t'\left(\frac{\partial u_r}{r\partial\theta}\frac{\partial^2 u_r}{r\partial\theta\partial r}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}\right)\right]\right\} \tag{12-1-19}$$

式(12-1-12)至式(12-1-14)是球坐标系下不可压缩湍流动量矩微分方程。当定常流时，去掉含$\frac{\partial}{\partial t}$项。

12.2 可压缩湍流微分方程组

由于是研究可压缩湍流运动，故要应用到球坐标系下应力公式：

$$p_{rr}=-p-\frac{2}{3}\mu\nabla\cdot\boldsymbol{V}+2\mu\frac{\partial u_r}{\partial r} \tag{12-2-1}$$

$$p_{\theta\theta}=-p-\frac{2}{3}\mu\nabla\cdot\boldsymbol{V}+2\mu\left(\frac{\partial v_\theta}{r\partial\theta}+\frac{u_r}{r}\right) \tag{12-2-2}$$

$$p_{\varphi\varphi}=-p-\frac{2}{3}\mu\nabla\cdot\boldsymbol{V}+2\mu\left(\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}+\frac{u_r}{r}+\cot\theta\frac{v_\theta}{r}\right) \tag{12-2-3}$$

$$p_{r\theta}=p_{\theta r}=\mu\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial u_r}{r\partial\theta}-\frac{v_\theta}{r}\right) \tag{12-2-4}$$

$$p_{\theta\varphi}=p_{\varphi\theta}=\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\varphi}+\frac{\partial w_\varphi}{r\partial\theta}-w_\varphi\frac{\cot\theta}{r}\right) \tag{12-2-5}$$

$$p_{\varphi r}=p_{r\varphi}=\mu\left(\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{\partial r}-\frac{w_\varphi}{r}\right) \tag{12-2-6}$$

式中：

$$\nabla\cdot\boldsymbol{V}=2\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+\frac{\partial v_\theta}{r\partial\theta}+\cot\theta\frac{v_\theta}{r}+\frac{\partial w_\varphi}{r\sin\theta\partial\varphi} \tag{12-2-7}$$

12.2.1 连续性微分方程

在流体运动中，取系统与控制体时，其大小可变，表明流体密度在其运动过程中是变化的。无坐标系下可压缩湍流质量守恒的微分方程就是式(3-7-1)：

$$\varphi\frac{\mathrm{D}\rho_s}{\mathrm{D}t}+(1-\varphi)\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho_s\nabla\cdot\boldsymbol{V}_s+\rho\nabla\cdot\boldsymbol{V}=0$$

因为流体是连续介质，它在运动过程一定保持连续性，式(3-7-1)成立，可以保持流体运动的连续性，故又称之为连续性微分方程。

现在将式(3-7-1)转化为球坐标系的可压缩湍流连续性微分方程，它作为独立微分方程配合其他微分方程来研究流体速度压力等的变化规律。

将式(3-7-1)转写为

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho+\varphi\frac{\mathrm{D}}{\mathrm{D}t}(\rho_s-\rho)+\rho_s\nabla\cdot\boldsymbol{V}_s+\rho\nabla\cdot\boldsymbol{V}=0 \tag{3-7-1''}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\rho T}{T_s}-\rho\right)+\frac{\rho T}{T_s}\nabla\cdot\boldsymbol{V}_s+\rho\nabla\cdot\boldsymbol{V}=0 \tag{a}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}+\varphi(\alpha-1)\frac{\mathrm{D}}{\mathrm{D}t}\rho+\rho(\alpha\nabla\cdot\boldsymbol{V}_s+\nabla\cdot\boldsymbol{V})=0 \tag{b}$$

φ，α 是参变量，全式除以 ρ，得

$$[1+\varphi(\alpha-1)]\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\nabla\cdot(\alpha\boldsymbol{V}_s+\boldsymbol{V})=0 \tag{c}$$

$$[1+\varphi(\alpha-1)]\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+(\alpha+1)\left(\frac{\partial u_r}{\partial r}+\frac{\partial v_\theta}{r\partial\theta}+\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\right)-\frac{\alpha\nu}{2}\left\{\frac{\partial}{\partial r}\left[\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]+\frac{\partial}{r\partial\theta}\left[\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right]+\frac{\partial}{r\sin\theta\partial\varphi}\left[\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]\right\} \tag{12-2-8}$$

式(12-2-8)就是球坐标系下可压缩湍流连续性微分方程。当定常流时，去掉含$\frac{\partial}{\partial t}$项。

12.2.2　动量微分方程

无坐标系下可压缩湍流动量微分方程式(3-7-2)：

$$\varphi\frac{\mathrm{D}(\rho_s\boldsymbol{V}_s)}{\mathrm{D}t}+\varphi_1\frac{\mathrm{D}(\rho\boldsymbol{V})}{\mathrm{D}t}+\rho_s\boldsymbol{V}_s(\nabla\cdot\boldsymbol{V}_s)+\rho\boldsymbol{V}(\nabla\cdot\boldsymbol{V})=\boldsymbol{f}[\varphi\rho_s+(1-\varphi)\rho]+\nabla\cdot\boldsymbol{P}$$

连续性微分方程另一个作用，就是用它化简其他方程的推导工作。依此将式(3-7-1)′代入上式

$$\frac{\mathrm{D}(\rho\boldsymbol{V})}{\mathrm{D}t}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}(\rho_s\boldsymbol{V}_s-\rho\boldsymbol{V})-\rho_s\boldsymbol{V}_s\left[\varphi\frac{\mathrm{D}\rho_s}{\mathrm{D}t}+(1-\varphi)\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\nabla\cdot\boldsymbol{V}\right]$$
$$=\boldsymbol{f}[\varphi\rho_s+(1-\varphi)\rho]+\nabla\cdot\boldsymbol{P} \tag{a}$$

$$\frac{\mathrm{D}(\rho\boldsymbol{V})}{\mathrm{D}t}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}(\rho_s\boldsymbol{V}_s-\rho\boldsymbol{V})+\rho\nabla\cdot\boldsymbol{V}(\boldsymbol{V}-\boldsymbol{V}_s)-\boldsymbol{V}_s\frac{\mathrm{D}\rho_s}{\mathrm{D}t}-\boldsymbol{V}_s\varphi\frac{\mathrm{D}}{\mathrm{D}t}(\rho_s-\rho)$$
$$=\boldsymbol{f}[\rho+\varphi(\rho_s-\rho)]+\nabla\cdot\boldsymbol{P} \tag{b}$$

$$\frac{\mathrm{D}(\rho\boldsymbol{V})}{\mathrm{D}t}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}\rho(\alpha\boldsymbol{V}_s-\boldsymbol{V})+\rho\boldsymbol{V}\cdot\nabla(\boldsymbol{V}-\boldsymbol{V}_s)-\boldsymbol{V}_s[1+\varphi(\alpha-1)]\frac{\mathrm{D}\rho}{\mathrm{D}t}$$
$$=[\varphi(\alpha-1)+1]\rho\boldsymbol{f}+\nabla\cdot\boldsymbol{P} \tag{c}$$

全式除以 ρ，则有

$$\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}+\boldsymbol{V}\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}(\alpha\boldsymbol{V}_s-\boldsymbol{V})+\varphi(\alpha\boldsymbol{V}_s-\boldsymbol{V})\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\boldsymbol{V}\cdot\nabla(\boldsymbol{V}-\boldsymbol{V}_s)-[1+\varphi(\alpha-1)]\boldsymbol{V}_s\frac{\mathrm{D}\rho}{\mathrm{D}t}$$
$$=[1+\varphi(\alpha-1)]\boldsymbol{f}+\nabla\cdot\boldsymbol{P} \tag{d}$$

$$\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}-(\beta\boldsymbol{V}_s-\boldsymbol{V})\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}(\alpha\boldsymbol{V}_s-\boldsymbol{V})+\varphi(\alpha\boldsymbol{V}_s-\boldsymbol{V})\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\boldsymbol{V}\cdot\nabla(\boldsymbol{V}-\boldsymbol{V}_s)$$
$$=\beta\boldsymbol{f}+\frac{1}{\rho}\left[\frac{\partial}{\partial r}(p_{rr}\boldsymbol{e}_r+p_{\theta r}\boldsymbol{e}_\theta+p_{\varphi r}\boldsymbol{e}_\varphi)+\frac{\partial}{r\partial\theta}(p_{r\theta}\boldsymbol{e}_r+p_{\theta\theta}\boldsymbol{e}_\theta+p_{\varphi\theta}\boldsymbol{e}_\varphi)+\frac{\partial}{r\sin\theta\partial\varphi}(p_{r\varphi}\boldsymbol{e}_r+p_{\theta\varphi}\boldsymbol{e}_\theta+p_{\varphi\varphi}\boldsymbol{e}_\varphi)\right] \tag{e}$$

令

$$\overline{U_r^a}=(\alpha-1)u_r-\frac{\alpha\nu}{2}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right) \tag{12-2-9}$$

$$\overline{V}_\theta^\alpha=(\alpha-1)v_\theta-\frac{\alpha\nu}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\tag{12-2-10}$$

$$\overline{W}_\varphi^\alpha=(\alpha-1)w_\varphi-\frac{\alpha\nu}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\tag{12-2-11}$$

$$\overline{U}_r^\beta=(\beta-1)u_r-\frac{\beta\nu}{2}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\tag{12-2-12}$$

$$\overline{V}_\theta^\beta=(\beta-1)v_\theta-\frac{\beta\nu}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\tag{12-2-13}$$

$$\overline{W}_\varphi^\beta=(\beta-1)w_\varphi-\frac{\beta\nu}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\tag{12-2-14}$$

由数学可知，球坐标系下：

$$\frac{\partial\boldsymbol{e}_r}{r\partial\theta}=\frac{\boldsymbol{e}_\theta}{r};\ \frac{\partial\boldsymbol{e}_r}{r\sin\theta\partial\varphi}=\frac{\boldsymbol{e}_\varphi}{r};\ \frac{\partial\boldsymbol{e}_\theta}{r\partial\theta}=-\frac{\boldsymbol{e}_r}{r};\ \frac{\partial\boldsymbol{e}_\theta}{r\sin\theta\partial\varphi}=\frac{\boldsymbol{e}_\varphi}{r}\cot\theta;\ \frac{\partial\boldsymbol{e}_\varphi}{r\sin\theta\partial\varphi}=-(\boldsymbol{e}_r+\boldsymbol{e}_\theta\cot\theta)\frac{1}{r}$$

其他均为零。

$$\begin{aligned}&\frac{\partial\boldsymbol{V}}{\partial t}-(\overline{U}_r^\beta\boldsymbol{e}_r+\overline{V}_\theta^\beta\boldsymbol{e}_\theta+\overline{W}_\varphi^\beta\boldsymbol{e}_\varphi)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\varphi\Big[\frac{\partial}{\partial t}(\overline{U}_r^\alpha\boldsymbol{e}_r+\overline{V}_\theta^\alpha\boldsymbol{e}_\theta+\overline{W}_\varphi^\alpha\boldsymbol{e}_\varphi)+u_r\frac{\partial}{\partial r}(\overline{U}_r^\alpha\boldsymbol{e}_r+\overline{V}_\theta^\alpha\boldsymbol{e}_\theta+\\&\overline{W}_\varphi^\alpha\boldsymbol{e}_\varphi)+v_\varphi\frac{\partial}{r\partial\theta}(\overline{U}_r^\alpha\boldsymbol{e}_r+\overline{V}_\theta^\alpha\boldsymbol{e}_\theta+\overline{W}_\varphi^\alpha\boldsymbol{e}_\varphi)+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}(\overline{U}_r^\alpha\boldsymbol{e}_r+\overline{V}_\theta^\alpha\boldsymbol{e}_\theta+\overline{W}_\varphi^\alpha\boldsymbol{e}_\varphi)\Big]+\varphi(\overline{U}_r^\alpha\boldsymbol{e}_r+\\&\overline{V}_\theta^\alpha\boldsymbol{e}_\theta+\overline{W}_\varphi^\alpha\boldsymbol{e}_\varphi)\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\overline{\boldsymbol{V}}\cdot\nabla(\boldsymbol{V}-\boldsymbol{V}_s)\\&=\beta\boldsymbol{f}+\frac{1}{\rho}\left\{\left(\boldsymbol{e}_r\frac{\partial p_{rr}}{\partial r}+\boldsymbol{e}_\theta\frac{\partial p_{\theta r}}{\partial r}+\boldsymbol{e}_\varphi\frac{\partial p_{\varphi r}}{\partial r}\right)+\left(\boldsymbol{e}_r\frac{\partial p_{r\theta}}{r\partial\theta}+p_{r\theta}\frac{\boldsymbol{e}_\theta}{r}+\boldsymbol{e}_\theta\frac{\partial p_{\theta\theta}}{r\partial\theta}-\frac{\boldsymbol{e}_r}{r}p_{\theta\theta}+\boldsymbol{e}_\varphi\frac{\partial p_{\varphi\theta}}{r\partial\theta}\right)+\right.\\&\left.\left[\boldsymbol{e}_r\frac{\partial p_{r\varphi}}{r\sin\theta\partial\varphi}+p_{r\varphi}\frac{\boldsymbol{e}_\varphi}{r}+\boldsymbol{e}_\theta\frac{\partial p_{\theta\varphi}}{r\sin\theta\partial\varphi}+\boldsymbol{e}_\theta p_{\theta\varphi}\frac{\cot\theta}{r}+\boldsymbol{e}_\varphi\frac{\partial p_{\varphi\varphi}}{r\sin\theta\partial\varphi}-(\boldsymbol{e}_r+\boldsymbol{e}_\theta\cot\theta)\frac{p_{\varphi\varphi}}{r}\right]\right\}\end{aligned}\tag{f}$$

式中：

$$\beta=1+\varphi(\alpha-1)\tag{12-2-15}$$

将式(f)进一步展开：

$$\begin{aligned}&\frac{\partial\boldsymbol{V}}{\partial t}-(\overline{U}_r^\beta\boldsymbol{e}_r+\overline{V}_\theta^\beta\boldsymbol{e}_\theta+\overline{W}_\varphi^\beta\boldsymbol{e}_\varphi)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\varphi\Big[\frac{\partial}{\partial t}(\overline{U}_r^\alpha\boldsymbol{e}_r+\overline{V}_\theta^\alpha\boldsymbol{e}_\theta+\overline{W}_\varphi^\alpha\boldsymbol{e}_\varphi)+u_r\Big(\boldsymbol{e}_r\frac{\partial\overline{U}_r^\alpha}{\partial r}+\overline{U}_r^\alpha\frac{\partial\boldsymbol{e}_r}{\partial r}+\\&\boldsymbol{e}_\theta\frac{\partial\overline{V}_\theta^\alpha}{\partial r}+\overline{V}_\theta^\alpha\frac{\partial\boldsymbol{e}_\theta}{\partial r}+\boldsymbol{e}_\varphi\frac{\partial\overline{W}_\varphi^\alpha}{\partial r}+\overline{W}_\varphi^\alpha\frac{\partial\boldsymbol{e}_\varphi}{\partial r}\Big)\Big]+v_\theta\Big(\boldsymbol{e}_r\frac{\partial\overline{U}_r^\alpha}{r\partial\theta}+\overline{U}_r^\alpha\frac{\partial\boldsymbol{e}_r}{r\partial\theta}+\boldsymbol{e}_\theta\frac{\partial\overline{V}_\theta^\alpha}{r\partial\theta}+\overline{V}_\theta^\alpha\frac{\partial\boldsymbol{e}_\theta}{r\partial\theta}+\boldsymbol{e}_\varphi\frac{\partial\overline{W}_\varphi^\alpha}{r\partial\theta}+\\&\overline{W}_\varphi^\alpha\frac{\partial\boldsymbol{e}_\varphi}{r\partial\theta}\Big)+w_\varphi\Big(\boldsymbol{e}_r\frac{\partial\overline{U}_r^\alpha}{r\sin\theta\partial\varphi}+\overline{U}_r^\alpha\frac{\partial\boldsymbol{e}_r}{r\sin\theta\partial\varphi}+\boldsymbol{e}_\theta\frac{\partial\overline{V}_\theta^\alpha}{r\sin\theta\partial\varphi}+\overline{V}_\theta^\alpha\frac{\partial\boldsymbol{e}_\theta}{r\sin\theta\partial\varphi}+\boldsymbol{e}_\varphi\frac{\partial\overline{W}_\varphi^\alpha}{r\sin\theta\partial\varphi}+\\&\overline{W}_\varphi^\alpha\frac{\partial\boldsymbol{e}_\varphi}{r\sin\theta\partial\varphi}\Big)\Big]+\varphi(\overline{U}_r^\alpha\boldsymbol{e}_r+\overline{V}_\theta^\alpha\boldsymbol{e}_\theta+\overline{W}_\varphi^\alpha\boldsymbol{e}_\varphi)\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\frac{\nu}{2}(\boldsymbol{V}\cdot\nabla)\Big[\frac{1}{u_r}\Big(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\Big)\boldsymbol{e}_r+\\&\frac{1}{v_\theta}\Big(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\Big)\boldsymbol{e}_\theta+\frac{1}{w_\varphi}\Big(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\Big)\boldsymbol{e}_\varphi\Big]\\&=\boldsymbol{f}+\frac{1}{\rho}\Big\{\boldsymbol{e}_r\frac{\partial}{\partial r}\Big(-p-\frac{2}{3}\mu\nabla\cdot\boldsymbol{V}+2\mu\frac{\partial u_r}{\partial r}\Big)+\boldsymbol{e}_\theta\frac{\partial}{\partial r}\Big[(1-\varphi^{\frac{2}{3}})\mu\Big(\frac{\partial u_r}{r\partial\theta}+\frac{\partial w_\varphi}{\partial r}-\frac{v_\theta}{r}\Big)+\end{aligned}$$

$$\varphi^{\frac{2}{3}}\rho\nu t'\left(\frac{\partial u_r}{r\partial\theta}\right)^2\Big]+e_\varphi\frac{\partial}{\partial r}\Big[(1-\varphi^{\frac{2}{3}})\mu\left(\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{\partial r}-\frac{w_\varphi}{r}\right)+\varphi^{\frac{2}{3}}\rho\nu t'\left(\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)^2\Big]+$$

$$e_r\frac{\partial}{r\partial\theta}\Big[(1-\varphi^{\frac{2}{3}})\mu\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial u_r}{r\partial\theta}-\frac{v_\theta}{r}\right)+\varphi^{\frac{2}{3}}\rho\nu t'\left(\frac{\partial v_\theta}{\partial r}\right)^2\Big]+\frac{e_\theta}{r}\Big[(1-\varphi^{\frac{2}{3}})\mu\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial u_r}{r\partial\theta}-\frac{v_\theta}{r}\right)+$$

$$\varphi^{\frac{2}{3}}\rho\nu t'\left(\frac{\partial v_\theta}{\partial r}\right)^2\Big]+e_\theta\frac{\partial}{r\partial\theta}\Big[-p-\frac{2}{3}\mu\nabla\cdot\mathbf{V}+2\mu\left(\frac{\partial v_\theta}{r\partial\theta}+\frac{u_r}{r}\right)\Big]-\frac{e_r}{r}\Big[-p-\frac{2}{3}\mu\nabla\cdot\mathbf{V}+$$

$$2\mu\left(\frac{\partial v_\theta}{r\partial\theta}+\frac{u_r}{r}\right)\Big]+e_\varphi\frac{\partial}{r\partial\theta}\Big[(1-\varphi^{\frac{2}{3}})\mu\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{r\partial\theta}-\frac{\cot\theta}{r}w_\varphi\right)+\varphi^{\frac{2}{3}}\rho\nu t'\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)^2\Big]+$$

$$e_r\frac{\partial}{r\sin\theta\partial\varphi}\Big[(1-\varphi^{\frac{2}{3}})\mu\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial u_r}{r\sin\theta\partial\varphi}-\frac{w_\varphi}{r}\right)+\varphi^{\frac{2}{3}}\rho\nu t'\left(\frac{\partial w_\varphi}{\partial r}\right)^2\Big]+\frac{e_\varphi}{r}\Big[(1-\varphi^{\frac{2}{3}})\mu$$

$$\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial u_r}{r\sin\theta\partial\varphi}-\frac{w_\varphi}{r}\right)+\varphi^{\frac{2}{3}}\rho\nu t'\left(\frac{\partial w_\varphi}{\partial r}\right)^2\Big]+e_\theta\frac{\partial}{r\sin\theta\partial\varphi}\Big[(1-\varphi^{\frac{2}{3}})\mu\left(\frac{\partial w_\varphi}{r\partial\theta}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\right.$$

$$\left.w_\varphi\frac{\cot\theta}{r}\right)+\varphi^{\frac{2}{3}}\rho\nu t'\left(\frac{\partial w_\varphi}{r\partial\theta}\right)^2\Big]+e_\theta\frac{\cot\theta}{r}\Big[(1-\varphi^{\frac{2}{3}})\mu\left(\frac{\partial w_\varphi}{r\partial\theta}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-w_\varphi\frac{\cot\theta}{r}\right)+$$

$$\varphi^{\frac{2}{3}}\rho\nu t'\left(\frac{\partial w_\varphi}{r\partial\theta}\right)^2\Big]+e_\varphi\frac{\partial}{r\sin\theta\partial\varphi}\Big[-p-\frac{2}{3}\mu\nabla\cdot\mathbf{V}+2\mu\left(\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}+\frac{u_r}{r}+\cot\theta\frac{v_\theta}{r}\right)\Big]-$$

$$(e_r+e_\theta\cot\theta)\frac{1}{2}\Big[-p-\frac{2}{3}\mu\nabla\cdot\mathbf{V}+2\mu\left(\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}+\frac{u_r}{r}+\cot\theta\frac{v_\theta}{r}\right)\Big]\Big\}\tag{g}$$

将式(g)按 e_r，e_θ，e_φ 三个方向列出

$$\frac{\partial u_r}{\partial t}-\left[(\beta-1)u_r-\frac{\beta\nu}{2}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\varphi\Big\{\frac{\partial}{\partial t}\Big[(\alpha-1)u_r-\frac{\alpha\nu}{2}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\right.$$

$$\left.\left.\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]+u_r\frac{\partial}{\partial r}\overline{U_r^\alpha}+v_\theta\left(\frac{\partial}{r\partial\theta}\overline{U_r^\alpha}+\frac{1}{r}\overline{U_r^\alpha}\right)+w_\varphi\left(\frac{\partial}{r\sin\theta\partial\varphi}\overline{U_r^\alpha}+\frac{1}{r}\overline{U_r^\alpha}\right)\Big\}+\varphi\,\overline{U_r^\alpha}\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\frac{\nu}{2}$$

$$\Big[u_r\frac{\partial}{\partial r}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+v_\theta\frac{\partial}{r\partial\theta}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\frac{v_\theta}{u_r}\frac{1}{r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+$$

$$w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\frac{w_\varphi}{u_r}\frac{1}{r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\Big]$$

$$=f_r-\frac{\partial p}{\rho\partial r}+\nu\Big[\frac{4}{3}\left(\frac{\partial^2u_r}{\partial r^2}+\frac{u_r}{r^2}-\frac{\partial u_r}{r\partial r}\right)+\frac{2}{3}\left(\frac{\partial u_r}{r^2\sin\theta\partial\varphi}+\frac{\partial u_r}{r^2\partial\theta}\right)-\frac{2}{3}\left(\frac{\partial^2u_r}{r\partial r\partial\theta}+\frac{\partial^2u_r}{r\sin\theta\partial\varphi\partial r}\right)\Big]+$$

$$\nu\Big[(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2u_r}{r\partial\theta\partial r}+\frac{\partial^2u_r}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2u_r}{r^2\partial\theta^2}+\frac{\partial^2u_r}{r^2\sin^2\partial\varphi^2}+\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r^2\sin\theta\partial\varphi}\right)+2\varphi^{\frac{2}{3}}t'$$

$$\left(\frac{\partial u_r}{r\partial\theta}\frac{\partial^2u_r}{r\partial\theta\partial r}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\frac{\partial^2u_r}{r\sin\theta\partial\varphi\partial r}\right)\Big]\tag{12-2-16}$$

$$\frac{\partial v_\theta}{\partial t}-\left[(\beta-1)v_\theta-\frac{\beta\nu}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right]\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\varphi\Big\{\frac{\partial}{\partial t}\Big[(\alpha-1)v_\theta-\frac{\alpha\nu}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\Big]+$$

$$u_r\frac{\partial}{\partial r}\Big[(\alpha-1)v_\theta-\frac{\alpha\nu}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\Big]+v_\theta\frac{\partial}{r\partial\theta}\Big[(\alpha-1)v_\theta-\frac{\alpha\nu}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\Big]-$$

$$\frac{v_\theta}{r}\overline{V_\theta^\alpha}+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}\Big[(\alpha-1)v_\theta-\frac{\alpha\nu}{2}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\Big]\Big\}+\varphi\Big[(\alpha-1)v_\theta-\frac{\alpha\nu}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\right.$$

$$\left.\left.\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right]\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\frac{\nu}{2}\left[u_r\frac{\partial}{\partial r}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+v_\theta\frac{\partial}{r\partial\theta}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)-\frac{1}{r}\right.$$

$$\left.\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+\frac{w_\varphi}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\frac{\cot\theta}{r}\right]$$

$$=f_\theta-\frac{\partial p}{\rho r\partial\theta}+\nu\left[\frac{4}{3}\frac{\partial^2 v_\theta}{r^2\partial\theta^2}+\frac{2}{3}(\csc^2\theta+\cot\theta)\frac{v_\theta}{r^2}+\frac{4}{3}\frac{\cot\theta}{r^2}\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\frac{2}{3}\cot\theta\left(\frac{\partial v_\theta}{r^2\partial\theta}+\frac{\partial v_\theta}{r\partial r}\right)-\right.$$

$$\left.\frac{2}{3}\left(\frac{\partial v_\theta}{r\partial\theta\partial r}+\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial\theta}\right)\right]+\nu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial r}+\frac{\partial^2 v_\theta}{r^2\sin^2\partial\varphi^2}+\frac{\partial^2 v_\theta}{r\partial\theta\partial r}+\right.$$

$$\left.\csc\theta\cot\theta\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}\right)+\nu\varphi^{\frac{2}{3}}t'\left[2\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial\theta}+\frac{\partial v_\theta}{\partial r}\frac{\partial^2 v_\theta}{r\partial\theta\partial r}\right)+\left(\frac{\partial v_\theta}{\partial r}\right)^2\right]$$

$$(12\text{-}2\text{-}16)'$$

$$\frac{\partial w_\varphi}{\partial t}-\left[(\beta-1)w_\varphi-\frac{\beta\nu}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\varphi\left\{\frac{\partial}{\partial t}\left[(\alpha-1)w_\varphi-\frac{\alpha\nu}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\right.\right.\right.$$

$$\left.\left.\frac{\partial w_\varphi}{r\partial\theta}\right)\right]+u_r\frac{\partial}{\partial r}\left[(\alpha-1)w_\varphi-\frac{\alpha\nu}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\right)\right]+v_\theta\frac{\partial}{r\partial\theta}\left[(\alpha-1)w_\varphi-\frac{\alpha\nu}{2}\frac{1}{w_\varphi}\right.$$

$$\left.\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}\left[(\alpha-1)w_\varphi-\frac{\alpha\nu}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]-(1+\cot\theta)\frac{1}{r}$$

$$\left.\left[(\alpha-1)w_\varphi-\frac{\alpha\nu}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]\right\}+\varphi\left[(\alpha-1)w_\varphi-\frac{\alpha\nu}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\frac{\nu}{2}$$

$$\left[u_r\frac{\partial}{\partial r}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)+v_\theta\frac{\partial}{r\partial\theta}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)-\right.$$

$$\left.(1+\cot\theta)\frac{1}{r}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]$$

$$=f_\varphi-\frac{\partial p}{\rho r\sin\theta\partial\varphi}+\nu\left[\frac{4}{3}\frac{\partial^2 w_\varphi}{r^2\sin^2\partial\varphi^2}-\frac{2}{3}\left(\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}+\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}\right)\right]+\nu(1-\varphi^{\frac{2}{3}})\left[\frac{\partial^2 w_\varphi}{\partial r^2}+\right.$$

$$\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}+(\csc\theta-\cot^2\theta)\csc\theta\frac{\partial w_\varphi}{r^2}-\cot\theta(\csc\theta+1)\frac{\partial w_\theta}{r\partial\theta}-\frac{1}{r}\frac{\partial w_\varphi}{\partial r}-$$

$$\left.(1+\cot\theta)\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}\right]+\nu\varphi^{\frac{2}{3}}t'\left[2\left(\frac{\partial w_\varphi}{\partial r}\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}\right)+\left(\frac{\partial w_\varphi}{\partial r}\right)^2+\left(\frac{\partial w_\varphi}{r\partial\theta}\right)^2\right]$$

$$(12\text{-}2\text{-}16)''$$

式(12-2-16)至式(12-2-16)″就是球坐标系可压缩湍流动量微分方程。当定常流时，去掉含$\frac{\partial}{\partial t}$项。

12.2.3 能量微分方程

将式(3-7-1)′代入式(3-7-4)为

$$\varphi\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho_s\left(e_s+\frac{v_s^2}{2}\right)\right]-\rho_s\left(e_s+\frac{v_s^2}{2}\right)\left[\varphi\frac{\mathrm{D}\rho_s}{\mathrm{D}t}+(1-\varphi)\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\nabla\cdot\boldsymbol{V}\right]+\varphi_1\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho\left(e+\frac{v^2}{2}\right)\right]+$$

$$\rho\left(e+\frac{v^2}{2}\right)\nabla\cdot\boldsymbol{V}=\nabla\cdot(\lambda\nabla T)+\rho q_{\mathrm{R}}+\boldsymbol{f}(\varphi\rho_s\boldsymbol{V}_s+\varphi_1\rho\boldsymbol{V})+\nabla\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{V}_s+\varphi_2\boldsymbol{V})\tag{3-7-4}'$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho\left(e+\frac{v^2}{2}\right)\right]+\varphi\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho_s\left(e_s+\frac{v_s^2}{2}\right)-\rho\left(e+\frac{v^2}{2}\right)\right]-\left(e_s+\frac{v_s^2}{2}\right)\frac{\mathrm{D}\rho}{\mathrm{D}t}-\left(e_s+\frac{v_s^2}{2}\right)\varphi\frac{\mathrm{D}}{\mathrm{D}t}(\rho_s-\rho)+$$
$$\rho\boldsymbol{V}\cdot\nabla\left[\left(e+\frac{v^2}{2}\right)-\left(e_s+\frac{v_s^2}{2}\right)\right]$$
$$=\nabla\cdot(\lambda\nabla T)+\rho q_{\mathrm{R}}+\boldsymbol{f}\cdot(\varphi\rho_s\boldsymbol{V}_s+\varphi_1\rho\boldsymbol{V})+\nabla\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{V}_s+\varphi_2\boldsymbol{V})\tag{3-7-4}''$$

利用以前有关的公式，处理 e_s，e，ρ_s，将式(3-7-4)″简化为

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho\left(C_pT+\frac{v^2}{2}\right)\right]+\varphi\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{\rho T}{T_s}\left(C_pT_s+\frac{v_s^2}{2}\right)-\rho\left(C_pT+\frac{v^2}{2}\right)\right]-\left(C_pT_s+\frac{v_s^2}{2}\right)\frac{\mathrm{D}\rho}{\mathrm{D}t}-$$
$$\left(C_pT_s+\frac{v_s^2}{2}\right)\varphi\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\rho T}{T_s}-\rho\right)+\rho\boldsymbol{V}\cdot\nabla\left[\left(C_pT+\frac{v^2}{2}\right)-\left(C_pT_s+\frac{v_s^2}{2}\right)\right]$$
$$=\nabla\cdot(\lambda\nabla T)+\rho q_{\mathrm{R}}+\boldsymbol{f}\cdot\left[\varphi\frac{\rho T}{T_s}\boldsymbol{V}_s+(1-\varphi)\rho\boldsymbol{V}\right]+\nabla\cdot\boldsymbol{P}\cdot[\varphi^{\frac{2}{3}}\boldsymbol{V}_s+(1-\varphi^{\frac{2}{3}})\boldsymbol{V}]\tag{3-7-4}'''$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\rho\left(C_pT+\frac{v^2}{2}\right)\right]+\varphi\frac{\mathrm{D}}{\mathrm{D}t}\rho\left(\alpha\frac{v_s^2}{2}-\frac{v^2}{2}\right)-\left(C_pnT+\frac{v_s^2}{2}\right)\frac{\mathrm{D}\rho}{\mathrm{D}t}-\left(C_pnT+\frac{v_s^2}{2}\right)\varphi\frac{\mathrm{D}}{\mathrm{D}t}(\alpha-1)$$
$$+\rho\boldsymbol{V}\cdot\nabla\left[C_pT(1-n)+\frac{v^2}{2}-\frac{v_s^2}{2}\right]$$
$$=\nabla\cdot(\lambda\nabla T)+\rho q_{\mathrm{R}}+\boldsymbol{f}\rho\cdot[\boldsymbol{V}+\varphi(\boldsymbol{V}_s-\boldsymbol{V})]+\nabla\cdot\boldsymbol{P}\cdot[\boldsymbol{V}+\varphi^{\frac{2}{3}}(\boldsymbol{V}_s-\boldsymbol{V})]\tag{3-7-4}''''$$

展开除以 ρ 后为

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(C_pT+\frac{v^2}{2}\right)+\left(C_pT+\frac{v^2}{2}\right)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}\left(\alpha\frac{v_s^2}{2}-\frac{v^2}{2}\right)+\varphi\left(\frac{\alpha v_s^2}{2}-\frac{v^2}{2}\right)\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}-$$
$$[1+\varphi(\alpha-1)]\left(nC_pT+\frac{v_s^2}{2}\right)\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}\boldsymbol{V}\cdot\nabla\left[C_p(1-n)T+\frac{v^2}{2}-\frac{v_s^2}{2}\right]$$
$$=\frac{1}{\rho}\nabla\cdot(\lambda\nabla T)+q_{\mathrm{R}}+\boldsymbol{f}[\boldsymbol{V}+\varphi(\alpha\boldsymbol{V}_s-\boldsymbol{V})]+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot[\boldsymbol{V}+\varphi^{\frac{2}{3}}(\boldsymbol{V}_s-\boldsymbol{V})]\tag{12-2-17}$$

进一步整理为

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(C_pT+\frac{v^2}{2}\right)+\left\{C_pT\{1-[1+\varphi(\alpha-1)]n\}+\frac{v^2}{2}-[1+\varphi(\alpha-1)]\frac{v_s^2}{2}\right\}\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+$$
$$\boldsymbol{V}\cdot\nabla\left[C_p(1-n)T+\frac{1}{2}(v^2-v_s^2)\right]$$
$$=\frac{1}{\rho}\nabla\cdot(\lambda T)+q_{\mathrm{R}}+\boldsymbol{f}\cdot[\boldsymbol{V}+\varphi(\alpha\boldsymbol{V}_s-\boldsymbol{V})]+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot[\boldsymbol{V}+\varphi^{\frac{2}{3}}(\boldsymbol{V}_s-\boldsymbol{V})]\tag{12-2-17}'$$

令

$$\zeta=1+\varphi(\alpha-1)\tag{12-2-18}$$

$$\xi=1-\zeta n\tag{12-2-19}$$

在具体问题，ζ，ξ 均为参变常数。故式(12-2-17)′可写成

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(C_pT+\frac{v^2}{2}\right)+\left(\xi C_pT+\frac{v^2}{2}-\zeta\frac{v_s^2}{2}\right)\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\boldsymbol{V}\cdot\nabla\left[C_p(1-n)T+\frac{1}{2}(v^2-v_s^2)\right]$$
$$=\frac{1}{\rho}\nabla\cdot(\lambda\nabla T)+q_{\mathrm{R}}+\boldsymbol{f}\cdot[\boldsymbol{V}+\varphi(\alpha\boldsymbol{V}_s-\boldsymbol{V})]+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot[\boldsymbol{V}+\varphi^{\frac{2}{3}}(\boldsymbol{V}_s-\boldsymbol{V})] \tag{12-2-17″}$$

涡旋微团速度 v_s 转化为连续相速度 v。

$$v^2=u_r^2+v_\theta^2+w_\varphi^2,\quad v_s^2=u_{sr}^2+v_{s\theta}^2+w_{s\varphi}^2$$

因为

$$u_{sr}^2=\left[u_r-\frac{\nu}{2u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]^2=u_r^2-\nu\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\frac{\nu^2}{4u_r^2}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)^2 \tag{1}$$

取近似值

$$u_{sr}^2=u_r^2-\nu\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right) \tag{1′}$$

同理

$$v_{s\theta}^2=v_\theta^2-\nu\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right) \tag{2}$$

$$w_{s\varphi}^2=w_\varphi^2-\nu\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right) \tag{3}$$

将式(1)′，式(2)，式(3)代入式(12-2-17)″左边

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[C_pT+\frac{1}{2}(u_r^2+v_\theta^2+w_\varphi^2)\right]+\left[\xi C_pT+(1-\zeta)\frac{1}{2}(u_r^2+v_\theta^2+w_\varphi^2)+\frac{\nu\zeta}{2}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}+\right.\right.$$
$$\left.\left.\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+C_p(1-n)\left(\frac{\partial T}{\partial r}+\frac{\partial T}{r\partial\theta}+\frac{\partial T}{r\sin\theta\partial\varphi}\right)+\frac{\nu}{2}\left[\frac{\partial}{\partial r}\left(\frac{\partial u_r}{r\partial\theta}+\right.\right.$$
$$\left.\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\frac{\partial}{r\partial\theta}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\frac{\partial}{r\sin\theta\partial\varphi}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\frac{\partial}{\partial r}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+$$
$$\frac{\partial}{r\partial\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+\frac{\partial}{r\sin\theta\partial\varphi}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+\frac{\partial}{\partial r}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)+\frac{\partial}{r\partial\theta}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)+$$
$$\left.\frac{\partial}{r\sin\theta\partial\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]$$
$$=\frac{\lambda}{\rho}\left(\frac{\partial^2T}{\partial r^2}+\frac{\partial^2T}{r^2\partial\theta^2}+\frac{\partial^2T}{r^2\sin^2\theta\partial\varphi^2}\right)+q_{\mathrm{R}}+f_r\left[u_r(1+\varphi(\alpha-1))-\frac{\alpha\nu}{2}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]+$$
$$f_\theta\left[v_\theta(1+\varphi(\alpha-1))-\frac{\alpha\nu}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right]+f_\varphi\left[w_\varphi(1+\varphi(\alpha-1))-\frac{\alpha\nu}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\right.\right.$$
$$\left.\left.\frac{\partial w_\varphi}{r\partial\theta}\right)\right]+\left\{-\frac{\partial p}{\rho\partial r}+\nu\left[\frac{4}{3}\left(\frac{\partial^2u_r}{\partial r^2}+\frac{u_r}{r^2}-\frac{\partial u_r}{r\partial r}\right)+\frac{2}{3}\left(\frac{\partial u_r}{r^2\sin\theta\partial\varphi}+\frac{\partial u_r}{r^2\partial\theta}\right)-\frac{2}{3}\left(\frac{\partial^2u_r}{r\partial r\partial\theta}+\right.\right.\right.$$
$$\left.\left.\frac{\partial^2u_r}{r\sin\theta\partial\varphi\partial r}\right)\right]+\nu\left[(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2u_r}{r\partial\theta\partial r}+\frac{\partial^2u_r}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2u_r}{r^2\partial\theta^2}+\frac{\partial^2u_r}{r^2\sin^2\partial\varphi^2}+\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r^2\sin\theta\partial\varphi}\right)+\right.$$

$$2\varphi^{\frac{2}{3}}t'\left(\frac{\partial u_r}{r\partial\theta}\frac{\partial^2 u_r}{r\partial\theta\partial r}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}\right)\Bigg]\Bigg\}\frac{\varphi^{\frac{2}{3}}\nu}{2}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)+\Bigg\{-\frac{\partial p}{\rho r\partial\theta}+\nu\cdot$$

$$\left[\frac{4}{3}\frac{\partial^2 v_\theta}{r^2\partial\theta^2}+\frac{2}{3}(\csc^2\theta+\cot\theta)\frac{v_\theta}{r^2}+\frac{4}{3}\frac{\cot\theta}{r^2}\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\frac{2}{3}\cot\theta\left(\frac{\partial v_\theta}{r^2\partial\theta}+\frac{\partial v_\theta}{r\partial r}\right)-\frac{2}{3}\left(\frac{\partial v_\theta}{r\partial\theta\partial r}+\right.\right.$$

$$\left.\left.\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial\theta}\right)\right]+\nu(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial r}+\frac{\partial^2 v_\theta}{r^2\sin^2\partial\varphi^2}+\frac{\partial^2 v_\theta}{r\partial\theta\partial r}+\csc\theta\cot\theta\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\right.$$

$$\left.\frac{\partial v_\theta}{r^2\partial\theta}\right)+\nu\varphi^{\frac{2}{3}}t'\left[2\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial\theta}+\frac{\partial v_\theta}{\partial r}\frac{\partial^2 v_\theta}{r\partial\theta\partial r}\right)+\left(\frac{\partial v_\theta}{\partial r}\right)^2\right]\Bigg\}\frac{\varphi^{\frac{2}{3}}\nu}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)+$$

$$\Bigg\{-\frac{\partial p}{\rho r\sin\theta\partial\varphi}+\nu\left[\frac{4}{3}\frac{\partial^2 w_\varphi}{r^2\sin^2\partial\varphi^2}-\frac{2}{3}\left(\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}+\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}\right)\right]+\nu(1-\varphi^{\frac{2}{3}})\left[\frac{\partial^2 w_\varphi}{\partial r^2}+\right.$$

$$\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}+(\csc\theta-\cot^2\theta)\csc\theta\frac{\partial w_\varphi}{r^2}-\cot\theta(\csc\theta+1)\frac{\partial w_\theta}{r\partial\theta}-\frac{1}{r}\frac{\partial w_\varphi}{\partial r}-$$

$$\left.(1+\cot\theta)\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}\right]+\nu\varphi^{\frac{2}{3}}t'\left[2\left(\frac{\partial w_\varphi}{\partial r}\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}\right)+\left(\frac{\partial w_\varphi}{\partial r}\right)^2+\left(\frac{\partial w_\varphi}{r\partial\theta}\right)^2\right]\Bigg\}$$

$$\frac{\varphi^{\frac{2}{3}}\nu}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\tag{12-2-20}$$

式(12-2-20)就是球坐标系下可压缩湍流能量微分方程。当定常流时，去掉式中含$\frac{\partial}{\partial t}$项。

12.2.4 动量矩微分方程

推导球坐标系下可压缩湍流动量矩微分方程，应依式(3-7-3)，首先将式(3-7-1)′代入式(3-7-3)，得

$$\varphi\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{r}\times\rho_s\boldsymbol{V}_s)+(\boldsymbol{r}\times\boldsymbol{V}_s)\left[\varphi\frac{\mathrm{D}\rho_s}{\mathrm{D}t}+(1-\varphi)\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\nabla\cdot\boldsymbol{V}\right]+\varphi_1\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{r}\times\rho\boldsymbol{V})+(\boldsymbol{r}\times\rho\boldsymbol{V})\nabla\cdot\boldsymbol{V}$$
$$=\boldsymbol{r}\times[\varphi\rho_s+(1-\varphi)\rho]\boldsymbol{f}+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P})\tag{3-7-3′}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho(\boldsymbol{r}\times\boldsymbol{V})+\varphi\frac{\mathrm{D}}{\mathrm{D}t}[\boldsymbol{r}\times(\rho_s\boldsymbol{V}_s-\rho\boldsymbol{V})]-(\boldsymbol{r}\times\boldsymbol{V}_s)\frac{\mathrm{D}\rho}{\mathrm{D}t}-(\boldsymbol{r}\times\boldsymbol{V}_s)\varphi\frac{\mathrm{D}}{\mathrm{D}t}+(\rho_s-\rho)+$$
$$\rho\boldsymbol{V}\cdot\nabla(\boldsymbol{r}\boldsymbol{V}-\boldsymbol{r}\boldsymbol{V}_s)=\boldsymbol{r}\times[\varphi\rho_s+(1-\varphi)\rho]\boldsymbol{f}+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P})\tag{3-7-3″}$$

令 $\rho_s=\frac{\rho T}{T_s}$，$\frac{T}{T_s}=\alpha$，然后全式除以 ρ，得

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{r}\times\boldsymbol{V})+(\boldsymbol{r}\times\boldsymbol{V})\frac{\mathrm{D}\rho}{\mathrm{D}t}+\varphi\frac{\mathrm{D}}{\mathrm{D}t}[\boldsymbol{r}\times(\alpha\boldsymbol{V}_s-\boldsymbol{V})]+\varphi[\boldsymbol{r}\times(\alpha\boldsymbol{V}_s-\boldsymbol{V})]\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}-[1-\varphi(\alpha-1)]$$
$$(\boldsymbol{r}\times\boldsymbol{V}_s)\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\boldsymbol{V}\cdot\nabla(\boldsymbol{r}\boldsymbol{V}-\boldsymbol{V}_s)=\boldsymbol{r}\times[1+\varphi(\alpha-1)]\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P})\tag{3-7-3‴}$$

现令

$$\eta_1=[1-\varphi(\alpha-1)]\tag{12-2-21}$$

$$\eta_2=[1+\varphi(\alpha-1)]\tag{12-2-22}$$

$$
\begin{aligned}
&\frac{D}{Dt}(\boldsymbol r\times\boldsymbol V)+\left[\boldsymbol r\times(\boldsymbol V-\eta_1\boldsymbol V_s)\frac{D\rho}{\rho Dt}+\varphi\frac{D}{Dt}[\boldsymbol r\times(\alpha\boldsymbol V_s-\boldsymbol V)]+\varphi[\boldsymbol r\times(\alpha\boldsymbol V_s-\boldsymbol V)]\frac{D\rho}{\rho Dt}+\right.\\
&\boldsymbol V\cdot\nabla(\boldsymbol r\boldsymbol V-\boldsymbol V_s)=\boldsymbol r\times\eta_2\boldsymbol f+\frac{1}{\rho}\nabla\cdot(\boldsymbol r\times\boldsymbol P)
\end{aligned}
\tag{12-2-23}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\boldsymbol r\times\boldsymbol V)+u_r\frac{\partial}{\partial r}(\boldsymbol r\times\boldsymbol V)+v_\theta\frac{\partial}{r\partial\theta}(\boldsymbol r\times\boldsymbol V)+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}(\boldsymbol r\times\boldsymbol V)+\Bigg\{\boldsymbol r\times\Bigg\{\Bigg[(1-\eta_1)u_r-\frac{\alpha\eta_1}{2}\\
&\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\Bigg]\boldsymbol e_r+\Bigg[(1-\eta_1)v_\theta-\frac{\alpha\eta_1}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\Bigg]\boldsymbol e_\theta+\Bigg[(1-\eta_1)w_\varphi-\frac{\alpha\eta_1}{2}\\
&\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\Bigg]\boldsymbol e_\varphi\Bigg\}\Bigg\}\frac{D\rho}{\rho Dt}+\varphi\Bigg\{\boldsymbol r\times\Bigg\{\Bigg[(\alpha-1)u_r-\frac{\alpha\nu}{2}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\Bigg]\boldsymbol e_r+\Bigg[(\alpha-1)v_\theta-\frac{\alpha\nu}{2}\\
&\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\Bigg]\boldsymbol e_\theta+\Bigg[(\alpha-1)w_\varphi-\frac{\alpha\nu}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\Bigg]\boldsymbol e_\varphi\Bigg\}\Bigg\}\frac{D\rho}{\rho Dt}+\varphi\frac{D}{Dt}\Bigg\{\boldsymbol r\times\Bigg\{\Bigg[(\alpha-1)u_r-\frac{\alpha\nu}{2}\\
&\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\Bigg]\boldsymbol e_r+\Bigg[(\alpha-1)v_\theta-\frac{\alpha\nu}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\Bigg]\boldsymbol e_\theta+\Bigg[(\alpha-1)w_\varphi-\frac{\alpha\nu}{2}\\
&\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\Bigg]\boldsymbol e_\varphi\Bigg\}\Bigg\}\frac{D\rho}{\rho Dt}+\boldsymbol V\cdot\nabla\Bigg\{\boldsymbol r\times\Bigg[\frac{\nu}{2u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\boldsymbol e_r+\frac{\nu}{2v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\boldsymbol e_\theta+\\
&\frac{\nu}{2w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\boldsymbol e_\varphi\Bigg]\Bigg\}=\boldsymbol r\times\eta_2\boldsymbol f+\frac{1}{\rho}\nabla\cdot(\boldsymbol r\times\boldsymbol P)
\end{aligned}
\tag{12-2-23$'$}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\boldsymbol r\times\boldsymbol V)+u_r\Big[\boldsymbol e_r\frac{\partial}{\partial r}(r_\theta w_\varphi-r_\varphi v_\theta)+\boldsymbol e_\theta\frac{\partial}{\partial r}(r_\varphi u_r-r_r w_\varphi)+\boldsymbol e_\varphi\frac{\partial}{\partial r}(r_r v_\theta-r_\theta u_r)\Big]+\\
&v_\theta\Big[\boldsymbol e_r\frac{\partial}{r\partial\theta}(r_\theta w_\varphi-r_\varphi v_\theta)+\frac{\boldsymbol e_\theta}{r}(r_\theta w_\varphi-r_\varphi v_\theta)+\boldsymbol e_\theta\frac{\partial}{r\partial\theta}(r_\varphi u_r-r_r w_\varphi)-\frac{\boldsymbol e_r}{r}(r_\varphi u_r-r_r w_\varphi)+\\
&\boldsymbol e_\varphi\frac{\partial}{r\partial\theta}(r_r v_\theta-r_\theta u_r)\Big]+w_\varphi\Big[\boldsymbol e_r\frac{\partial}{r\sin\theta\partial\varphi}(r_\theta w_\varphi-r_\varphi v_\theta)+\frac{\boldsymbol e_\varphi}{r}(r_\theta w_\varphi-r_\varphi v_\theta)+\boldsymbol e_\theta\frac{\partial}{r\sin\theta\partial\varphi}\\
&(r_\varphi u_r-r_r w_\varphi)+\frac{\boldsymbol e_\varphi}{r}\cot\theta(r_\varphi u_r-r_r w_\varphi)+\boldsymbol e_\varphi\frac{\partial}{r\sin\theta\partial\varphi}(r_r v_\theta-r_\theta u_r)-(\boldsymbol e_r+\cot\theta\boldsymbol e_\theta)\frac{1}{r}(r_r v_\theta-r_\theta u_r)\Big]+\\
&\big[R_\varphi^\theta(\eta_1)\boldsymbol e_r+R_r^\varphi(\eta_1)\boldsymbol e_\theta+R_\theta^r(\eta_1)\boldsymbol e_\varphi\big]\frac{D\rho}{\rho Dt}+\varphi\big[R_\varphi^\theta(\alpha)\boldsymbol e_r+R_r^\varphi(\alpha)\boldsymbol e_\theta+R_\theta^r(\alpha)\boldsymbol e_\varphi\big]\frac{D\rho}{\rho Dt}+\\
&\varphi\Big\{\frac{\partial}{\partial t}\big[R_\varphi^\theta(\alpha)\boldsymbol e_r+R_r^\varphi(\alpha)\boldsymbol e_\theta+R_\theta^r(\alpha)\boldsymbol e_\varphi\big]\Big\}+\varphi\Big\{u_r\Big[\boldsymbol e_r\frac{\partial}{\partial r}R_\varphi^\theta(\alpha)+\boldsymbol e_\theta\frac{\partial}{\partial r}R_r^\varphi(\alpha)+\boldsymbol e_\varphi\frac{\partial}{\partial r}R_\theta^r(\alpha)\Big]+\\
&v_\theta\Big[\boldsymbol e_r\frac{\partial}{r\partial\theta}R_\varphi^\theta(\alpha)+\frac{\boldsymbol e_\theta}{r}R_\varphi^\theta(\alpha)+\boldsymbol e_\theta\frac{\partial}{r\partial\theta}R_r^\varphi(\alpha)-\frac{\boldsymbol e_r}{r}R_r^\varphi(\alpha)+\boldsymbol e_\varphi\frac{\partial}{r\partial\theta}R_\theta^r(\alpha)\Big]+w_\varphi\Big[\boldsymbol e_r\frac{\partial}{r\sin\theta\partial\varphi}\\
&R_\varphi^\theta(\alpha)+\frac{\boldsymbol e_\varphi}{r}R_\varphi^\theta(\alpha)+\boldsymbol e_\theta\frac{\partial}{r\sin\theta\partial\varphi}R_r^\varphi(\alpha)+\frac{\boldsymbol e_r}{r}\cot\theta R_r^\varphi(\alpha)+\boldsymbol e_\varphi\frac{\partial}{r\sin\theta\partial\varphi}R_\theta^r(\alpha)-(\boldsymbol e_r+\boldsymbol e_\theta\cot\theta)\\
&\frac{1}{r}R_\theta^r(\alpha)\Big]\Big\}+u_r\Big(\boldsymbol e_r\frac{\partial}{\partial r}A_\varphi^\theta+\boldsymbol e_\theta\frac{\partial}{\partial r}A_r^\varphi+\boldsymbol e_\varphi\frac{\partial}{\partial r}A_\theta^r\Big)+v_\theta\Big(\boldsymbol e_r\frac{\partial}{r\partial\theta}A_\varphi^\theta+\frac{\boldsymbol e_\theta}{r}A_\varphi^\theta+\boldsymbol e_\theta\frac{\partial}{r\partial\theta}A_r^\varphi-\\
&\frac{\boldsymbol e_r}{r}A_r^\varphi+\boldsymbol e_\varphi\frac{\partial}{r\partial\theta}A_\theta^r\Big)+w_\varphi\Big[\boldsymbol e_r\frac{\partial}{r\sin\theta\partial\varphi}A_\varphi^\theta+\frac{\boldsymbol e_\varphi}{r}A_\varphi^\theta+\boldsymbol e_\theta\frac{\partial}{r\sin\theta\partial\varphi}A_r^\varphi+\boldsymbol e_\varphi\frac{\cot\theta}{r}A_r^\varphi+\boldsymbol e_\varphi\frac{\partial}{r\sin\theta\partial\varphi}\\
&A_\theta^r-A_\theta^r(\boldsymbol e_r+\boldsymbol e_\theta\cot\theta)\frac{1}{r}\Big]=\boldsymbol r\times\eta_2\boldsymbol f+\frac{1}{\rho}\nabla\cdot(\boldsymbol r\times\boldsymbol P)
\end{aligned}
\tag{12-2-23$''$}
$$

式中：

$$R_\varphi^\theta(\eta_1)=r_\theta\left[(1-\eta_1)w_\varphi-\frac{\nu\eta_1}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]-r_\varphi\left[(1-\eta_1)v_\theta-\frac{\nu\eta_1}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right] \tag{1}$$

$$R_r^\varphi(\eta_1)=r_\varphi\left[(1-\eta_1)u_r-\frac{\nu\eta_1}{2}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]-r_r\left[(1-\eta_1)w_\varphi-\frac{\nu\eta_1}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right] \tag{2}$$

$$R_\theta^r(\eta_1)=r_r\left[(1-\eta_1)v_\theta-\frac{\nu\eta_1}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right] \tag{3}$$

$$R_\varphi^\theta(\alpha)=r_\theta\left[(\alpha-1)w_\varphi-\frac{\nu\alpha}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right]-r_\varphi\left[(\alpha-1)v_\theta-\frac{\nu\alpha}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right] \tag{4}$$

$$R_r^\varphi(\alpha)=r_\varphi\left[(\alpha-1)u_r-\frac{\nu\alpha}{2}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right]-r_r\left[(\alpha-1)w_\varphi-\frac{\nu\alpha}{2}\frac{1}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right] \tag{5}$$

$$R_\theta^r(\alpha)=r_r\left[(\alpha-1)v_\theta-\frac{\nu\alpha}{2}\frac{1}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right]-r_\theta\left[(\alpha-1)u_r-\frac{\nu\alpha}{2}\frac{1}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right] \tag{6}$$

$$A_\varphi^\theta=\frac{\nu}{2}\left[\frac{r_\theta}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)-\frac{r_\varphi}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)\right] \tag{7}$$

$$A_r^\varphi=\frac{\nu}{2}\left[\frac{r_\varphi}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)-\frac{r_r}{w_\varphi}\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\right)\right] \tag{8}$$

$$A_\theta^r=\frac{\nu}{2}\left[\frac{r_r}{v_\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\right)-\frac{r_\theta}{u_r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)\right] \tag{9}$$

将式(12-2-23)″按力矩转轴 e_r，e_θ，e_φ 列出：

$$\frac{\partial}{\partial t}(r_\theta w_\varphi-r_\varphi v_\theta)+u_r\frac{\partial}{\partial r}(r_\theta w_\varphi-r_\varphi v_\theta)+v_\theta\left[\frac{\partial}{r\partial\theta}(r_\theta w_\varphi-r_\varphi v_\theta)+\frac{1}{r}(r_\theta w_\varphi-r_\varphi v_\theta)\right]+$$
$$w_\varphi\left[\frac{\partial}{r\sin\theta\partial\varphi}(r_\theta w_\varphi-r_\varphi v_\theta)+\frac{1}{r}(r_\theta w_\varphi-r_\varphi v_\theta)\right]+\left[R_\varphi^\theta(\eta_1)+\varphi R_\varphi^\theta(\alpha)\right]\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\varphi\left\{\frac{\partial}{\partial t}R_\varphi^\theta(\alpha)+\right.$$
$$\left.u_r\frac{\partial}{\partial r}R_\varphi^\theta(\alpha)+v_\theta\left[\frac{\partial}{r\partial\theta}R_\varphi^\theta(\alpha)+\frac{1}{r}R_\varphi^\theta(\alpha)\right]+w_\varphi\left[\frac{\partial}{r\sin\theta\partial\varphi}R_\varphi^\theta(\alpha)+\frac{1}{r}R_\varphi^\theta(\alpha)\right]\right\}+u_r\frac{\partial}{\partial r}A_\varphi^\theta+$$
$$v_\theta\left(\frac{\partial}{r\partial\theta}A_\varphi^\theta+\frac{1}{r}A_\varphi^\theta\right)+w_\varphi\left(\frac{\partial}{r\sin\theta\partial\varphi}A_\varphi^\theta+\frac{1}{r}A_\varphi^\theta\right)$$
$$=(r_\theta f_\varphi-r_\varphi f_\theta)\eta_2+r_\varphi\frac{\partial p}{\rho r\partial\theta}-r_\theta\frac{\partial p}{\rho r\sin\theta\partial\varphi}+\nu\left\{r_\theta\left\{\frac{4}{3}\frac{\partial^2 w_\varphi}{r^2\sin^2\partial\varphi^2}-\frac{2}{3}\left(\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi}+\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}\right)+\right.\right.$$
$$(1-\varphi^{\frac{2}{3}})\left[\frac{\partial^2 w_\varphi}{\partial r^2}+\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}+(\csc\theta-\cot^2\theta)\csc\theta\frac{w_\varphi}{r^2}-\cot\theta(\csc\theta+1)\right.$$
$$\left.\frac{\partial w_\theta}{r\partial\theta}-\frac{1}{r}\frac{\partial w_\varphi}{\partial r}-(1+\cot\theta)\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}\right]+\varphi^{\frac{2}{3}}t'\left[2\left(\frac{\partial w_\varphi}{\partial r}\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}\right)+\right.$$
$$\left.\left.\left(\frac{\partial w_\varphi}{\partial r}\right)^2+\left(\frac{\partial w_\varphi}{r\partial\theta}\right)^2\right]\right\}-r_\varphi\left\{\frac{4}{3}\frac{\partial^2 v_\theta}{r^2\partial\theta^2}+\frac{2}{3}(\csc^2\theta+\cot\theta)\frac{v_\theta}{r^2}+\frac{4}{3}\frac{\cot\theta}{r^2}\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\frac{2}{3}\cot\theta\left(\frac{\partial v_\theta}{r^2\partial\theta}+\right.\right.$$

$$\left.\frac{\partial v_\theta}{r\partial r}\right)-\frac{2}{3}\left(\frac{\partial v_\theta}{r\partial\theta\partial r}+\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial\theta}\right)+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial r}+\frac{\partial^2 v_\theta}{r^2\sin^2\partial\varphi^2}+\frac{\partial^2 v_\theta}{r\partial\theta\partial r}+\right.$$

$$\left.\left.\left.\csc\theta\cot\theta\,\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}\right)+\varphi^{\frac{2}{3}}t'\left[2\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\frac{\partial^2 v_\theta}{r^2\sin\theta^2\partial\varphi\partial\theta}+\frac{\partial v_\theta}{\partial r}\frac{\partial^2 v_\theta}{r\partial\theta\partial r}\right)+\left(\frac{\partial v_\theta}{\partial r}\right)^2\right]\right\}\right\}$$

$$(12\text{-}2\text{-}24)$$

$$\frac{\partial}{\partial t}(r_\varphi u_r-r_r w_\varphi)+u_r\,\frac{\partial}{\partial r}(r_\varphi u_r-r_r w_\varphi)+v_\theta\left[\frac{\partial}{r\partial\theta}(r_\varphi u_r-r_r w_\varphi)-\frac{1}{r}(r_\varphi u_r-r_r w_\varphi)\right]+$$

$$w_\varphi\left[\frac{\partial}{r\sin\theta\partial\varphi}(r_\varphi u_r-r_r w_\varphi)+\frac{\cot\theta}{r}(r_\varphi u_r-r_r w_\varphi)\right]+\left[R_r^\varphi(\eta_1)+\varphi R_r^\varphi(\alpha)\right]\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\varphi\left\{\frac{\partial}{\partial t}R_r^\varphi(\alpha)+\right.$$

$$\left.u_r\,\frac{\partial}{\partial r}R_r^\varphi(\alpha)+v_\theta\left[\frac{\partial}{r\partial\theta}R_r^\varphi(\alpha)-\frac{1}{r}R_r^\varphi(\alpha)\right]+w_\varphi\left[\frac{\partial}{r\sin\theta\partial\varphi}R_r^\varphi(\alpha)+\frac{\cot\theta}{r}R_r^\varphi(\alpha)\right]\right\}+u_r\,\frac{\partial}{\partial r}A_r^\varphi+$$

$$v_\theta\left(\frac{\partial}{r\partial\theta}A_r^\varphi+\frac{1}{r}A_r^\varphi\right)+w_\varphi\left(\frac{\partial}{r\sin\theta\partial\varphi}A_r^\varphi+\frac{\cot\theta}{r}A_r^\varphi\right)$$

$$=(r_\varphi f_r-r_r f_\varphi)\eta_2+r_r\,\frac{\partial p}{\rho r\sin\theta\partial\varphi}-r_\varphi\,\frac{\partial p}{\rho\partial r}+\nu\left\{r_\varphi\left[\frac{4}{3}\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{u_r}{r^2}-\frac{\partial u_r}{r\partial r}\right)+\frac{2}{3}\left(\frac{\partial u_r}{r^2\sin\theta\partial\varphi}+\right.\right.\right.$$

$$\left.\frac{\partial u_r}{r^2\partial\theta}\right)-\frac{2}{3}\left(\frac{\partial^2 u_r}{r\partial r\partial\theta}+\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}\right)+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 u_r}{r\partial\theta\partial r}+\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 u_r}{r^2\partial\theta^2}+\frac{\partial^2 u_r}{r^2\sin^2\theta\partial\varphi^2}+\right.$$

$$\left.\left.\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r^2\sin\theta\partial\varphi}\right)+2\varphi^{\frac{2}{3}}t'\left(\frac{\partial u_r}{r\partial\theta}\frac{\partial^2 u_r}{r\partial\theta\partial r}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}\right)\right]-r_r\left\{\left[\frac{4}{3}\frac{\partial^2 w_\varphi}{r^2\sin^2\theta\partial\varphi^2}-\right.\right.$$

$$\left.\frac{2}{3}\left(\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}-\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}\right)\right]+(1-\varphi^{\frac{2}{3}})\left[\frac{\partial^2 w_\varphi}{\partial r^2}+\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}+\right.$$

$$\left.(\csc\theta-\cot^2\theta)\csc\theta\,\frac{w_\varphi}{r^2}-\cot\theta(\csc\theta+1)\frac{\partial w_\varphi}{r\partial\theta}-\frac{1}{r}\frac{\partial w_\varphi}{\partial r}-(1+\cot\theta)\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}\right]+$$

$$\left.\left.\varphi^{\frac{2}{3}}t'\left[2\left(\frac{\partial w_\varphi}{\partial r}\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\frac{\partial w_\varphi}{r\partial\theta}\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}\right)+\left(\frac{\partial w_\varphi}{\partial r}\right)^2+\left(\frac{\partial w_\varphi}{r\partial\theta}\right)^2\right]\right\}\right\}\qquad(12\text{-}2\text{-}25)$$

$$\frac{\partial}{\partial t}(r_r v_\theta-r_\theta u_r)+u_r\,\frac{\partial}{\partial r}(r_r v_\theta-r_\theta u_r)+v_\theta\,\frac{\partial}{r\partial\theta}(r_r v_\theta-r_\theta u_r)+w_\varphi\left[\frac{\partial}{r\sin\theta\partial\varphi}(r_r v_\theta-r_\theta u_r)-\right.$$

$$\left.(1+\cot\theta)\frac{1}{r}(r_r v_\theta-r_\theta u_r)\right]+\left[R_\theta^r(\eta_1)+\varphi R_\theta^r(\alpha)\right]\frac{\mathrm{D}\rho}{\rho\mathrm{D}t}+\varphi\left\{\frac{\partial}{\partial t}R_\theta^r(\alpha)+u_r\,\frac{\partial}{\partial r}R_\theta^r(\alpha)+\right.$$

$$\left.v_\theta\,\frac{\partial}{r\partial\theta}R_\theta^r(\alpha)+w_\varphi\left[\frac{\partial}{r\sin\theta\partial\varphi}R_\theta^r(\alpha)-(1+\cot\theta)\frac{1}{r}R_\theta^r(\alpha)\right]\right\}+u_r\,\frac{\partial}{\partial r}A_\theta^r\varphi+v_\theta\,\frac{\partial}{r\partial\theta}A_\theta^r+$$

$$w_\varphi\left[\frac{\partial}{r\sin\theta\partial\varphi}A_\theta^r-(1+\cot\theta)\frac{1}{r}A_\theta^r\right]$$

$$=(r_r f_\theta-r_\theta f_r)\eta_2+r_\theta\,\frac{\partial p}{\rho\partial r}-r_r\,\frac{\partial p}{\rho r\partial\theta}+\nu\left\{r_r\left\{\frac{4}{3}\frac{\partial^2 v_\theta}{r^2\partial\theta^2}+\frac{2}{3}(\csc^2\theta+\cot\theta)\frac{v_\theta}{r^2}+\frac{4}{3}\frac{\cot\theta}{r^2}-\right.\right.$$

$$\frac{2}{3}\cot\theta\left(\frac{\partial v_\theta}{r^2\partial\theta}+\frac{\partial v_\theta}{r\partial r}\right)-\frac{2}{3}\left(\frac{\partial v_\theta}{r\partial\theta\partial r}+\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial\theta}\right)+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial r}+\right.$$

$$\left.\frac{\partial^2 v_\theta}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2 v_\theta}{r\partial\theta\partial r}+\csc\theta\cot\theta\,\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}\right)+\varphi^{\frac{2}{3}}t'\left[2\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial\theta}+\right.\right.$$

$$\left.\frac{\partial v_\theta}{\partial r}\frac{\partial^2 v_\theta}{r\partial\theta\partial r}\right)+\left(\frac{\partial v_\theta}{\partial r}\right)^2\Bigg]\Bigg\}-r_\theta\Bigg[\frac{4}{3}\frac{\partial^2 u_r}{\partial r^2}+\frac{u_r}{r^2}-\frac{\partial u_r}{r\partial r}+\frac{2}{3}\left(\frac{\partial u_r}{r^2\sin\theta\partial\varphi}+\frac{\partial u_r}{r^2\partial\theta}\right)-\frac{2}{3}\left(\frac{\partial^2 u_r}{r\partial r\partial\theta}+\right.$$

$$\left.\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}\right)+(1-\varphi^{\frac{2}{3}})\left(\frac{\partial^2 u_r}{r\partial\theta\partial r}+\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 u_r}{r^2\partial\theta^2}+\frac{\partial^2 u_r}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r^2\sin\theta\partial\varphi}\right)+$$

$$\varphi^{\frac{2}{3}}t'\left(\frac{\partial u_r}{r\partial\theta}\frac{\partial^2 u_r}{r\partial\theta\partial r}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}\right)\Bigg]\Bigg\} \tag{12-2-26}$$

式(12-2-24)～式(12-2-26)是球坐标系可压缩湍流动量矩微分方程。当定常流时，去掉含$\frac{\partial}{\partial t}$项。

12.3　不可压缩层流微分方程组

依无坐标系下不可压缩质量守恒方程式(3-4-1)

$$\frac{\partial\rho}{\partial t}+\varphi^{\frac{2}{3}}\rho_s\nabla\cdot\boldsymbol{v}_s+\varphi_2\rho\nabla\cdot\boldsymbol{V}_s=0$$

当层流时，$\varphi=0$，$\varphi^{\frac{2}{3}}=0$，$\rho_s=0$，则上式为

$$\frac{\partial\rho}{\partial t}+\rho\nabla\cdot\boldsymbol{V}=0 \tag{12-3-1}$$

当不可压缩时，$\rho=$常数，则为

$$\nabla\cdot\boldsymbol{V}=0 \tag{12-3-2}$$

式(12-3-2)为球坐标系下的方程。

12.3.1　连续性微分方程

$$\nabla\cdot\boldsymbol{V}=2\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+\frac{v_\theta}{r}\cot\theta+\frac{\partial v_\theta}{r\partial\theta}+\frac{\partial w_\varphi}{r\sin\theta\partial\varphi} \tag{12-3-3}$$

12.3.2　动量微分方程

依式(3-4-2)，当层流时，$\varphi=0$，$\varphi^{\frac{2}{3}}=0$，则式(3-4-2)变为

$$\frac{\partial}{\partial t}\rho\boldsymbol{V}+\rho\nabla\cdot\boldsymbol{V}\boldsymbol{V}=\boldsymbol{f}\rho+\nabla\cdot\boldsymbol{P} \tag{12-3-4}$$

当 $\rho=$常数时，除以 ρ

$$\frac{\partial\boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{12-3-5}$$

将式(12-3-5)转变为球坐标系下层流不可压缩动量微分方程

$$\frac{\partial\boldsymbol{V}}{\partial t}+u_r\frac{\partial\boldsymbol{V}}{\partial r}+v_\theta\frac{\partial\boldsymbol{V}}{r\partial\theta}+w_\varphi\frac{\partial\boldsymbol{V}}{r\sin\theta\partial\varphi}=\boldsymbol{f}+(\nabla\cdot\boldsymbol{P})\frac{1}{\rho} \tag{a}$$

$$\frac{\partial\boldsymbol{V}}{\partial t}+u_r\frac{\partial}{\partial r}(u_r\boldsymbol{e}_r+v_\theta\boldsymbol{e}_\theta+w_\varphi\boldsymbol{e}_\varphi)+v_\theta\frac{\partial}{r\partial\theta}(u_r\boldsymbol{e}_r+v_\theta\boldsymbol{e}_\theta+w_\varphi\boldsymbol{e}_\varphi)+w_\varphi\frac{\partial}{r\sin\theta\partial\varphi}(u_r\boldsymbol{e}_r+v_\theta\boldsymbol{e}_\theta$$

$$+w_\varphi e_\varphi)=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot\mathbf{P} \tag{b}$$

$$\frac{\partial\mathbf{V}}{\partial t}+u_r\left(e_r\frac{\partial u_r}{\partial r}+e_\theta\frac{\partial v_\theta}{\partial r}+e_\varphi\frac{\partial w_\varphi}{\partial r}\right)+v_\theta\left(e_r\frac{\partial u_r}{r\partial\theta}+e_\theta\frac{u_r}{r}+e_\theta\frac{\partial v_\theta}{r\partial\theta}-\frac{e_r}{r}v_\theta+e_\varphi\frac{\partial w_\varphi}{r\partial\theta}\right)+$$
$$w_\varphi\left[e_r\frac{\partial u_r}{r\sin\theta\partial\varphi}+e_\varphi\frac{u_r}{r}+e_\varphi\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}+e_\varphi\frac{\cot\theta}{r}v_\theta+e_\varphi\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}-(e_r+e_\theta\cot\theta)\frac{w_\varphi}{r}\right]$$
$$=\boldsymbol{f}+\frac{1}{\rho}\left[\left(e_r\frac{\partial p_{rr}}{\partial r}+e_\theta\frac{\partial p_{r\theta}}{\partial r}+e_\varphi\frac{\partial p_{r\varphi}}{\partial r}\right)+\left(e_r\frac{\partial p_{\theta r}}{r\partial\theta}+p_{\theta r}\frac{e_\theta}{r}+e_\theta\frac{\partial p_{\theta\theta}}{r\partial\theta}-\frac{e_r}{r}p_{\theta\theta}+e_\varphi\frac{\partial p_{\theta\varphi}}{r\partial\theta}\right)+\right.$$
$$\left.\left(e_r\frac{\partial p_{\varphi r}}{r\sin\theta\partial\varphi}+e_\varphi\frac{1}{r}p_{\varphi r}+e_\theta\frac{\partial p_{\varphi\theta}}{r\sin\theta\partial\varphi}+e_\varphi\cot\theta\frac{1}{r}p_{\varphi\theta}-(e_r+e_\theta\cot\theta)\frac{1}{r}p_{\varphi\varphi}\right)\right] \tag{c}$$

$$\frac{\partial\mathbf{V}}{\partial t}+u_r\left(e_r\frac{\partial u_r}{\partial r}+e_\theta\frac{\partial v_\theta}{\partial r}+e_\varphi\frac{\partial w_\varphi}{\partial r}\right)+v_\theta\left(e_r\frac{\partial u_r}{r\partial\theta}+e_\theta\frac{u_r}{r}+e_\theta\frac{\partial v_\theta}{r\partial\theta}-e_r\frac{v_\theta}{r}+e_\varphi\frac{\partial w_\varphi}{r\partial\theta}\right)+$$
$$w_\varphi\left[e_r\frac{\partial u_r}{r\sin\theta\partial\varphi}+e_\varphi\frac{u_r}{r}+e_\theta\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+e_\varphi\cot\theta\frac{v_\theta}{r}+e_\varphi\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}-(e_r+e_\theta\cot\theta)\frac{w_\varphi}{r}\right]$$
$$=\boldsymbol{f}+\frac{1}{\rho}\left\{e_r\frac{\partial}{\partial r}\left(-p+2\mu\frac{\partial u_r}{\partial r}\right)+e_\theta\frac{\partial}{\partial r}\mu\left(\frac{\partial v_\theta}{\partial r}+\frac{\partial u_r}{r\partial\theta}-\frac{v_\theta}{r}\right)+e_\varphi\frac{\partial}{\partial r}\mu\left(\frac{\partial w_\varphi}{\partial r}+\frac{\partial u_r}{r\sin\theta\partial\varphi}-\frac{w_\varphi}{r}\right)+\right.$$
$$e_r\frac{\partial\mu}{r\partial\theta}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)+e_\theta\frac{\mu}{r}\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)+e_\theta\frac{\partial}{r\partial\theta}\left[-p+2\mu\left(\frac{\partial v_\theta}{r\partial\theta}+\frac{u_r}{r}\right)\right]-$$
$$e_r\frac{1}{r}\left[-p+2\mu\left(\frac{\partial v_\theta}{r\partial\theta}+\frac{u_r}{r}\right)\right]+e_\varphi\frac{\partial\mu}{r\partial\theta}\left(\frac{\partial w_\varphi}{r\partial\theta}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\frac{\cot\theta}{r}w_\varphi\right)+e_r\frac{\partial}{r\sin\theta\partial\varphi}$$
$$\left(\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{\partial r}-\frac{w_\varphi}{r}\right)+\frac{1}{r}\left(\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{\partial r}-\frac{w_\varphi}{r}\right)+\frac{e_\varphi}{r}\left(\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{\partial r}-\frac{w_\varphi}{r}\right)+e_\theta$$
$$\frac{\partial}{r\sin\theta\partial\varphi}\mu\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{r\partial\theta}-\cot\theta\frac{w_\varphi}{r}\right)+e_\varphi\cot\theta\frac{\mu}{r}\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{r\partial\theta}-\cot\theta\frac{w_\varphi}{r}\right)+e_\varphi$$
$$\frac{\partial}{r\sin\theta\partial\varphi}\left[-p+2\mu\left(\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}+\frac{u_r}{r}+\cot\theta\frac{v_\theta}{r}\right)\right]-(e_r+e_\theta\cot\theta)\frac{1}{r}\left[-p+2\mu\left(\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}+\right.\right.$$
$$\left.\left.\left.\frac{u_r}{r}+\cot\theta\frac{v_\theta}{r}\right)\right]\right\} \tag{d}$$

将式(d)按 e_r，e_θ，e_φ 三个方向集合：

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+v_\theta\left(\frac{\partial u_r}{r\partial\theta}+\frac{u_r}{r}\right)+w_\varphi\left(\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{u_r}{r}\right)$$
$$=f_r-\frac{\partial p}{\rho\partial r}+\nu\left[2\frac{\partial^2u_r}{\partial r^2}+\frac{\partial^2u_r}{r^2\partial\theta^2}+\frac{\partial^2u_r}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2u_r}{r\partial\theta\partial r}+\frac{\partial^2u_r}{r\sin\theta\partial\varphi\partial r}+3\left(\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r\sin\theta\partial\varphi}\right)-\right.$$
$$\left.(3+2\cot\theta)\frac{u_r}{r^2}\right] \tag{12-3-6}$$

$$\frac{\partial v_\theta}{\partial t}+u_r\frac{\partial v_\theta}{\partial r}+v_\theta\left(\frac{\partial v_\theta}{r\partial\theta}-\frac{v_\theta}{r}\right)+w_\varphi\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+\cot\theta\frac{v_\theta}{r}\right)$$
$$=f_\theta-\frac{\partial p}{\rho r\partial\theta}+\nu\left[\frac{\partial^2v_\theta}{\partial r^2}+2\frac{\partial^2v_\theta}{r^2\partial\theta^2}+\frac{\partial^2v_\theta}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2v_\theta}{r\partial\theta\partial r}+\frac{\partial^2v_\theta}{r^2\sin\theta\partial\varphi\partial\theta}+\frac{\partial v_\theta}{r\partial r}+3\left(\cot\theta\frac{\partial v_\theta}{r^2\sin\theta\partial\varphi}-\right.\right.$$
$$\left.\left.\frac{\partial v_\theta}{r^2\partial\theta}\right)-2\left(1+\cot\theta\frac{v_\theta}{r^2}\right)\right] \tag{12-3-7}$$

$$\frac{\partial w_\varphi}{\partial t}+u_r\frac{\partial w_\varphi}{\partial r}+v_\theta\frac{\partial w_\varphi}{r\partial\theta}+w_\varphi\left[\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}+(1+\cot\theta)\frac{w_\varphi}{r}\right]$$
$$=f_\varphi-\frac{\partial p}{\rho r\sin\theta\partial\varphi}+\nu\left[\frac{\partial^2 w_\varphi}{\partial r^2}+\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+2\frac{\partial^2 w_\varphi}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}-3(1+\cot\theta)\cdot\right.$$
$$\left.\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}-\cot^2\theta\frac{w_\varphi}{r^2}\right]\tag{12-3-8}$$

式(12-3-6)至式(12-3-8)就是球坐标系下不可压缩层流运动动量微分方程。当定常流时，去掉含$\frac{\partial}{\partial t}$项。

12.3.3　能量微分方程

由无坐标系下不可压缩层流运动能量方程式(3-4-4)

$$\frac{\partial}{\partial t}\left(\varphi\frac{v_s^2}{2}+\varphi_1\frac{v^2}{2}\right)+\nabla\cdot\left(\varphi^{\frac{2}{3}}\frac{v_s^2}{2}\boldsymbol{V}_s+\varphi_2\frac{v^2}{2}\boldsymbol{V}\right)$$
$$=\frac{1}{\rho}\nabla\cdot(\lambda\nabla T)+q_R+\boldsymbol{f}\cdot(\varphi\boldsymbol{V}_s+\varphi_1\boldsymbol{V})+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{V}_s+\varphi_2\boldsymbol{V})$$

式(3-4-4)为出发点。当层流时，$\varphi=0$，$\varphi^{\frac{2}{3}}=0$，则上式为

$$\frac{\partial}{\partial t}\left(\frac{v^2}{2}\right)+\nabla\cdot\frac{v^2}{2}\boldsymbol{V}=\frac{1}{\rho}\nabla\cdot(\lambda\nabla T)+q_R+\boldsymbol{f}\cdot\boldsymbol{V}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot\boldsymbol{V}\tag{12-3-9}$$

将式(12-3-9)写成

$$\frac{\partial}{\partial t}\left(\frac{v^2}{2}\right)+\boldsymbol{V}\cdot\nabla\frac{v^2}{2}=\frac{1}{\rho}\nabla\cdot(\lambda\nabla T)+q_R+\boldsymbol{f}\cdot\boldsymbol{V}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P}\cdot\boldsymbol{V}\tag{12-3-10}$$

将式(12-3-10)转变为球坐标系下层流不可压缩运动能量方程：

$$v\frac{\partial v}{\partial t}+u_r v\frac{\partial v}{\partial r}+v_\theta v\frac{\partial v}{r\partial\theta}+w_\varphi v\frac{\partial v}{r\sin\theta\partial\varphi}$$
$$=f_r u_r+f_\theta v_\theta+f_\varphi w_\varphi+\frac{\lambda}{\rho}\left(\frac{\partial^2 T}{\partial r^2}+\frac{\partial^2 T}{r^2\partial\theta^2}+\frac{\partial^2 T}{r^2\sin^2\theta\partial\varphi^2}\right)+q_R-\frac{1}{\rho}\left(u_r\frac{\partial p}{\partial r}+v_\theta\frac{\partial p}{r\partial\theta}+w_\varphi\right.$$
$$\left.\frac{\partial p}{r\sin\theta\partial\varphi}\right)+\nu\left\{u_r\left[2\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial^2 u_r}{r^2\partial\theta^2}+\frac{\partial^2 u_r}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2 u_r}{r\partial\theta\partial r}+\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}+3\left(\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r^2\sin\theta\partial\varphi}\right)-\right.\right.$$
$$\left.(3+2\cot\theta)\frac{u_r}{r^2}\right]+v_\theta\left[\frac{\partial^2 v_\theta}{\partial r^2}+2\frac{\partial^2 v_\theta}{r^2\partial\theta^2}+\frac{\partial^2 v_\theta}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2 v_\theta}{r\partial\theta\partial r}+\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi}+\frac{\partial v_\theta}{r\partial r}+\right.$$
$$\left.3\left(\cot\theta\frac{\partial v_\theta}{r^2\sin\theta\partial\varphi}-\frac{\partial v_\theta}{r^2\partial\theta}\right)-2(1+\cot\theta)\cot\theta\frac{v_\theta}{r^2}\right]+w_\varphi\left[\frac{\partial^2 w_\varphi}{\partial r^2}+\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+2\frac{\partial^2 w_\varphi}{r^2\sin^2\theta\partial\varphi^2}+\right.$$
$$\left.\left.\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}-3(1+\cot\theta)\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}-\cot^2\theta\frac{w_\varphi}{r^2}\right]\right\}\tag{12-3-11}$$

式(12-3-11)就是球坐标系下层流不可压缩运动能量微分方程。当定常流时，去掉式中含$\frac{\partial}{\partial t}$项。式中：

$$v=\sqrt{u_r^2+v_\theta^2+w_\varphi^2}$$

12.3.4 动量矩微分方程

依无坐标系下不可压缩流体运动动量矩微分方程式(3-4-3)

$$\frac{\partial}{\partial t}[\boldsymbol{r}\times\rho(\varphi\boldsymbol{V}_s+\varphi_1\boldsymbol{V})]+\nabla\cdot\boldsymbol{V}_s(\boldsymbol{r}\times\varphi^{\frac{2}{3}}\rho\boldsymbol{V}_s)+\nabla\cdot\boldsymbol{V}(\boldsymbol{r}\times\varphi_2\rho\boldsymbol{V})$$
$$=(\boldsymbol{r}\times\boldsymbol{f})\rho+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P})$$

当层流时，$\varphi=0$，$\varphi^{\frac{2}{3}}=0$，则上式为

$$\frac{\partial}{\partial t}(\boldsymbol{r}\times\rho\boldsymbol{V})+\nabla\cdot\boldsymbol{V}(\boldsymbol{r}\times\rho\boldsymbol{V})=(\boldsymbol{r}\times\boldsymbol{f})\rho+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{12-3-12}$$

因为是不可压缩层流运动，$\rho=$常量，全式除以 ρ，则

$$\frac{\partial}{\partial t}(\boldsymbol{r}\times\boldsymbol{V})+\boldsymbol{V}\cdot\nabla(\boldsymbol{r}\times\boldsymbol{V})=\boldsymbol{r}\times\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{12-3-13}$$

又因 $\boldsymbol{r}$ 是参变量，不参加微积分运动，则式(12-3-13)为

$$\boldsymbol{r}\times\frac{\partial\boldsymbol{V}}{\partial t}+\boldsymbol{r}\times(\boldsymbol{V}\cdot\nabla)=\boldsymbol{r}\times\boldsymbol{f}+\frac{1}{\rho}\boldsymbol{r}\times\nabla\cdot\boldsymbol{P} \tag{12-3-14}$$

式(12-3-14)仍然是不可压缩层流运动无坐标系方程。现在，将它变为球坐标系下对应的方程。利用式(12-3-6)至式(12-3-8)结果，根据力矩定义，可以方便得到结果如下：

$$r_\theta\frac{\partial w_\varphi}{\partial t}-r_\varphi\frac{\partial v_\theta}{\partial t}+r_\theta\left\{u_r\frac{\partial w_\varphi}{\partial r}+v_\theta\frac{\partial w_\varphi}{r\partial\theta}+w_\varphi\left[\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}-(1+\cot\theta)\frac{w_\varphi}{r}\right]\right\}-r_\varphi\left[u_r\frac{\partial v_\theta}{\partial r}+\right.$$
$$\left.v_\theta\left(\frac{\partial v_\theta}{r\partial\theta}-\frac{v_\theta}{r}\right)+w_\varphi\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+\cot\theta\frac{v_\theta}{r}\right)\right]$$
$$=r_\theta f_\varphi-r_\varphi f_\theta+\frac{1}{\rho}\left(\frac{\partial p}{r\partial\theta}-\frac{\partial p}{r\sin\theta\partial\varphi}\right)+\nu\left\{r_\theta\left[\frac{\partial^2 w_\varphi}{\partial r^2}+\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+2\frac{\partial^2 w_\varphi}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\right.\right.$$
$$\left.\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}-3(1+\cot\theta)\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}-\cot^2\theta\frac{w_\varphi}{r^2}\right]-r_\varphi\left[\frac{\partial^2 v_\theta}{\partial r^2}+2\frac{\partial^2 v_\theta}{r^2\partial\theta^2}+\frac{\partial^2 v_\theta}{r^2\sin^2\theta\partial\varphi}+\right.$$
$$\left.\left.\frac{\partial^2 v_\theta}{r\partial\theta\partial r}+\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi}+\frac{\partial v_\theta}{r\partial\theta}+3\left(\frac{\cot\theta}{r^2}\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\frac{\partial v_\theta}{r^2\partial\theta}\right)-2(1+\cot\theta)\frac{v_\theta}{r^2}\right]\right\} \tag{12-3-15}$$

$$r_\varphi\frac{\partial u_r}{\partial t}-r_r\frac{\partial w_\varphi}{\partial t}+r_\varphi\left[u_r\frac{\partial u_r}{\partial r}+v_\theta\left(\frac{\partial u_r}{r\partial\theta}+\frac{u_r}{r}\right)+w_\varphi\left(\frac{\partial u_z}{r\sin\theta\partial\varphi}+\frac{u_r}{r}\right)\right]-r_r\left\{u_r\frac{\partial w_\varphi}{\partial r}+\right.$$
$$\left.v_\theta\frac{\partial w_\varphi}{r\partial\theta}+w_\varphi\left[\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}-(1+\cot\theta)\frac{w_\varphi}{r}\right]\right\}$$
$$=r_\varphi f_r-r_r f_\varphi+\frac{1}{\rho}\left(\frac{\partial p}{r\sin\theta\partial\varphi}-\frac{\partial p}{\partial r}\right)+\nu\left\{r_\varphi\left[2\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial^2 u_r}{r^2\partial\theta^2}+\frac{\partial^2 u_r}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2 u_r}{r\partial\theta\partial r}+\right.\right.$$
$$\left.\frac{\partial^2 u_r}{r\sin\theta\partial\varphi\partial r}+3\left(\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r^2\sin\theta\partial\varphi}\right)-(3+2\cot\alpha)\frac{u_r}{r^2}\right]-r_r\left[\frac{\partial^2 w_\varphi}{\partial r^2}+\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+2\frac{\partial^2 w_\varphi}{r^2\sin^2\theta\partial\varphi^2}+\right.$$
$$\left.\left.\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}-3(1+\cot\theta)\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}-\cot^2\theta\frac{w_\varphi}{r^2}\right]\right\} \tag{12-3-16}$$

$$r_r\frac{\partial v_\theta}{\partial t}-r_\theta\frac{\partial u_r}{\partial t}+r_r\left[u_r\frac{\partial v_\theta}{\partial r}+v_\theta\left(\frac{\partial v_\theta}{r\partial\theta}-\frac{v_\theta}{r}\right)+w_\varphi\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+\cot\theta\frac{v_\theta}{r}\right)\right]-r_\theta\left[u_r\frac{\partial u_r}{\partial r}+\right.$$
$$\left.v_\theta\left(\frac{\partial u_r}{r\partial\theta}+\frac{u_r}{r}\right)+w_\varphi\left(\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{u_r}{r}\right)\right]$$
$$=r_rf_\theta-r_\theta f_r+\frac{1}{\rho}\left(\frac{\partial p}{\partial r}-\frac{\partial p}{r\partial\theta}\right)+\nu\left\{r_r\left[\frac{\partial^2v_\theta}{\partial r^2}+2\frac{\partial^2v_\theta}{r^2\partial\theta^2}+\frac{\partial^2v_\theta}{r^2\sin^2\theta\partial\varphi^2}+\frac{\partial^2v_\theta}{r\partial\theta\partial r}+\frac{\partial^2v_\theta}{r^2\sin\theta\partial\varphi\partial\theta}+\right.\right.$$
$$\left.\frac{\partial v_\theta}{r\partial r}+3\left(\cot\theta\frac{\partial v_\theta}{r^2\sin\theta\partial\varphi}-\frac{\partial v_\theta}{r^2\partial\theta}\right)-2(1+\cot\theta)\frac{v_\theta}{r^2}\right]-r_\theta\left[2\frac{\partial^2u_r}{\partial r^2}+\frac{\partial^2u_r}{r^2\partial\theta^2}+\frac{\partial^2u_r}{r^2\sin^2\theta\partial\varphi^2}+\right.$$
$$\left.\left.\frac{\partial^2u_r}{r\partial\theta\partial r}+\frac{\partial^2u_r}{r\sin\theta\partial\varphi\partial r}+3\left(\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r^2\sin\theta\partial\varphi}\right)-(3+2\cot\theta)\frac{u_r}{r^2}\right]\right\}\qquad(12\text{-}3\text{-}17)$$

式(12-3-15)至式(12-3-17)就是球坐标系下层流不可压缩动量矩微分方程。当定常流时，去掉含$\frac{\partial}{\partial t}$项。

12.4　可压缩层流微分方程组

由无坐标系流体运动质量守恒微分方程式(3-7-1)：

$$\varphi\frac{D\rho_s}{Dt}+(1-\varphi)\frac{D\rho}{Dt}+\rho_s\nabla\cdot\boldsymbol{V}_s+\rho\nabla\cdot\boldsymbol{V}=0$$

当层流时，$\varphi=0$，$\rho_s=0$，则为

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\boldsymbol{V}=0\qquad(12\text{-}4\text{-}1)$$

式(12-4-1)是可压缩层流运动无坐标系质量守恒微分方程。从运动学角度看，它为连续性微分方程，现将其变为球坐标系连续性微分方程。

12.4.1　连续性微分方程

$$\frac{\partial\rho}{\partial t}+u_r\frac{\partial\rho}{\partial r}+v_\theta\frac{\partial\rho}{r\partial\theta}+w_\varphi\frac{\partial\rho}{r\sin\theta\partial\varphi}+\rho\left(2\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+\frac{\partial v_\theta}{r\partial\theta}+\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}+\cot\theta\frac{v_\theta}{r}\right)=0$$
$$(12\text{-}4\text{-}2)$$

12.4.2　动量微分方程

依式(3-7-2)：

$$\varphi\frac{D(\rho_s\boldsymbol{V}_s)}{Dt}+(1-\varphi)\frac{D(\rho\boldsymbol{V})}{Dt}+\rho_s\boldsymbol{V}_s\nabla\cdot\boldsymbol{V}_s+\rho\boldsymbol{V}(\nabla\cdot\boldsymbol{V})=\boldsymbol{f}[\varphi\rho_s+(1-\varphi)\rho]+\nabla\cdot\boldsymbol{P}$$

当层流时，$\rho_s=0$，$\varphi=0$，$v_s=0$，则上式为

$$\frac{D(\rho\boldsymbol{V})}{Dt}+\rho\boldsymbol{V}\nabla\cdot\boldsymbol{V}=\rho\boldsymbol{f}+\nabla\cdot\boldsymbol{P}\qquad(12\text{-}4\text{-}3)$$

将式(12-4-1)代入式(12-4-3)，全式除以 ρ，则

$$\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}=\boldsymbol{f}+\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{12-4-4}$$

式(12-4-4)是层流运动无坐标系下动量微分方程，现将其变为球坐标系下的对应方程。

$$\frac{\partial \boldsymbol{V}}{\partial t}+u_r\frac{\partial \boldsymbol{V}}{\partial r}+v_\theta\frac{\partial \boldsymbol{V}}{r\partial\theta}+w_\varphi\frac{\partial \boldsymbol{V}}{r\sin\theta\partial\varphi}$$

$$=\boldsymbol{f}+\frac{1}{\rho}\Big[\boldsymbol{e}_r\frac{\partial p_{rr}}{\partial r}+\boldsymbol{e}_\theta\frac{\partial p_{\theta r}}{\partial r}+\boldsymbol{e}_\varphi\frac{\partial p_{\varphi r}}{\partial r}+\boldsymbol{e}_r\frac{\partial p_{\theta r}}{r\partial\theta}+p_{\theta r}\frac{\boldsymbol{e}_\theta}{r}+\boldsymbol{e}_\theta\frac{\partial p_{\theta\theta}}{r\partial\theta}-\frac{\boldsymbol{e}_r}{r}p_{\theta\theta}+\boldsymbol{e}_\varphi\frac{\partial p_{\theta\varphi}}{r\partial\theta}+$$

$$\boldsymbol{e}_r\frac{\partial p_{\varphi r}}{r\sin\theta\partial\varphi}+p_{\varphi r}\frac{\boldsymbol{e}_\varphi}{r}+\boldsymbol{e}_\theta\frac{\partial p_{\varphi\theta}}{r\sin\theta\partial\varphi}+\boldsymbol{e}_\varphi\cot\theta\frac{1}{r}p_{\varphi\theta}+\boldsymbol{e}_\varphi\frac{\partial p_{\varphi\varphi}}{r\sin\theta\partial\varphi}-(\boldsymbol{e}_r+\boldsymbol{e}_\theta\cot\theta)\frac{1}{r}p_{\varphi\varphi}\Big] \tag{12-2-4}'$$

$$\frac{\partial \boldsymbol{V}}{\partial t}+u_r\Big[\frac{\partial}{\partial r}(u_r\boldsymbol{e}_r+v_\theta\boldsymbol{e}_\theta+w_\varphi\boldsymbol{e}_\varphi)\Big]+v_\theta\Big[\frac{\partial}{r\partial\theta}(u_r\boldsymbol{e}_r+v_\theta\boldsymbol{e}_\theta+w_\varphi\boldsymbol{e}_\varphi)\Big]+w_\varphi\Big[\frac{\partial}{r\sin\theta\partial\varphi}(u_r\boldsymbol{e}_r+v_\theta\boldsymbol{e}_\theta+w_\varphi\boldsymbol{e}_\varphi)\Big]$$

$$=\boldsymbol{f}+\frac{1}{\rho}\Big\{\frac{\partial}{\partial r}\Big[-p-\frac{2}{3}\mu\Big(2\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+\frac{\partial v_\theta}{r\partial\theta}+\cot\theta\frac{v_\theta}{r}+\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\Big)+2\mu\frac{\partial u_r}{\partial r}\Big]+\frac{\partial}{\partial r}\mu\Big(\frac{\partial u_r}{r\partial\theta}+$$

$$\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\Big)+\frac{\partial\mu}{\partial r}\Big(\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{\partial r}-\frac{w_\varphi}{r}\Big)+\frac{1}{r}\mu\Big(\frac{\partial u_r}{r\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\Big)+\frac{\partial}{r\partial\theta}\Big[-p-\frac{2}{3}\mu\cdot$$

$$\Big(2\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+\frac{\partial v_\theta}{r\partial\theta}+\cot\theta\frac{v_\theta}{r}+\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\Big)+2\mu\Big(\frac{\partial v_\theta}{r\partial\theta}+\frac{u_r}{r}\Big)\Big]-\frac{1}{r}\Big[-p-\frac{2}{3}\mu\Big(2\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+$$

$$\frac{\partial v_\theta}{r\partial\theta}+\cot\theta\frac{v_\theta}{r}+\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\Big)+2\mu\Big(\frac{\partial v_\theta}{r\partial\theta}+\frac{u_r}{r}\Big)\Big]+\boldsymbol{e}_\varphi\frac{\partial\mu}{r\partial\theta}\Big(\frac{\partial w_\varphi}{r\partial\theta}+\frac{\partial v_\theta}{r\sin\theta\partial\varphi}-\cot\theta\frac{w_\varphi}{r}\Big)+$$

$$\frac{\partial\mu}{r\sin\theta\partial\varphi}\Big(\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{\partial r}-\frac{w_\varphi}{r}\Big)+\frac{1}{r}\mu\Big(\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{\partial r}-\frac{w_\varphi}{r}\Big)+\frac{\partial\mu}{r\sin\theta\partial\varphi}\Big(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+$$

$$\frac{\partial w_\varphi}{r\partial\theta}-w_\varphi\frac{\cot\theta}{r}\Big)+\frac{\cot\theta}{r}\Big(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{r\partial\theta}-w_\varphi\frac{\cot\theta}{r}\Big)+\frac{\partial}{r\sin\theta\partial\varphi}\Big[-p-\frac{2}{3}\mu\Big(2\frac{u_r}{r}+$$

$$\frac{\partial u_r}{\partial r}+\frac{\partial v_\theta}{r\partial\theta}+\cot\theta\frac{v_\theta}{r}+\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\Big)+2\mu\Big(\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}+\frac{u_r}{r}+\cot\theta\frac{v_\theta}{r}\Big)\Big]-(1+\cot\theta)\frac{1}{r}\Big[-p-\frac{2}{3}\mu\cdot$$

$$\Big(2\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+\frac{\partial v_\theta}{r\partial\theta}+\cot\theta\frac{v_\theta}{r}+\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}\Big)+2\mu\Big(\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}+\frac{u_r}{r}+\cot\theta\frac{v_\theta}{r}\Big)\Big]\Big\} \tag{12-4-4}''$$

将式(12-4-4)″按 $\boldsymbol{e}_r$，$\boldsymbol{e}_\theta$，$\boldsymbol{e}_\varphi$ 三个方向列出：

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+v_\theta\Big(\frac{\partial u_r}{r\partial\theta}+\frac{u_r}{r}\Big)+w_\varphi\Big(\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{u_r}{r}\Big)$$

$$=f_r-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\Big\{\frac{4}{3}\frac{\partial^2u_r}{\partial r^2}+\frac{\partial^2u_r}{r^2\partial\theta^2}+\frac{\partial^2u_r}{r^2\sin^2\theta\partial\varphi^2}+\frac{1}{3}\Big(\frac{\partial^2u_r}{r\partial\theta\partial r}+\frac{\partial^2u_r}{r^2\sin\theta\partial\varphi}\Big)+\frac{5}{3}\Big(\frac{\partial u_r}{r^2\partial\theta}+$$

$$\frac{\partial u_r}{r^2\sin\theta\partial\varphi}\Big)+\Big[\frac{2}{3}(1+\cot\theta)-2\Big]\frac{\partial u_r}{r\partial r}+\Big[\frac{4}{3}(1+\cot\theta)-\frac{8}{3}\Big]\frac{u_r}{r^2}\Big\} \tag{12-4-5}$$

$$\frac{\partial v_\theta}{\partial t}+u_r\frac{\partial v_\theta}{\partial r}+v_\theta\Big(\frac{\partial v_\theta}{r\partial\theta}-\frac{v_\theta}{r}\Big)+w_\varphi\Big(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+\frac{\cot\theta}{r}v_\theta\Big)$$

$$=f_\theta-\frac{\partial p}{\rho r\partial\theta}+\nu\Big[\frac{\partial^2v_\theta}{\partial r^2}+\frac{4}{3}\frac{\partial^2v_\theta}{r^2\partial\theta^2}+\frac{\partial^2v_\theta}{r^2\sin^2\theta\partial\varphi^2}+\frac{1}{3}\Big(\frac{\partial^2v_\theta}{r\partial\theta\partial r}+\frac{\partial^2v_\theta}{r^2\sin\theta\partial\varphi\partial\theta}\Big)+\Big(3+\frac{4}{3}\cot\theta\Big)\cdot$$

$$\left.\frac{\partial v_\theta}{r^2\partial\theta}-\cot\theta\frac{\partial v_\theta}{r\partial r}+\cot^2\theta\frac{v_\theta}{r^2}\right] \tag{12-4-6}$$

$$\frac{\partial w_\varphi}{\partial t}+u_r\frac{\partial w_\varphi}{\partial r}+v_\theta\frac{\partial w_\varphi}{r\partial\theta}+w_\varphi\left[\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}-(1+\cot\theta)\frac{w_\varphi}{r}\right]$$

$$=f_\varphi-\frac{\partial p}{\rho r\sin\theta\partial\varphi}+\nu\left[\frac{\partial^2 w_\varphi}{\partial r^2}+\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+\frac{4}{3}\frac{\partial^2 w_\varphi}{r^2\sin^2\theta\partial\varphi^2}+\frac{1}{3}\left(\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}\right)\right.$$

$$\left.-\frac{7}{3}(1+\cot\theta)\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{r\partial r}+(\csc^2\theta-\cot\theta)\frac{w_\varphi}{r^2}\right] \tag{12-4-7}$$

式(12-4-5)至式(12-4-7)是球坐标系下可压缩层流运动动量微分方程。当定常流时，去掉含$\frac{\partial}{\partial t}$项。

12.4.3 动量矩微分方程

由式(3-7-3)：

$$\varphi\frac{\mathrm{D}(\boldsymbol{r}\times\rho_s\boldsymbol{V}_s)}{\mathrm{D}t}+(\boldsymbol{r}\times\rho_s\boldsymbol{V}_s)\nabla\cdot\boldsymbol{V}_s+\varphi_1\frac{\mathrm{D}(\boldsymbol{r}\times\rho\boldsymbol{V})}{\mathrm{D}t}+(\boldsymbol{r}\times\rho\boldsymbol{V})\nabla\cdot\boldsymbol{V}$$

$$=\boldsymbol{r}\times[\varphi\rho_s+(1-\varphi)\rho]+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P})$$

当层流时，$\boldsymbol{V}_s=0$，$\varphi=0$，$\varphi_1=1-\varphi$，$\rho_s=0$，则上式为

$$\frac{\mathrm{D}(\boldsymbol{r}\times\rho\boldsymbol{V})}{\mathrm{D}t}+(\boldsymbol{r}\times\rho\boldsymbol{V})\nabla\cdot\boldsymbol{V}=\boldsymbol{r}\times\boldsymbol{f}\rho+\nabla\cdot(\boldsymbol{r}\times\boldsymbol{P}) \tag{12-4-8}$$

将式(12-4-1)代入式(12-4-8)，并除以 ρ，则

$$\boldsymbol{r}\times\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}=\boldsymbol{r}\times\boldsymbol{f}+\boldsymbol{r}\times\frac{1}{\rho}\nabla\cdot\boldsymbol{P} \tag{12-4-9}$$

利用式(12-4-5)至式(12-4-7)的成果，可得

$$r_\theta\frac{\partial w_\varphi}{\partial t}-r_\varphi\frac{\partial v_\theta}{\partial t}+r_\theta\left\{u_r\frac{\partial w_\varphi}{\partial r}+v_\theta\frac{\partial w_\varphi}{r\partial\theta}+w_\varphi\left[\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}-(1+\cot\theta)\frac{w_\varphi}{r}\right]\right\}+r_\varphi\left[u_r\frac{\partial v_\theta}{\partial r}+\right.$$

$$\left.v_\theta\left(\frac{\partial v_\theta}{r\partial\theta}-\frac{v_\theta}{r}\right)+w_\varphi\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi}+\frac{\cot\theta}{r}v_\theta\right)\right]$$

$$=r_\theta f_\varphi-r_\varphi f_\theta+\frac{1}{\rho}\left(r_\varphi\frac{\partial p}{r\partial\theta}-r_\theta\frac{\partial p}{r\sin\theta\partial\varphi}\right)+\nu\left\{r_\theta\left[\frac{\partial^2 w_\varphi}{\partial r^2}+\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+\frac{4}{3}\frac{\partial^2 w_\varphi}{r^2\sin^2\theta\partial\varphi^2}+\frac{1}{3}\left(\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\right.\right.\right.$$

$$\left.\left.\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}\right)-\frac{7}{3}(1+\cot\theta)\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{r\partial r}+(\csc^2\theta-\cot\theta)\frac{w_\varphi}{r^2}\right]-r_\varphi\left[\frac{\partial^2 v_\theta}{\partial r^2}+\frac{4}{3}\frac{\partial^2 v_\theta}{r^2\partial\theta^2}+\right.$$

$$\left.\left.\frac{\partial^2 v_\theta}{r^2\sin^2\theta\partial\varphi^2}+\frac{1}{3}\left(\frac{\partial^2 v_\theta}{r\partial\theta\partial r}+\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial\theta}\right)+\left(3+\frac{4}{3}\cot\theta\right)\frac{\partial v_\theta}{r^2\partial\theta}-\cot\theta\frac{\partial v_\theta}{r\partial r}+\cot^2\theta\frac{v_\theta}{r^2}\right]\right\}$$

$$\tag{12-4-10}$$

$$r_\varphi\frac{\partial u_r}{\partial t}-r_r\frac{\partial w_\varphi}{\partial t}+r_\varphi\left[u_r\frac{\partial u_r}{\partial r}+v_\theta\left(\frac{\partial u_r}{r\partial\theta}+\frac{u_r}{r}\right)+w_\varphi\left(\frac{\partial u_r}{r\sin\theta\partial\varphi}+\frac{u_r}{r}\right)\right]-r_r\left\{u_r\frac{\partial w_\varphi}{\partial r}+\right.$$

$$\left.v_\theta\frac{\partial w_\varphi}{r\partial\theta}+w_\varphi\left[\frac{\partial w_\varphi}{r\sin\theta\partial\varphi}-(1+\cot\theta)\frac{w_\varphi}{r}\right]\right\}$$

$$= r_\varphi f_r - r_r f_\varphi + \frac{1}{\rho}\left(\frac{\partial p}{r\sin\theta\partial\varphi} - \frac{\partial p}{\partial r}\right) + \nu\left\{ r_\varphi\left\{\frac{4}{3}\frac{\partial^2 u_r}{\partial r^2} + \frac{\partial^2 u_r}{r^2\partial\theta^2} + \frac{\partial^2 u_r}{r^2\sin^2\theta\partial\varphi^2} + \frac{1}{3}\left(\frac{\partial^2 u_r}{r\partial\theta\partial r} + \frac{\partial^2 u_r}{r^2\sin\theta\partial\varphi}\right) + \frac{5}{3}\left(\frac{\partial u_r}{r^2\partial\theta} + \frac{\partial u_r}{r^2\sin\theta\partial\varphi}\right) + \left[\frac{2}{3}(1+\cot\theta) - 2\right]\frac{\partial u_r}{r\partial r} + \left[\frac{4}{3}(1+\cot\theta) - \frac{8}{3}\right]\frac{u_r}{r^2}\right\} - r_r\left[\frac{\partial^2 w_\varphi}{\partial r^2} + \frac{\partial^2 w_\varphi}{r^2\partial\theta^2} + \frac{4}{3}\frac{\partial^2 w_\varphi}{r^2\sin^2\theta\partial\varphi^2} + \frac{1}{3}\left(\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r} + \frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}\right) - \frac{7}{3}(1+\cot\theta)\cdot \frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi} + \frac{\partial w_\varphi}{r\partial r} + (\csc^2\theta - \cot\theta)\frac{w_\varphi}{r^2}\right]\right\} \tag{12-4-11}$$

$$r_r\frac{\partial v_\theta}{\partial t} - r_\theta\frac{\partial u_r}{\partial t} + r_r\left[u_r\frac{\partial v_\theta}{\partial r} + v_\theta\left(\frac{\partial v_\theta}{r\partial\theta} - \frac{v_\theta}{r}\right) + w_\varphi\left(\frac{\partial v_\theta}{r\sin\theta\partial\varphi} + \frac{\cot\theta}{r}v_\theta\right)\right] - r_\theta\left[u_r\frac{\partial u_r}{\partial r} + v_\theta\left(\frac{\partial u_r}{r\partial\theta} + \frac{u_r}{r}\right) + w_\varphi\left(\frac{\partial u_r}{r\sin\theta\partial\varphi} + \frac{u_r}{r}\right)\right]$$

$$= r_r f_\theta - r_\theta f_r + \frac{1}{\rho}\left(\frac{\partial p}{\partial r} - \frac{\partial p}{r\partial\theta}\right) + \nu\left\{ r_r\left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{4}{3}\frac{\partial^2 v_\theta}{r^2\partial\theta^2} + \frac{\partial^2 v_\theta}{r^2\sin^2\theta\partial\varphi^2} + \frac{1}{3}\left(\frac{\partial^2 v_\theta}{r\partial\theta\partial r} + \frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial\theta}\right) + \left(3 + \frac{4}{3}\cot\theta\right)\frac{\partial v_\theta}{r^2\partial\theta} - \cot\theta\frac{\partial v_\theta}{r\partial r} + \cot^2\theta\frac{v_\theta}{r^2}\right] - r_\theta\left\{\frac{4}{3}\frac{\partial^2 u_r}{\partial r^2} + \frac{\partial^2 u_r}{r^2\partial\theta^2} + \frac{\partial^2 u_r}{r^2\sin^2\theta\partial\varphi^2} + \frac{1}{3}\left(\frac{\partial^2 u_r}{r\partial\theta\partial r} + \frac{\partial^2 u_r}{r^2\sin\theta\partial\varphi}\right) + \frac{5}{3}\left(\frac{\partial u_r}{r^2\partial\theta} + \frac{\partial u_r}{r^2\sin\theta\partial\varphi}\right) + \left[\frac{2}{3}(1+\cot\theta) - 2\right]\frac{\partial u_r}{r\partial r} + \left[\frac{4}{3}(1+\cot\theta) - \frac{8}{3}\right]\frac{u_r}{r^2}\right\}\right\} \tag{12-4-12}$$

式(12-4-10)至式(12-4-12)是球坐标系下层流可压缩运动动量矩微分方程。当定常流时，去掉含$\frac{\partial}{\partial t}$项。

12.4.4 能量微分方程

由式(3-7-4)：

$$\varphi\frac{D}{Dt}\left[\rho_s\left(e_s + \frac{v_s^2}{2}\right)\right] + \rho_s\left(e_s + \frac{v_s^2}{2}\right)\nabla\cdot\boldsymbol{V}_s + \varphi_1\frac{D}{Dt}\left[\rho\left(e + \frac{v^2}{2}\right)\right] + \rho\left(e + \frac{v^2}{2}\right)\nabla\cdot\boldsymbol{V}$$

$$= \nabla\cdot(\lambda\nabla T) + q_R\rho + \boldsymbol{f}(\varphi\rho_s\boldsymbol{V}_s + \varphi_1\rho\boldsymbol{V}) + \nabla\cdot\boldsymbol{P}\cdot(\varphi^{\frac{2}{3}}\boldsymbol{V}_s + \varphi_2\boldsymbol{V})$$

当层流时，$\rho_s = 0$，$\varphi = 0$，$\varphi^{\frac{2}{3}} = 0$，$v_s = 0$，$\varphi_1 = 1 - \varphi$，代入式(3-7-4)，则有

$$\frac{D}{Dt}\left[\rho\left(e + \frac{v^2}{2}\right)\right] + \rho\left(e + \frac{v^2}{2}\right)\nabla\cdot\boldsymbol{V} = \nabla\cdot(\lambda\nabla T) + \rho q_R + \boldsymbol{f}\rho\boldsymbol{V} + \nabla\cdot\boldsymbol{P}\boldsymbol{V} \tag{12-4-13}$$

将式(12-4-1)代入式(12-4-13)，并除以 ρ，则有

$$\frac{D}{Dt}\left(e + \frac{v^2}{2}\right) = \frac{1}{\rho}\nabla\cdot(\lambda\nabla T) + q_R + \boldsymbol{f}\boldsymbol{V} + \frac{1}{\rho}\nabla\cdot\boldsymbol{P}\boldsymbol{V} \tag{12-4-14}$$

将分子能量 $e = C_pT$ 代入，并利用球坐标系可压缩层流运动动量微分方程式(12-4-5)至式(12-4-7)，则得

$$\frac{D}{Dt}\left(C_pT + \frac{v^2}{2}\right) = \frac{\lambda}{\rho}\left(\frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{r^2\partial\theta^2} + \frac{\partial^2 T}{r^2\sin^2\theta\partial\varphi^2}\right) + q_R + f_r u_r + f_\theta v_\theta + f_\varphi w_\varphi - \frac{1}{\rho}\left(\frac{\partial p}{\partial r}u_r + \right.$$

$$\frac{\partial p}{r\partial\theta}v_\theta+\frac{\partial p}{r\sin\theta\partial\varphi}w_\varphi\Big)+\nu\Big\{\Big\{\frac{4}{3}\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial^2 u_r}{r^2\partial\theta^2}+\frac{\partial^2 u_r}{r^2\sin^2\theta\partial\varphi^2}+\frac{1}{3}\Big(\frac{\partial^2 u_r}{r\partial\theta\partial r}+$$

$$\frac{\partial^2 u_r}{r^2\sin\theta\partial\varphi}\Big)+\frac{5}{3}\Big(\frac{\partial u_r}{r^2\partial\theta}+\frac{\partial u_r}{r^2\sin\theta\partial\varphi}\Big)+\Big[\frac{2}{3}(1+\cot\theta)-2\Big]\frac{\partial u_r}{r\partial r}+\Big[\frac{4}{3}(1+$$

$$\cot\theta)-\frac{8}{3}\Big]\frac{\partial u_r}{r^2}\Big\}u_r+\Big[\frac{\partial^2 v_\theta}{\partial r^2}+\frac{4}{3}\frac{\partial^2 v_\theta}{r^2\partial\theta^2}+\frac{\partial^2 v_\theta}{r^2\sin^2\theta\partial\varphi^2}+\frac{1}{3}\Big(\frac{\partial^2 v_\theta}{r\partial\theta\partial r}+$$

$$\frac{\partial^2 v_\theta}{r^2\sin\theta\partial\varphi\partial\theta}\Big)+\Big(3+\frac{4}{3}\cot\theta\Big)\frac{\partial v_\theta}{r^2\partial\theta}-\cot\theta\frac{\partial v_\theta}{r\partial r}+\cot^2\theta\frac{v_\theta}{r^2}\Big]v_\theta+\Big[\frac{\partial^2 w_\varphi}{\partial r^2}+$$

$$\frac{\partial^2 w_\varphi}{r^2\partial\theta^2}+\frac{4}{3}\frac{\partial^2 w_\varphi}{r^2\sin^2\theta\partial\varphi^2}+\frac{1}{3}\Big(\frac{\partial^2 w_\varphi}{r\sin\theta\partial\varphi\partial r}+\frac{\partial^2 w_\varphi}{r^2\sin\theta\partial\varphi\partial\theta}\Big)-\frac{2}{3}(1+\cot\theta)\cdot$$

$$\frac{\partial w_\varphi}{r^2\sin\theta\partial\varphi}+\frac{\partial w_\varphi}{r\partial r}+(\csc^2\theta-\cot\theta)\frac{w_\varphi}{r^2}\Big]w_\varphi\Big\}\tag{12-4-15}$$

式(12-4-15)是球坐标系可压缩层流运动能量微分方程。当定常流时，去掉含$\frac{\partial}{\partial t}$项。球坐标系中，$r_r=r$，$r_\theta=r\sin\theta$，$r_\varphi=(r\sin\theta)\varphi$。具体计算时应注意。

第 13 章　平面近壁流

一提到平板近壁流，人们定会想到附面层，它是研究一维流动、二维变化的，而近壁流是研究一维流动、一维变化的。附面层研究的尺寸比较小，如以平板层流附层面的厚度 δ 为例，其公式为

$$\delta \approx 5.0\frac{x}{\sqrt{Re}}$$

若 x 很长，则 δ 也很大，则失去薄层的意义。附面层湍流问题至今未得到合理的解决。而平板近壁，无论层流与湍流均能得到较好的解决。附面层与近壁流共同的特点是在流场中均存在着理想流体运动区。

本章研究的问题是有压管道直径很大，壁面影响达不到轴线，也就是说，边层流界面产生的涡旋在未达到管轴线之前就消失了，使得管中心附近存在着理想流体运动区。在明渠水很深、很宽的情况下，渠底、侧壁边层流界面上产生的涡旋均未达到中心及水面，使得明渠近水面附近存在着理想流体运动区。这样，整个流场分为理想流体运动与实际流体运动区。将实际流体运动区称为近壁流。

第 5 章有压管道与第 6 章明渠的研究，它的前提是整个断面流体运动均为实际流体，不存在理想流体，所以也就没有近壁流的提法。在那里研究的是管道与明渠都是充分发展的，没有进口段问题。同样，本章讨论的问题也是充分发展的流动，不涉及进口段问题。

13.1　平板近壁流流态判别准数

雷诺数 Re 是判别实际流体运动是层流还是湍流的准数。近壁是实际流体运动，当然可用雷诺数判别其流态。具体判别公式如下。

矩形明渠、梯形明渠、大直径管道，它们的近壁流态判别准数均可共用

$$Re=\frac{v'_0\delta}{\nu} \tag{13-1-1}$$

式中：v'_0——近壁流层的断面平均速度；

δ——近壁流层厚度。

δ 实际是水力半径。梯形明渠、矩形明渠、管道三种情况下的近壁流水力半径均为 δ。如图 13-1 所示。

根据水力半径的定义

$$R=\frac{\omega}{\chi} \tag{13-1-2}$$

式中：ω——实际流体通过的断面；

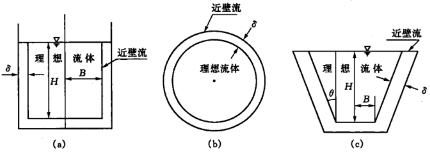


图 13-1　平板近壁流示意图

χ——实际流体固体接触的周长。

对圆形：

$$R=\frac{\delta\pi D}{\pi D}=\delta \tag{13-1-3}$$

对矩形明渠：

$$R=\frac{\delta(H+B)}{H+B+\delta}=\delta \tag{13-1-4}$$

对梯形明渠：

$$R=\frac{2\delta H\sqrt{1+\tan\theta}+\delta(2B+\delta\cot\theta)}{2(\delta\cot\theta+B+\sqrt{1+\tan^{2}\theta})} \tag{13-1-5}$$

综上所述，三种平板近壁，其层流与湍流判别条件为

$$\frac{v'_0R}{\nu}>580\quad(\text{湍流}) \tag{13-1-6}$$

$$\frac{v'_0R}{\nu}<580\quad(\text{层流}) \tag{13-1-7}$$

13.2　明渠层流近壁流

只要明渠流动为层流，无论是渠底和侧壁，其近壁流层流控制微分方程均相同。如图 13-2 所示。

$$g\sin\alpha=\nu\frac{\mathrm{d}u^{2}}{\mathrm{d}z^{2}} \tag{13-2-1}$$

边界条件

$$u\big|_{z=0}=0 \tag{13-2-2}$$

$$\left.\frac{\mathrm{d}u}{\mathrm{d}z}\right|_{z=\delta_0}=0 \tag{13-2-3}$$

这是一组有未知边界条件式(13-2-3)的特殊二阶线性常微分方程。用一般方法无法求解它。

为此，从无因次分析入手，运用“逆算法”求解之。令

$$\frac{z}{\delta_0}=Z,\qquad \frac{u}{u_0}=U$$

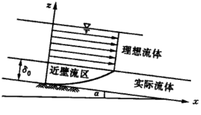


图 13-2　近壁层流示意图

将它们代入方程组后，得

$$K=\frac{\mathrm{d}^2U}{\mathrm{d}Z^2} \tag{13-2-4}$$

$$u\big|_{Z=0}=0 \tag{13-2-5}$$

$$\left.\frac{\mathrm{d}U}{\mathrm{d}Z}\right|_{Z=1}=0 \tag{13-2-6}$$

$$\frac{\mathrm{d}^2U}{\mathrm{d}Z^2}<0 \tag{13-2-7}$$

式中：

$$K=\frac{g\sin\alpha\delta_0^2}{\nu u_0} \tag{13-2-8}$$

式(13-2-4)至式(13-2-8)是平板层流近壁流无因次控制运动方程组。依它们确定其断面速度分布如下。

选形函数：

$$U'=2Z-Z^2 \tag{13-2-9}$$

写出待定函数：

$$U=\eta U'=\eta(2Z-Z^2) \tag{13-2-10}$$

将式(13-2-10)代入式(13-2-4)，确定其参变常数 η：

$$\eta=\frac{g\sin\alpha(2Z-Z^2)}{2\nu u_0} \tag{13-2-11}$$

将式(13-2-11)代入式(13-2-10)得无因次速度分布

$$U=\frac{g\sin\alpha\delta_0^2}{2\nu u_0}(2Z-Z^2) \tag{13-2-12}$$

有因次速度分布

$$u=\frac{g\sin\alpha}{2\nu}(2\delta_0Z-Z^2) \tag{13-2-13}$$

确定近壁流厚度 δ_0。由式(13-2-12)，当 $Y=1$ 时，$U=1$，则得

$$\delta_0=\sqrt{\frac{2\nu u_0}{g\sin\alpha}} \tag{13-2-14}$$

确定通过层流近壁断面 1m 宽的流量

$$Q=\int_0^{\delta_0}u\,\mathrm{d}z=\int_0^{\delta_0}\frac{g\sin\alpha}{2\nu}(2\delta_0z-z^2)\mathrm{d}z=\frac{g\sin\alpha}{2\nu}\left(\delta_0^3-\frac{\delta_0^3}{3}\right)=\frac{g\sin\alpha}{3\nu}\delta_0^3 \tag{13-2-15}$$

确定近壁流断面平均速度

$$v'_0=\frac{Q}{\delta_0}=\frac{g\sin\alpha\delta_0^2}{3\nu} \tag{13-2-16}$$

确定近壁流内边层流界面位置 y_B。将 v'_0 代入式(13-2-13)

$$\frac{g\sin\alpha\delta_0^2}{3\nu}=\frac{g\sin\alpha}{2\nu}(2\delta_0 z_B-z_B^2)$$

$$\frac{2}{3}\delta_0^2=2\delta_0 z_B-z_B^2$$

$$z_B^2=2\delta_0 z_B+\frac{2}{3}\delta_0^2=0$$

$$z_B=\frac{2\delta_0\pm\sqrt{(2\delta_0)^2-4\times\frac{2}{3}\delta_0^2}}{2}$$

$$z_B=\delta_0\left(1\pm\sqrt{\frac{1}{3}}\right)=\delta_0(1-0.577)$$

$$z_B=0.423\delta_0 \tag{13-2-17}$$

将式(13-2-14)代入式(13-2-17)

$$z_B=0.423\sqrt{\frac{2\nu u_0}{g\sin\alpha}} \tag{13-2-18}$$

式(13-2-18)与式(13-2-17)说明，知道近壁流厚度，就可以知道边层流的位置，即厚度。而近壁流厚度 δ_0 是可以计算出来的。如图 13-3 所示。

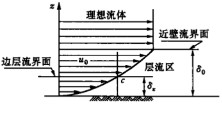


图 13-3　近壁流中边层流示意图

13.3　明渠湍流近壁流

明渠湍流近壁流如图 13-4 所示，其控制微分方程可由式(4-4-11)，结合其流动情况，简化为

$$g\sin\alpha=\nu(1-\varphi^{\frac{2}{3}})\frac{d^2u}{dz^2}+\varphi^{\frac{2}{3}}6\nu t\frac{du}{dz}\frac{d^2u}{dz^2} \tag{13-3-1}$$

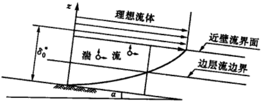


图 13-4　平板湍流近壁流示意图

边界条件

$$u\big|_{z=0}=0 \tag{13-3-2}$$

$$\left.\frac{\mathrm{d}u}{\mathrm{d}z}\right|_{y=\delta_0^*}=0 \tag{13-3-3}$$

将方程组无因次化，令

$$\frac{u}{u_0}=U,\qquad \frac{z}{\delta_0^*}=Z$$

式中：u_0——明渠理想流体速度；

δ_0^*——明渠湍流近壁流厚度。

$$K_1=\left[(1-\varphi^{\frac{2}{3}})+K_2\varphi^{\frac{2}{3}}\frac{\mathrm{d}U}{\mathrm{d}Z}\right]\frac{\mathrm{d}^2U}{\mathrm{d}Z^2} \tag{13-3-4}$$

边界条件

$$U|_{Z=0}=0 \tag{13-3-5}$$

$$\left.\frac{\mathrm{d}U}{\mathrm{d}Z}\right|_{Z=1}=0 \tag{13-3-6}$$

式中

$$K_1=\frac{g\sin\alpha\delta_0^{*2}}{\nu u_0} \tag{13-3-7}$$

$$K_2=\frac{6tu_0}{\delta_0^*} \tag{13-3-8}$$

式(13-3-4)至式(13-3-6)是一组明渠近壁湍流运动控制方程。它是非线性二阶常微分方程。

确定近壁湍流断面速度分布。根据图13-4，明渠近壁湍流速度分布的特点，选形函数U'为

$$U'=2Z-Z^2 \tag{a}$$

则所求函数U为

$$U=\eta U'=\eta(2Z-Z^2) \tag{b}$$

将式(b)代入式(13-3-4)，整理为

$$K_1=2(1-\varphi^{\frac{2}{3}})\eta+4\varphi^{\frac{2}{3}}K_2(1-Z)\eta^2 \tag{c}$$

取近似解

$$\eta=\frac{K_1}{2(1-\varphi^{\frac{2}{3}})} \tag{d}$$

将式(d)代入式(b)，得

$$U=\frac{K_1(2Z-Z^2)}{2(1-\varphi^{\frac{2}{3}})} \tag{13-3-9}$$

将无因次速度转回有因次速度

$$u=\frac{\gamma\sin\alpha(2\delta_0^*Z-Z^2)}{2\mu(1-\varphi^{\frac{2}{3}})} \tag{13-3-10}$$

确定近壁流厚度 δ_0^*。当 $u=u_0$，$z=\delta_0^*$，代入式(13-3-10)，则有

$$\delta_0^*=\sqrt{\frac{2\mu(1-\varphi^{\frac{2}{3}})u_0}{\gamma\sin\alpha}} \tag{13-3-11}$$

确定近壁流区通过的流量 Q：

$$Q=\int_0^{\delta_0^*}\frac{\gamma\sin\alpha(2\delta_0^* z-z^2)}{2\mu(1-\varphi^{\frac{2}{3}})}\mathrm{d}z=\frac{\gamma\sin\alpha\delta_0^{*3}}{3\mu(1-\varphi^{\frac{2}{3}})} \tag{13-3-12}$$

确定近壁流断面平均速度 v_0^*：

$$v_0^*=\frac{\gamma\sin\alpha\delta_0^{*3}}{1\cdot\delta_0^*\cdot 3\mu(1-\varphi^{\frac{2}{3}})}=\frac{\gamma\sin\alpha\delta_0^{*2}}{3\mu(1-\varphi^{\frac{2}{3}})} \tag{13-3-13}$$

确定平均速度 v_0^* 与理想流体速度 u_0 的关系

$$v_0^*=\frac{\gamma\sin\alpha\delta_0^{*2}}{3\mu(1-\varphi^{\frac{2}{3}})}=\frac{\gamma\sin\alpha}{3\mu(1-\varphi^{\frac{2}{3}})}\cdot\frac{2\mu(1-\varphi^{\frac{2}{3}})u_0}{\gamma\sin\alpha}=\frac{2}{3}u_0 \tag{13-3-14}$$

确定边层流界面位置：将式(13-3-14)代入式(13-3-10)，得

$$\frac{2}{3}u_0=\frac{\gamma\sin\alpha(2\delta_0^* z_B-z_B^2)}{2\mu(1-\varphi^{\frac{2}{3}})}$$

$$\frac{4}{3}u_0\mu(1-\varphi^{\frac{2}{3}})=\gamma\sin\alpha(2\delta_0^* z_B-z_B^2)$$

$$\frac{\frac{4}{3}\mu u_0(1-\varphi^{\frac{2}{3}})}{\gamma\sin\alpha}=2\delta_0^* z_B-z_B^2$$

将式(13-3-11)代入得

$$\frac{\frac{4}{3}\mu u_0(1-\varphi^{\frac{2}{3}})}{\gamma\sin\alpha}=2\sqrt{\frac{2\mu(1-\varphi^{\frac{2}{3}})u_0}{\gamma\sin\alpha}}z_B-z_B^2$$

写为

$$z_B^2-2\sqrt{\frac{2\mu(1-\varphi^{\frac{2}{3}})u_0}{\gamma\sin\alpha}}z_B+\frac{4}{3}\frac{\mu u_0(1-\varphi^{\frac{2}{3}})}{\gamma\sin\alpha}=0$$

取其近似解为

$$z_B=\frac{2}{3}\sqrt{\frac{\mu u_0(1-\varphi^{\frac{2}{3}})}{2\gamma\sin\alpha}} \tag{13-3-15}$$

式(13-3-15)就是明渠近壁湍流时，其中边层流界面位置 z_B。

确定明渠近壁湍流边层流界面上产生的涡旋强度与大小，根据涡旋定义式，利用式(13-3-10)，有

$$\omega_y^0\Big|_{z=z_B}=\frac{1}{2}\frac{\partial u}{\partial z}\Big|_{z=z_B}=\frac{1}{2}\left\{\frac{\gamma\sin\alpha(2\delta_0^*-2z_B)}{2\mu(1-\varphi^{\frac{2}{3}})}\right\}=\frac{\gamma\sin\alpha}{2\mu(1-\varphi^{\frac{2}{3}})}(\delta_0^*-z_B)$$

$$=\frac{\gamma\sin\alpha}{2\mu(1-\varphi^{\frac{2}{3}})}\sqrt{\frac{2\mu(1-\varphi^{\frac{2}{3}})u}{\gamma\sin\alpha}}-\frac{2}{3}\sqrt{\frac{\mu u_0(1-\varphi^{\frac{2}{3}})}{2\gamma\sin\alpha}}=\frac{\sqrt{2}}{3}\sqrt{\frac{\gamma\sin\alpha u_0}{\mu(1-\varphi^{\frac{2}{3}})}} \tag{13-3-16}$$

涡旋直径 d_s：

$$d_s\big|_{z=z_B}=2\sqrt{\frac{10\nu}{\omega\big|_{z=z_B}}}=2\left[\frac{30\nu(\mu(1-\varphi^{\frac{2}{3}}))^{\frac{1}{2}}}{\sqrt{2}(\gamma\sin\alpha)^{\frac{1}{2}}}\right]^{\frac{1}{2}} \tag{13-3-17}$$

确定涡旋在近壁湍流内边层流界面上涡旋横向运动分速度 $u^*\big|_{z=z_B}$，由式(4-3-29)

$$u_{zy}^{*0}=3\nu t\left(\frac{\partial u}{\partial z}\right)^2u^{-1} \tag{13-3-18}$$

将式(13-3-16)与式(13-3-14)代入

$$u_{zy}^{*0}\big|_{z=z_B}=3\nu t\left(\frac{2}{3}\sqrt{\frac{\gamma\sin\alpha u_0}{\mu(1-\varphi^{\frac{2}{3}})}}\right)^2v_0^{*-1}=2\nu t\frac{\gamma\sin\alpha}{\mu(1-\varphi^{\frac{2}{3}})}=\frac{2\nu t\gamma\sin\alpha}{\mu(1-\varphi^{\frac{2}{3}})} \tag{13-3-19}$$

式中：ν——流体运动黏性系数；

γ——流体相对密度；

μ——流体动力黏性系数；

t——准定常流时间。

确定明渠近壁湍流涡旋体积分数 φ。依据涡旋体积分数的定义，结合近壁湍流特点，其表达式为

$$\varphi=\frac{\text{涡旋体积}}{\text{流量}}=\frac{\eta f\frac{\pi}{6}d_s^3}{\text{米}\cdot\delta_0^*\cdot v_0^*} \tag{13-3-20}$$

式中，f 为涡旋产生频率；η 按 4.2 节规定，结合现在情况

$$\eta=\frac{v_0^*\cdot 1\cdot\text{米}}{4d_s^2} \tag{13-3-21}$$

将 f，η 均代入式(13-3-20)，则有

$$\varphi=\frac{[\text{秒}]\pi u_{zy}^{*0}\big|_{z=z_B}}{48\delta_0^*} \tag{13-3-22}$$

将式(13-3-19)与式(13-3-11)代入

$$\varphi=\frac{[\text{秒}]\pi\nu t\gamma\sin\alpha}{36\delta_0^*\mu(1-\varphi^{\frac{2}{3}})}=\frac{[\text{秒}]\pi\nu t\gamma\sin\alpha}{36\mu(1-\varphi^{\frac{2}{3}})}\sqrt{\frac{\gamma\sin\alpha}{2\mu(1-\varphi^{\frac{2}{3}})u_0}}=\frac{[\text{秒}]\pi\nu t}{50.91\sqrt{u_0}}\left[\frac{\gamma\sin\alpha}{\mu(1-\varphi^{\frac{2}{3}})}\right]^{\frac{3}{2}} \tag{13-3-23}$$

例 13-1 一梯形明渠如图 13-5 所示，渠底坡 $i=0.0001$，边坡系数 $m=1.5$，粗糙度 $n=0.03$，渠底宽 $b=40\text{m}$，水深 $h_0=2.5\text{m}$。确定其近壁湍流涡旋体积分数 φ，近壁流厚度 δ^*，平均速度 v_0^*，边层流界面位置 z_B，以及产生的涡旋强度 $\omega\big|_{z=z_B}$ 和大小。已知 $\mu=$

$1.14\times10^{-3}\mathrm{N\cdot s/m^2}$，$\nu=1.141\times10^{-6}\mathrm{m^2/s}$。

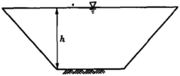


图 13-5 梯形明渠

解 计算前准备工作：断面面积

$A=(b+mh_0)h_0=(40+1.5\times2.5)2.5=109.4\mathrm{m}^2$

湿周

$\chi=b+2h_0\sqrt{1+m^2}=40+2\times2.5\sqrt{1+1.5^2}=49.01\mathrm{m}$

水力半径

$$R=\frac{A}{\chi}=\frac{109.4}{49.01}=2.23\mathrm{m}$$

谢才系数

$$c=\frac{1}{n}R^{\frac{1}{6}}=\frac{1}{0.03}\times2.23^{\frac{1}{6}}=38.1\mathrm{m}^{\frac{1}{2}}/\mathrm{s}$$

断面平均速度

$$v=c\sqrt{Ri}=38.1\times\sqrt{2.23\times0.0001}=0.568\mathrm{m/s}$$

即 $v\approx u_0$。

确定涡旋体积分数 φ。由式(13-3-23)，将它分为两式

$$y_1=\frac{50.91\varphi}{\pi}=16.205\varphi \tag{a}$$

$$y_2=\frac{[秒]1.141\times10^{-6}T_K}{\sqrt{0.568}}\left[\frac{1000\times0.0001}{1.14(1-\varphi^{\frac{2}{3}})}\right]^{\frac{3}{2}}=\frac{0.0393T_K}{(1-\varphi^{\frac{2}{3}})^{\frac{3}{2}}}\approx\frac{0.04T_K}{(1-\varphi^{\frac{2}{3}})^{\frac{3}{2}}} \tag{b}$$

设不同的 φ 值计算 y_1 与 y_2，结果列入表 13-1。

表 13-1

y_1	φ	$1-\varphi^{\frac{2}{3}}$	$(1-\varphi^{\frac{2}{3}})^{\frac{3}{2}}$	$\frac{0.04}{(1-\varphi^{\frac{2}{3}})^{\frac{3}{2}}}$	$y_2(T_K=1\mathrm{s})$
0.016	0.001	0.99	0.985	0.040	0.040
0.032	0.002	0.98	0.972	0.041	0.041
0.081	0.005	0.97	0.955	0.042	0.042
0.162	0.01	0.95	0.932	0.043	0.043
0.810	0.05	0.86	0.803	0.050	0.050

将表 13-1 的结果绘图，如图 13-6，得其交点，即 φ_K，对应的时间即为 T_K，$\varphi_K=0.0025$，$T_K=1\mathrm{s}$。

确定近壁流厚度 δ_0^*。依式(13-3-11)

$$\delta_0^*=\sqrt{\frac{2\mu(1-\varphi^{\frac{2}{3}})u_0}{\gamma\sin\alpha}}=\sqrt{\frac{2\times1.14\times10^{-3}(1-0.0025^{\frac{2}{3}})0.568}{1000\times0.001}}=\sqrt{\frac{12.73}{10^3}}$$

$$=\sqrt{0.01273}=0.113\mathrm{m}$$

确定近壁湍流断面速度分布

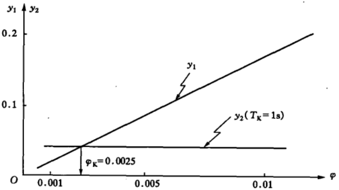


图 13-6　例 13-1 计算结果用图

$$u=\frac{\gamma\sin\alpha\left(\delta_0^* z-z^2\right)}{1.996\mu} \tag{c}$$

确定近壁流断面平均速度 v_0^*，由式(13-3-14)

$$v_0^*=\frac{2}{3}u_0=\frac{2\times0.568}{3}=0.379\text{m/s}$$

确定边层流位置 z_B，依式(13-3-15)

$$z_B=\frac{2}{3}\sqrt{\frac{\mu u_0(1-\varphi^{\frac{2}{3}})}{2\gamma\sin\alpha}}=\frac{2}{3}\sqrt{\frac{1.14\times10^{-3}\times0.568(1-0.0025^{\frac{2}{3}})}{2\times1000\times0.0001}}=\frac{2}{3}\sqrt{\frac{1.295\times0.983}{0.2\times10^3}}$$

$$=\frac{2}{3}\sqrt{0.00637}=0.053\text{m}$$

确定边层流界面上产生的涡旋强度及大小，依式(13-3-16)

$$\omega\big|_{z=z_B}=\frac{2}{3}\sqrt{\frac{\gamma\sin\alpha u_0}{\mu(1-\varphi^{\frac{2}{3}})}}=\frac{2}{3}\sqrt{\frac{1000\times0.0001\times0.568}{1.14\times10^{-3}\times0.983}}=\frac{2}{3}\sqrt{51.967}=4.81\text{rad/s}$$

涡旋直径

$$d_s=2\sqrt{\frac{10\nu}{\omega}}=\frac{2}{10^4}\sqrt{\frac{10\times1.141}{0.048}}=3\text{mm}$$

确定边层流界面上涡旋横向运动速度 u^*。由式(13-3-19)

$$u^*\big|_{z=z_B}=\frac{4}{3}\nu T_K\frac{\gamma\sin\alpha}{\mu(1-\varphi^{\frac{2}{3}})}=\frac{2\times1.141\times10^{-6}\times1\times1000\times0.0001}{1.14\times10^{-3}\times0.983}=\frac{0.2282}{1.121\times10^3}$$

$$=0.204\times10^{-3}=0.000204\text{m/s}=0.2\text{mm/s}$$

验证是否为湍流，依式(13-1-6)

$$Re=\frac{v_0^*\delta_0^*}{\nu}=\frac{0.4\times0.113}{1.141\times10^{-6}}=37500>580$$

确为湍流。

13.4　有压管道平板近壁层流运动

这里讨论的问题，对圆形、矩形即非圆形管道均适用。如图 13-7 所示。其运动控制微分方程由式(4-4-11)并结合它的情况，为

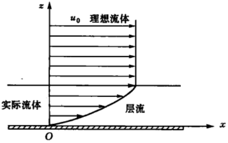


图 13-7　平板近壁层流示意图

$$\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}=\nu\frac{\mathrm{d}^2u}{\mathrm{d}z^2} \tag{13-4-1}$$

$$u\big|_{z=0}=0 \tag{13-4-2}$$

$$\frac{\mathrm{d}u}{\mathrm{d}z}\bigg|_{z=\delta^0}=0 \tag{13-4-3}$$

将式(13-4-1)至式(13-4-3)无因次化，为此取

$$\frac{z}{\delta^0}=Z,\qquad \frac{u}{u_0}=U$$

将它们代入方程组，得

$$K_1=\frac{\mathrm{d}^2U}{\mathrm{d}Z^2} \tag{13-4-4}$$

$$K_1=\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\delta_0^2}{u_0} \tag{13-4-5}$$

$$U\big|_{Z=0}=0 \tag{13-4-6}$$

$$\frac{\mathrm{d}U}{\mathrm{d}Z}\bigg|_{Z=1}=0 \tag{13-4-7}$$

$$\frac{\mathrm{d}^2U}{\mathrm{d}Z^2}<0 \tag{13-4-8}$$

根据图 13-7 所示及边界条件，选形函数 U'

$$U'=2Z-Z^2 \tag{a}$$

待定函数

$$U=\eta(2Z-Z^2) \tag{b}$$

将式(b)代入式(13-2-4)，得

$$\eta = \frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\delta_0^2}{u_0} \tag{c}$$

将式(c)代入式(b)

$$U = \frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\delta_0^2}{u_0}(2Z - Z^2) \tag{13-4-9}$$

将它转回有因次速度分布

$$u = \frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}(2\delta_0 z - z^2) \tag{13-4-10}$$

令式(13-4-10)中 $u = u_0$，则可确定近壁流厚度 δ_0。

13.5 有压管道近壁湍流运动

有压管道近壁湍流运动如图 13-8 所示，其运动控制微分方程由式(4-4-11)结合图 13-8 和边界条件，简化为

$$\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = \nu(1+\varphi^{\frac{2}{3}}) + \varphi^{\frac{2}{3}}6\nu t\left(\frac{\mathrm{d}u}{\mathrm{d}z}\right)\frac{\mathrm{d}^2u}{\mathrm{d}z^2} \tag{13-5-1}$$

$$u\big|_{z=0} = 0 \tag{13-5-2}$$

$$\left.\frac{\mathrm{d}u}{\mathrm{d}z}\right|_{z=\delta_0^*} = 0 \tag{13-5-3}$$

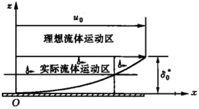


图 13-8 有压管道平板近壁湍流

将式(13-5-1)至式(13-5-3)无因次化，为此取

$$\frac{z}{\delta^{*0}} = Z,\ \frac{u}{u_0} = U$$

将其代入式(13-5-1)，得

$$K_1 = (1-\varphi^{\frac{2}{3}})\frac{\mathrm{d}^2U}{\mathrm{d}Z^2} + K_2\varphi^{\frac{2}{3}}\frac{\mathrm{d}U}{\mathrm{d}Z}\frac{\mathrm{d}^2U}{\mathrm{d}Z^2} \tag{13-5-4}$$

$$U\big|_{Z=0} = 0 \tag{13-5-5}$$

$$\left.\frac{\mathrm{d}U}{\mathrm{d}Z}\right|_{Z=1} = 0 \tag{13-5-6}$$

式中

$$K_1 = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\delta_0^{*2}}{u_0} \tag{d}$$

$$K_2 = \frac{6tu_0}{\delta_0^*} \tag{e}$$

结合图 13-8 和边界条件，选形函数 U' 为

$$U' = 2Z - Z^2 \tag{f}$$

待定函数

$$U = \eta(2Z - Z^2) \tag{g}$$

将式(g)代入式(13-5-4)，有

$$K_1=2(1-\varphi^{\frac{2}{3}})\eta+4\varphi^{\frac{2}{3}}K_2(1-Z)\eta^2 \tag{h}$$

取近似

$$\eta=\frac{K_1}{2(1-\varphi^{\frac{2}{3}})}=\frac{\frac{\mathrm{d}p}{\mathrm{d}x}\delta_0^{*2}}{2\mu(1-\varphi^{\frac{2}{3}})u_0} \tag{i}$$

则

$$U=\frac{\delta_0^{*2}}{2\mu(1-\varphi^{\frac{2}{3}})}\frac{\mathrm{d}p}{\mathrm{d}x}(2Z-Z^2)\frac{1}{u_0} \tag{13-5-7}$$

有因次速度分布

$$u=\frac{\frac{\mathrm{d}p}{\mathrm{d}x}}{2\mu(1-\varphi^{\frac{2}{3}})}(2\delta_0^* z-z^2) \tag{13-5-8}$$

确定近壁流厚度 δ_0^*，为此将 $u=u_0^*$，$z=z_B$ 代入式(13-5-8)

$$\frac{(1-\varphi^{\frac{2}{3}})2\mu u_0^*}{\frac{\mathrm{d}p}{\mathrm{d}x}}=\delta_0^{*2},\quad \delta_0^*=\sqrt{\frac{2\mu u_0(1-\varphi^{\frac{2}{3}})}{\frac{\mathrm{d}p}{\mathrm{d}x}}} \tag{13-5-9}$$

确定近壁流在一米宽通过流量 q

$$q=\int_0^{\delta_0^*}\frac{\frac{\mathrm{d}p}{\mathrm{d}x}}{2\mu(1-\varphi^{\frac{2}{3}})}(2\delta_0^* z-z^2)\mathrm{d}z=\frac{[米]}{3\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\delta_0^{*3}}{1-\varphi^{\frac{2}{3}}} \tag{13-5-10}$$

确定近壁流通过的断面平均速度 v_0^*

$$v_0^*=\frac{q}{\delta^*\cdot[米]}=\frac{1}{3}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\delta_0^{*2}}{\mu(1-\varphi^{\frac{2}{3}})} \tag{13-5-11}$$

确定理想流体速度与平均速度关系，将式(13-5-9)代入式(13-5-11)

$$v_0^*=\frac{2}{3}u_0 \tag{13-5-12}$$

确定近壁流，边层流位置 z_B，为此将式(13-5-9)代入(13-5-12)，并令 $u=v_0^*$，则有

$$\frac{2\mu v_0^*(1-\varphi^{\frac{2}{3}})}{\frac{\mathrm{d}p}{\mathrm{d}x}}=2\sqrt{\frac{2\mu u_0(1-\varphi^{\frac{2}{3}})}{\frac{\mathrm{d}p}{\mathrm{d}x}}}z_B-z_B^2 \tag{j}$$

将式(13-5-12)代入式(j)

$$\frac{4}{3}\frac{\mu u_0(1-\varphi^{\frac{2}{3}})}{\frac{\mathrm{d}p}{\mathrm{d}x}}=2\sqrt{\frac{2\mu u_0(1-\varphi^{\frac{2}{3}})}{\frac{\mathrm{d}p}{\mathrm{d}x}}}z_B-z_B^2 \tag{k}$$

解式(k)

$$z_B=0.423\sqrt{\frac{2\mu u_0(1-\varphi^{\frac{2}{3}})}{\frac{\mathrm{d}p}{\mathrm{d}x}}} \tag{k}$$

将式(13-5-9)代入式(k)

$$z_B=0.423\delta_0^* \tag{13-5-13}$$

确定边层流界面上产生的涡旋强度及大小

$$\omega\big|_{z=z_B}=\frac{1}{2}\frac{\partial u}{\partial z}=\frac{\frac{\mathrm{d}p}{\mathrm{d}x}}{2\mu(1-\varphi^{\frac{2}{3}})}(\delta_0^*-z_B) \tag{l}$$

将式(13-5-13)代入式(l)

$$\omega\big|_{z=z_B}=\frac{\mathrm{d}p}{\mathrm{d}x}\frac{0.577\delta_0^*}{2\mu(1-\varphi^{\frac{2}{3}})} \tag{l$'$}$$

将式(13-5-9)代入式(l)′

$$\omega\big|_{z=z_B}=0.204\sqrt{\frac{\frac{\mathrm{d}p}{\mathrm{d}x}u_0}{\mu(1-\varphi^{\frac{2}{3}})}} \tag{13-5-14}$$

涡旋直径

$$d_s=2\sqrt{\frac{10\nu}{\omega\big|_{z=z_B}}}=14\sqrt{\frac{\nu\mu(1-\varphi^{\frac{2}{3}})}{\frac{\mathrm{d}p}{\mathrm{d}x}u_0}} \tag{13-5-15}$$

确定近壁流内边层流界面上涡旋横向运动速度 u_{zy}^{*0}，由式(4-3-29)并将(13-5-9)，式(13-5-12)，(13-5-13)各式先后代入，则有

$$u_{zy}^{*0}\big|_{z=z_B}=3\nu t\left(\frac{\partial u}{\partial z}\right)^2u^{-1}=3\nu t\left[\frac{\frac{\mathrm{d}p}{\mathrm{d}x}(\delta_0^*-z_B)}{\mu(1-\varphi^{\frac{2}{3}})}\right]^2v_0^{*-1}=\frac{3(0.577)^2 3}{2u_0}\left[\frac{\frac{\mathrm{d}p}{\mathrm{d}x}}{\mu(1-\varphi^{\frac{2}{3}})}\right]^2\delta_0^{*2}\nu t$$

$$=\frac{2.996\nu t}{\mu(1-\varphi^{\frac{2}{3}})}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{13-5-16}$$

确定近壁流涡旋体积分数 φ，将式(13-5-16)代入式(13-3-22)

$$\varphi=\frac{[秒]2.996\pi\nu t}{48\delta_0^*\mu(1-\varphi^{\frac{2}{3}})}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{m}$$

将式(13-5-9)代入式(m)

$$\varphi=\frac{[秒]2.996\pi\nu t}{1.414\times 48\sqrt{u_0}}\left[\frac{\frac{\mathrm{d}p}{\mathrm{d}x}}{\mu(1-\varphi^{\frac{2}{3}})}\right]^{\frac{3}{2}} \tag{13-5-17}$$

例 13-2 一有压圆形管道，直径 1m，长为 5000m，通过水流速度为 $v_0=u_0=3\text{m/s}$，运

动黏性系数 $\nu=1.141\times10^{-6}\mathrm{m^2/s}$，动力黏性系数 $\mu=1.14\times10^{-3}\mathrm{N\cdot s/m^2}$，测得水头损失 $h_f=45.41\mathrm{m}$，确定近壁流厚度 δ_0^*，边层流界面位置 z_B，产生的涡旋强度及大小，涡旋在垂直边层流界面方向的速度 u^*，确定涡旋体积分数 φ_K，准定常流时间 $t=T_K$。

分析　$\Delta p=\gamma h_f=\dfrac{45.41\times1000}{1000}=45.41\mathrm{N/m^2}$

$$\frac{\mathrm{d}p}{\mathrm{d}x}=\frac{45.41}{m^3}\mathrm{N}$$

$$Re=\frac{v_0D}{\nu}=\frac{3\times1\times10^{-6}}{1.141}=2.629\times10^{-6}>2320$$

解　首先确定涡旋体积分数 φ_K，准定常 T_K，依式(13-5-17)

$$y_1=1.414\times48\varphi=67.872\varphi \tag{a}$$

$$y_2=\frac{[\text{秒}]2.996\pi\nu T_K}{\sqrt{u_0}}\left[\frac{\dfrac{\mathrm{d}p}{\mathrm{d}x}}{\mu(1-\varphi^{\frac{2}{3}})}\right]^{\frac{3}{2}}$$

$$=\frac{[\text{秒}]2.996\times3.1416\times1.141\times10^{-6}T_K}{\sqrt{3}}\left[\frac{45.41}{1.14\times10^{-3}(1-\varphi^{\frac{2}{3}})}\right]^{\frac{3}{2}}$$

$$=\frac{6.20\times10^{-6}T_K\times7950000}{(1-\varphi^{\frac{2}{3}})^{1.5}}=\frac{6.2\times7.95T_K}{(1-\varphi^{\frac{2}{3}})^{1.5}}=\frac{49.29T_K}{(1-\varphi^{\frac{2}{3}})^{1.5}} \tag{b}$$

设不同 φ，计算 y_1 与 y_2，列表于 13-2。

表 13-2　　y_1 与 y_2 计算结果

y_1	φ	$1-\varphi^{\frac{2}{3}}$	$(1-\varphi^{\frac{2}{3}})^{1.5}$	y_2		
				$T_K=0.01\mathrm{s}$	$T_K=0.05\mathrm{s}$	$T_K=0.1\mathrm{s}$
0.678	0.01	0.954	0.932	0.531	2.65	5.30
3.378	0.05	0.864	0.803	0.616	3.08	6.16
6.78	0.10	0.758	0.660	0.740	3.70	7.40
10.17	0.15	0.720	0.611	0.809	4.05	8.10
13.56	0.20	0.658	0.534	0.923	4.60	9.02

将表 13-2 中数据绘图，如图 13-9，分析结果，$T_K=0.1\mathrm{s}$ 合理。对应 $\varphi_K=0.11$，这样其他所求问题均可以解决。选确定近壁流厚度 δ_0^*，由式(13-5-9)

$$\delta_0^*=\sqrt{\frac{2\mu u_0(1-\varphi^{\frac{2}{3}})}{\dfrac{\mathrm{d}p}{\mathrm{d}x}}}=\sqrt{\frac{2\times1.14\times10^{-3}\times3(1-0.11^{\frac{2}{3}})}{45.41}}=0.204\sqrt{136230}$$

$$=0.204\times352.173=71.84\mathrm{rad/s}$$

涡旋直径

$$d_s=2\gamma_s=2\sqrt{\frac{10\nu}{\omega|_{z=z_B}}}=\frac{2}{10^2}\sqrt{\frac{1.141}{71.84\times10}}=\frac{2}{10^2}\sqrt{\frac{1.141}{718.4}}=\frac{2}{10^2}\sqrt{0.0016}$$

$$=\frac{2\times 0.04}{10^4}=\frac{0.08}{10^2}=0.0008\text{m}=0.8\text{mm}$$

确定边层流界面上涡旋横向运动速度 $u^*_{zy^0}$，由式(13-5-16)，$t=T_K=0.1\text{s}$

$$u^*\big|_{zy^0}=\frac{2.996\nu t}{\mu(1-\varphi^{\frac{2}{3}})}\frac{\mathrm{d}p}{\mathrm{d}x}=\frac{2.996\times 1.141\times 10^{-6}\times 45.41}{1.14\times 10^{-3}(1-0.11^{\frac{2}{3}})}=\frac{15.51\times 10^{-3}}{1.10}=0.014\text{m/s}$$

验证近壁流为湍流

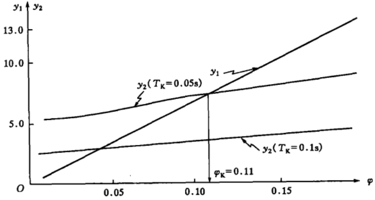


图 13-9 例 13-2 计算结果

$$R_0=\frac{v_0^*\delta_0^*}{\nu}=\frac{\frac{2}{3}\times 3\times 0.012\times 10^6}{1.141}=12000>580$$

后　记

综述：阐明湍流形成机理；建立“边层流”“近壁流”理论；导出湍流“第一输运公式”“第二输运公式”；推出直角、柱、球三种坐标系下不可压缩与可压缩的连续性、动量、动量矩、能量微分方程；对能用直角坐标系研究的一些实际问题进行求解计算，得出相应的速度分布、涡旋产生地带、涡旋运动速度、涡旋直径、涡旋体积分数，使问题得到本质的、深层次的解决。这也为其他有关学科的深入研究打下了理论基础。

在研究湍流运动过程中，遇到理想流体与实际流体分界问题，也就是数学中的定义域问题；另一个是非线性偏微分方程如何求解问题。对这两个问题，花费两年左右的时间才创立出解决它们的方法。

为何取名《工程紊（湍）流力学》？因为本书导出的理论公式已应用到许多工程实际问题，如管道、管道进口段、明渠、附面层等问题上，均得到可以参考的结果。

书中未列参考资料，原因是著者的思路与处理问题的方法均与他人不同，无法参阅别人的结果。

关于没有实验验证理论的问题说明：由于著者已离休多年，无法组织人力、无法筹集资金组织实验。希望有条件的国内外单位或专家进行这方面的工作。

本书最适合湍流科研工作者，让他们看到与自己不同思路的研究方法。

感谢东北大学领导左良副校长、教务处沈峰满副处长的大力支持，使得本书得以顺利出版。

著　者

2009 年 8 月